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Multi-Grounded Partitions and Character Formulas

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Abstract. We introduce a new generalisation of partitions, multi-grounded partitions, related to ground state paths indexed by dominant weights of Lie algebras. We use these to express characters of irreducible highest weight modules of Kac–Moody algebras of affine type as generating functions for multi-grounded partitions, and obtain new non-specialised character formulas.

Keywords: integer partitions, coloured partitions, representations of affine Lie algebras, character formulas, crystal bases

1 Introduction and statement of results

Let $\mathfrak{g}$ be a Kac–Moody affine Lie algebra and let $\mathfrak{h}^*$ be the dual of its Cartan subalgebra. Let P^+ be the set of dominant integral weights, and $L(\lambda)$ an irreducible highest weight $\mathfrak{g}$-module of highest weight $\lambda \in P^+$. Then the character of $L(\lambda)$ is defined as

$$\mathrm{ch}(L(\lambda)) = \sum_{\mu \in \mathfrak{h}^*} \dim L(\lambda)_\mu \cdot e^\mu,$$

where e is a formal exponential, and $\dim L(\lambda)_\mu$ is the dimension of the weight space $L(\lambda)_\mu$. Character formulas have been widely studied, starting with the famous Weyl–Kac character formula [8]:

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}, \tag{1.1}$$

where W is the Weyl group of $\mathfrak{g}$, Δ^+ the set of positive roots of $\mathfrak{g}$, $\mathrm{sgn}(w)$ the signature of w, $\rho \in \mathfrak{h}^*$ the Weyl vector, and $\mathfrak{g}_\alpha$ the α root space of $\mathfrak{g}$.

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Equation (1.1) is beautiful, but it is not so well suited to compute characters in practice. Moreover, even though by definition $e^{-\lambda}\mathrm{ch}(L(\lambda))$ is a series with positive coefficients in the $e^{-\alpha_i}$'s, this positivity is not explicit from the formulation in (1.1). We now briefly explain what solutions have been given to work around these issues, and present a new method which allows us to give simple non-specialised character formulas using perfect crystals and a new generalisation of integer partitions.

The first solution to obtain simple character formulas is to perform certain specialisations, *i.e.* for each of the simple roots α_i, applying the transformations $e^{-\alpha_i} \to q^{s_i}$ for some integer s_i. Using this method, it is possible to transform the Weyl–Kac character formula into infinite products. In particular, the principal specialisation, where $e^{-\alpha_i} \to q$ for all i, has been widely exploited in the theory of partition identities related to representations of affine Lie algebras, see for example [2, 6, 14, 15]. Lepowsky and Milne [12] were the first to expose the connection by noting that up to a $(q;q^2)_\infty$ factor, the principal specialisation of the Weyl–Kac character formula for level 3 standard modules of the affine Lie algebra $A_1^{(1)}$ is the product side of the Rogers–Ramanujan identities:

$$\sum_{n\geq 0} \frac{q^{n(n+i)}}{(q;q)_n} = \frac{1}{(q^{1+i},q^{4-i};q^5)_\infty}, \quad i \in \{0,1\}.$$

Here and in the whole paper, we use the standard q-series notation: for $n \in \mathbb{N} \cup \{\infty\}$ and $j \in \mathbb{N}$, $(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$ and $(a_1,\dots,a_j;q)_n := (a_1;q)\cdots(a_j;q)$.

Lepowsky and Wilson [13] later gave an interpretation of the sum side by constructing a basis of these standard modules using vertex operators. Their method has then led to the discovery of many new q-series and partition identities, see, *e.g.*, [2, 15, 17, 19]. However, without performing a specialisation, it is in general difficult to reduce the Weyl–Kac character formula to obtain a combinatorially simple character formula.

On the other hand, Bartlett and Warnaar [1] gave non-specialised formulas with explicitly positive coefficients for the characters of certain highest weight modules of the affine Lie algebras $C_n^{(1)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$ as sums using Hall–Littlewood polynomials. This led them to generalisations for the Macdonald identities for $B_n^{(1)}$, $C_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$. Using Macdonald–Koornwinder theory, Rains and Warnaar [18] found additional character formulas for these Lie algebras, together with new Rogers–Ramanujan type identities.

In a different direction, Kang, Kashiwara, Misra, Miwa, Nakashima, and Nakayashiki [9, 10] introduced the theory of perfect crystals to study the irreducible highest weight modules over quantum affine algebras. The proved the so-called "(KMN)^2 crystal base character formula" [9],

$$\mathrm{ch}(L(\lambda)) = \sum_{\mathfrak{p}\in\mathcal{P}(\lambda)} e^{\mathrm{wt}\,\mathfrak{p}},$$

which expresses the character $\mathrm{ch}(L(\lambda))$ as a series indexed by λ-paths (see Section 2). Here the weight $\mathrm{wt}\mathfrak{p}$ is computed using the energy function of a perfect crystal.

Primc [16] was the first to use this character formula to give new Rogers–Ramanujan type identities related to the level 1 standard modules of $A_1^{(1)}$ and $A_2^{(1)}$. The product side came from the principally specialised Weyl–Kac character formula again, while the sum side came from the principally specialised (KMN)^2 crystal base character formula. In a couple of previous papers [3, 4], we generalised Primc's identities to $A_{n-1}^{(1)}$ for all n, and managed to avoid doing a specialisation. To do this, we established a bijection between λ-paths and a new generalisation of partitions, in the case where the ground state path is constant.

But the case of standard modules whose ground state paths are not constant was still open, and the goal of [5], of which this article is an extended abstract, was to extend our method to treat this case as well. To do so, we introduced so-called "multi-grounded partitions". We explain their connection with crystals and characters in Section 2. Our main result is a character formula using generating functions for multi-grounded partitions (see Theorem 2.7). We proved it by transforming the (KMN)^2 crystal base character formula using a bijection with λ-paths. One big advantage of this formula is that one does not need to perform a specialisation, and that, being generating functions for combinatorial objects, the series always have obviously positive coefficients. Moreover, as we shall see on an example (Theorem 1.1) in Section 3, these generating functions are relatively easy to compute in practice.

In [5], we used multi-grounded partitions to compute new character formulas for irreducible highest weight level one modules of classical Lie algebras. All these formulas are non-specialised, with obviously positive coefficients, and are either infinite products or sums of two infinite products.

Let $G = G(x_1, \dots, x_n)$ be a power series in several variables $x_1, \dots, x_n$. For $k \leq n$, we denote by $\mathcal{E}_{x_1,\dots,x_k}(G)$ the sub-series of G where we only keep the terms in which the sum of the powers of $x_1, \dots, x_k$ is even. Note that if G has only positive coefficients, then the same is true for $\mathcal{E}_{x_1,\dots,x_k}(G)$ for all k. There is a simple formula to obtain $\mathcal{E}_{x_1,\dots,x_k}(G)$ from G:

$$\mathcal{E}_{x_1,\dots,x_k}(G) = \frac{1}{2}\Big(G(x_1, \dots, x_k, x_{k+1}, \dots, x_n) + G(-x_1, \dots, -x_k, x_{k+1}, \dots, x_n)\Big). \tag{1.2}$$

Our character formulas are the following.

Theorem 1.1. *Let $n \geq 3$, and let $\Lambda_0, \dots, \Lambda_n$ be the fundamental weights and $\alpha_0, \dots, \alpha_n$ be the simple roots of $A_{2n-1}^{(2)}$. Let $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$ be the null root. Set*

$$q = e^{-\delta/2}, \quad c_i = e^{\alpha_i + \cdots + \alpha_{n-1} + \alpha_n/2} \text{ for all } i \in \{1, \dots, n\},$$

$$\text{and} \quad \Pi_{a,b} = \prod_{k=a}^{b} (-c_k q^2; q^2)_\infty (-c_k^{-1}; q^2)_\infty.$$

We have

$$e^{-\Lambda_0}\mathrm{ch}(L(\Lambda_0)) = \mathcal{E}_{c_1,\dots,c_n}\left((q^2;q^4)_\infty \Pi_{1,n}\right), \tag{1.3}$$
$$e^{-\Lambda_1}\mathrm{ch}(L(\Lambda_1)) = \mathcal{E}_{c_1,\dots,c_n}\left((q^2;q^4)_\infty(-c_1q^3;q^2)_\infty(-c_1^{-1}q^{-1};q^2)_\infty \Pi_{2,n}\right).$$

The next theorem concerns the Lie algebra $B_n^{(1)}$. Note that the second author proved a character formula for $L(\Lambda_n)$, another level 1 module, in [11].

Theorem 1.2. *Let $n \geq 3$, and let $\Lambda_0,\dots,\Lambda_n$ be the fundamental weights and $\alpha_0,\dots,\alpha_n$ be the simple roots of $B_n^{(1)}$. Let $\delta = \alpha_0+\alpha_1+2\alpha_2+\dots+2\alpha_n$ be the null root. Set*

$$q = e^{-\delta/2}, \quad c_0 = 1, \quad c_i = e^{\alpha_i+\dots+\alpha_{n-1}+\alpha_n} \text{ for all } i \in \{1,\dots,n\},$$
$$\text{and} \quad \Pi_{a,b} = \prod_{k=a}^{b}(-c_kq^2;q^2)_\infty(-c_k^{-1};q^2)_\infty.$$

We have

$$e^{-\Lambda_0}\mathrm{ch}(L(\Lambda_0)) = \mathcal{E}_{c_0,c_1,\dots,c_n}\left((-c_0q;q^2)_\infty \Pi_{1,n}\right),$$
$$e^{-\Lambda_1}\mathrm{ch}(L(\Lambda_1)) = \mathcal{E}_{c_0,c_1,\dots,c_n}\left((-c_0q;q^2)_\infty(-c_1q^3;q^2)_\infty(-c_1^{-1}q^{-1};q^2)_\infty \Pi_{2,n}\right).$$

We conclude with the four level 1 standard modules of $D_n^{(1)}$.

Theorem 1.3. *Let $n \geq 4$, and let $\Lambda_0,\dots,\Lambda_n$ be the fundamental weights and $\alpha_0,\dots,\alpha_n$ be the simple roots of $D_n^{(1)}$. Let $\delta = \alpha_0+\alpha_1+2\alpha_2+\dots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$ is the null root. Set*

$$q = e^{-\delta/2}, \quad c_i = e^{\alpha_i+\dots+\alpha_{n-2}+\alpha_{n-1}/2+\alpha_n/2} \text{ for all } i \in \{1,\dots,n\},$$
$$\text{and} \quad \Pi_{a,b} = \prod_{k=a}^{b}(-c_kq^2;q^2)_\infty(-c_k^{-1};q^2)_\infty.$$

We have

$$e^{-\Lambda_0}\mathrm{ch}(L(\Lambda_0)) = \mathcal{E}_{c_1,\dots,c_n}\Pi_{1,n},$$
$$e^{-\Lambda_1}\mathrm{ch}(L(\Lambda_1)) = \mathcal{E}_{c_1,\dots,c_n}\left((-c_1q^3;q^2)_\infty(-c_1^{-1}q^{-1};q^2)_\infty \Pi_{2,n}\right),$$
$$e^{-\Lambda_{n-1}}\mathrm{ch}(L(\Lambda_{n-1})) = \mathcal{E}_{c_1,\dots,c_n}\left((-c_n;q^2)_\infty(-c_n^{-1}q^2;q^2)_\infty \Pi_{1,n-1}\right),$$
$$e^{-\Lambda_n}\mathrm{ch}(L(\Lambda_n)) = \mathcal{E}_{c_1,\dots,c_n}\left((-c_nq^2;q^2)_\infty(-c_n^{-1};q^2)_\infty \Pi_{1,n-1}\right).$$

Some more formulas of the same kind, but involving only products (not sums of products), can be found in Wakimoto's book [20, pages 50–58].

2 Perfect crystals and multi-grounded partitions

In this section, we define multi-grounded partitions and make the connection with characters of Lie algebra modules. We start by briefly recalling the theory of perfect crystals [7]. Let $\mathfrak{g}$ be an affine Kac–Moody algebra with simple positive roots $\alpha_0, \cdots, \alpha_n$ and with null root $\delta = d_0\alpha_0 + \cdots + d_n\alpha_n$. For $\lambda \in \overline{P}^+$, let $\mathcal{B}(\lambda)$ be the crystal graph of a crystal basis of $L(\lambda)$. For an integer level $\ell \geq 1$ and a weight $\lambda \in \overline{P}_\ell^+$, Kashiwara *et al.* [9, Section 1.4] define the notion of a *perfect crystal $\mathcal{B}$ of level ℓ*, an *energy function* $H\colon \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$, and a particular element

$$\mathfrak{p}_\lambda = (g_k)_{k=0}^{\infty} = \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0 \in \mathcal{B}^\infty,$$

called the *ground state path of weight λ*. From this they consider all elements of the form

$$\mathfrak{p} = (p_k)_{k=0}^{\infty} = \cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0 \in \mathcal{B}^\infty,$$

which satisfy $p_k = g_k$ for large enough k. Such elements are called λ-paths; their collective set is denoted $\mathcal{P}(\lambda)$. The crystal $\mathcal{B}(\lambda)$ can then be realised on the set of λ-paths, in particular the affine weight function is given by the following theorem.

Theorem 2.1 ($(\mathrm{KMN})^2$ crystal base character formula [9])**.** *Let $\lambda \in \overline{P}_\ell^+$, let H be an energy function on $\mathcal{B} \otimes \mathcal{B}$, and let $\mathfrak{p} = (p_k)_{k=0}^{\infty} \in \mathcal{P}(\lambda)$. Then the weight of $\mathfrak{p}$ and the character of the irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-module $L(\lambda)$ are given by the following expressions:*

$$\mathrm{wt}\mathfrak{p} = \lambda + \sum_{k=0}^{\infty} \left((\overline{\mathrm{wt}} p_k - \overline{\mathrm{wt}} g_k) - \frac{\delta}{d_0} \sum_{j=k}^{\infty} (H(p_{j+1} \otimes p_j) - H(g_{j+1} \otimes g_j)) \right),$$

$$\mathrm{ch}(L(\lambda)) = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{\mathrm{wt}\mathfrak{p}}.$$

To make the connection between character formulas and partitions, we introduce the concept of *multi-grounded partitions*.

Definition 2.2. Let $\mathcal{C}$ be a set of colours, and let $\mathbb{Z}_\mathcal{C} = \{k_c : k \in \mathbb{Z}, c \in \mathcal{C}\}$ be the set of integers coloured with the colours of $\mathcal{C}$. Let $\succ$ be a binary relation defined on $\mathbb{Z}_\mathcal{C}$. A *generalised coloured partition* with relation $\succ$ is a finite sequence $(\pi_0, \ldots, \pi_s)$ of coloured integers, such that for all $i \in \{0, \ldots, s-1\}$, $\pi_i \succ \pi_{i+1}$.

In the following, if $\pi = (\pi_0, \ldots, \pi_s)$ is a generalised coloured partition, then $c(\pi_i) \in \mathcal{C}$ denotes the colour of the part π_i. The quantity $|\pi| = \pi_0 + \cdots + \pi_s$ is the weight of π, and $C(\pi) = c(\pi_0) \cdots c(\pi_s)$ is its colour sequence.

Definition 2.3. Let $\mathcal{C}$ be a set of colors, $\mathbb{Z}_{\mathcal{C}}$ the set of integers coloured with colours in $\mathcal{C}$, and $\succ$ a binary relation defined on $\mathbb{Z}_{\mathcal{C}}$. Suppose that there exist some colors $c_{g_0},\dots,c_{g_{t-1}}$ in $\mathcal{C}$ and **unique** coloured integers $u^{(0)}_{c_{g_0}},\dots,u^{(t-1)}_{c_{g_{t-1}}}$ such that

$$u^{(0)}+\cdots+u^{(t-1)}=0, \tag{2.1}$$

$$u^{(0)}_{c_{g_0}}\succ u^{(1)}_{c_{g_1}}\succ\cdots\succ u^{(t-1)}_{c_{g_{t-1}}}\succ u^{(0)}_{c_{g_0}}. \tag{2.2}$$

Then a *multi-grounded partition* with ground $c_{g_0},\dots,c_{g_{t-1}}$ and relation $\succ$ is a non-empty generalised coloured partition $\pi=(\pi_0,\dots,\pi_{s-1},u^{(0)}_{c_{g_0}},\dots,u^{(t-1)}_{c_{g_{t-1}}})$ with relation $\succ$, such that $(\pi_{s-t},\cdots,\pi_{s-1})\neq(u^{(0)}_{c_{g_0}},\dots,u^{(t-1)}_{c_{g_{t-1}}})$ in terms of coloured integers. Let $\mathcal{P}^{\succ}_{c_{g_0}\cdots c_{g_{t-1}}}$ denote the set of multi-grounded partitions with ground $g_0,\dots,g_{t-1}$ and relation $\succ$.

Example 2.4. Let us consider the set of colours $\mathcal{C}=\{c_1,c_2,c_3\}$, the matrix

$$M=\begin{pmatrix}2&2&2\\0&0&2\\-2&0&2\end{pmatrix},$$

and define the relation $\succ$ on $\mathbb{Z}_{\mathcal{C}}$ by $k_{c_b}\succ k'_{c_{b'}}$ if and only if $k-k'\geq M_{b,b'}$.

If we choose $(g_0,g_1)=(1,3)$, the pair $(u^{(0)},u^{(1)})=(1,-1)$ is the unique pair satisfying (2.1) and (2.2). The generalised coloured partitions $(3_{c_3},3_{c_2},3_{c_1},-1_{c_3},1_{c_1},-1_{c_3})$ and $(1_{c_3},3_{c_1},1_{c_3},3_{c_1},-1_{c_3},1_{c_1},-1_{c_3})$ are examples of multi-grounded partitions with ground c_1,c_3 and relation $\succ$, while $(1_{c_1},-1_{c_3},1_{c_1},-1_{c_3})$ and $(2_{c_1},1_{c_1},-1_{c_3})$ are not.

Let $\mathcal{B}$ be a perfect crystal of level ℓ, and let $\lambda\in\overline{P}^+_\ell$ be a level ℓ dominant classical weight with ground state path $\mathfrak{p}_\lambda=(g_k)_{k\geq 0}$. The fact that the set P_ℓ is finite implies the periodicity of the sequence $(g_i)_{i\geq 0}$. Let us set t to be the *period* of the ground state path, *i.e.* the smallest non-negative integer k such that $g_k=g_0$. Let H be an energy function on $\mathcal{B}\otimes\mathcal{B}$, which is uniquely determined by fixing its value on a particular $b_0\otimes b'_0\in\mathcal{B}\otimes\mathcal{B}$.

We now define the function H_λ, for all $b,b'\in\mathcal{B}\otimes\mathcal{B}$, by

$$H_\lambda(b\otimes b'):=H(b\otimes b')-\frac{1}{t}\sum_{k=0}^{t-1}H(g_{k+1}\otimes g_k)\,. \tag{2.3}$$

Thus we have

$$\sum_{k=0}^{t-1}H_\lambda(g_{k+1}\otimes g_k)=0.$$

The function H_λ satisfies all the properties of energy functions, except that it only has integer values when t divides $\sum_{k=0}^{t-1}H(g_{k+1}\otimes g_k)$.

Note that for any energy function H, we always have

$$\sum_{k=0}^{t-1}(k+1)H_\lambda(g_{k+1}\otimes g_k) = \sum_{k=0}^{t-1}(k+1)H(g_{k+1}\otimes g_k) - \frac{t+1}{2}\sum_{k=0}^{t-1}H(g_{k+1}\otimes g_k) \in \frac{1}{2}\mathbb{Z}.$$

The quantity above is an integer as soon as t is odd, and is equal to 0 when $t=1$. Thus we can choose a suitable divisor D of $2t$ such that $DH_\lambda(\mathcal{B}\otimes\mathcal{B}) \subset \mathbb{Z}$ and $\frac{1}{t}\sum_{k=0}^{t-1}(k+1)DH_\lambda(g_{k+1}\otimes g_k) \in \mathbb{Z}$. In the whole paper, D always denotes such an integer.

Let us now consider the set of colours $\mathcal{C}_\mathcal{B}$ indexed by $\mathcal{B}$, and let us define, for the remainder of this paper, the relations $\gtrdot$ and $\gg$ on $\mathbb{Z}_{\mathcal{C}_\mathcal{B}}$ by

$$k_{c_b} \gtrdot k'_{c_{b'}} \text{ if and only if } k-k' = DH_\lambda(b'\otimes b),$$
$$k_{c_b} \gg k'_{c_{b'}} \text{ if and only if } k-k' \geq DH_\lambda(b'\otimes b).$$

We can define multi-grounded partitions associated with these relations, as can be seen in the next proposition.

Proposition 2.5. *The set $\mathcal{P}^{\gtrdot}_{c_{g_0}\cdots c_{g_{t-1}}}$ (resp. $\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$) of multi-grounded partitions with ground $c_{g_0},\dots,c_{g_{t-1}}$ and relation $\gtrdot$ (resp. $\gg$) is the set of non-empty generalised coloured partitions $\pi = (\pi_0,\cdots,\pi_{s-1},u^{(0)}_{c_{g_0}},\dots,u^{(t-1)}_{c_{g_{t-1}}})$ with relation $\gtrdot$ (resp. $\gg$), such that $(\pi_{s-t},\cdots,\pi_{s-1}) \neq (u^{(0)}_{c_{g_0}},\dots,u^{(t-1)}_{c_{g_{t-1}}})$, and for all $k\in\{0,\dots,t-1\}$,*

$$u^{(k)} = -\frac{1}{t}\sum_{j=0}^{t-1}(j+1)DH_\lambda(g_{j+1}\otimes g_j) + \sum_{j=k}^{t-1}DH_\lambda(g_{j+1}\otimes g_j). \tag{2.4}$$

Moreover, for any positive integer d, we denote by ${}^{d}\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ the set of multi-grounded partitions $\pi = (\pi_0,\cdots,\pi_{s-1},u^{(0)}_{c_{g_0}},\dots,u^{(t-1)}_{c_{g_{t-1}}})$ of $\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ such that for all $k\in\{0,\dots,s-1\}$, $\pi_k - \pi_{k+1} - DH_\lambda(p_{k+1}\otimes p_k) \in d\mathbb{Z}_{\geq 0}$, where $c(\pi_k) = c_{p_k}$ and $\pi_s = u^{(0)}_{c_{g_0}}$. Finally, let ${}^{d}_{t}\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ (resp. ${}_{t}\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$, ${}_{t}\mathcal{P}^{\gtrdot}_{c_{g_0}\cdots c_{g_{t-1}}}$) denote the set of multi-grounded partitions of ${}^{d}\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ (resp. $\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$, $\mathcal{P}^{\gtrdot}_{c_{g_0}\cdots c_{g_{t-1}}}$) with number of parts divisible by t.

Example 2.6. Assume that for all $b,b'\in\mathcal{B}$, $DH_\lambda(b'\otimes b) = M_{b,b'}$ given in Example 2.4. Then ${}^{2}\mathcal{P}^{\gg}_{c_1,c_3}$ is the set of multi-grounded partitions $\pi = (\pi_0,\cdots,\pi_{s-1},1_{c_1},-1_{c_3})$ of $\mathcal{P}^{\gg}_{c_1,c_3}$ such that for all $k\in\{0,\dots,s-1\}$, $\pi_k-\pi_{k+1}-DH_\lambda(p_{k+1}\otimes p_k)\in 2\mathbb{Z}_{\geq 0}$. Given that all the values in the matrix M are even and that these multi-grounded partitions always end with $(1_{c_1},-1_{c_3})$, the multi-grounded partitions in ${}^{2}\mathcal{P}^{\gg}_{c_1,c_3}$ only have *odd* parts. For example, the multi-grounded partition $(7_{c_1},5_{c_2},3_{c_3},1_{c_2},1_{c_1},-1c_{c_3})$ belongs to ${}^{2}\mathcal{P}^{\gg}_{c_1,c_3}$. It also belongs to ${}^{2}_{2}\mathcal{P}^{\gg}_{c_1,c_3}$ as its number of parts is divisible by 2.

Now that we have introduced all the relevant notation, we can state our main theorem, which makes the connection between perfect crystals and multi-grounded partitions.

Theorem 2.7. *Setting $q = e^{-\delta/(d_0 D)}$ and $c_b = e^{\overline{\mathrm{wt}} b}$ for all $b \in \mathcal{B}$, we have $c_{g_0} \cdots c_{g_{t-1}} = 1$, and the character of the irreducible highest weight $U_q(\mathfrak{g})$-module $L(\lambda)$ is given by the following expressions:*

$$\sum_{\mu \in {}_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}} C(\pi) q^{|\pi|} = e^{-\lambda} \mathrm{ch}(L(\lambda)),$$

$$\sum_{\pi \in {}^d_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}} C(\pi) q^{|\pi|} = \frac{e^{-\lambda} \mathrm{ch}(L(\lambda))}{(q^d; q^d)_\infty}.$$

Theorem 2.7 is proved in detail in [5], using the (KMN)^2 crystal base character formula and the two following bijections.

Proposition 2.8. *Define a map $\phi \colon \mathcal{P}(\lambda) \to \mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ by $\phi(\mathfrak{p}_\lambda) = (u^{(0)}_{c_{g_0}}, \ldots, u^{(t-1)}_{c_{g_{t-1}}})$ and for all $\mathfrak{p} = (p_k)_{k \geq 0} \in \mathcal{P}(\lambda)$ different from $\mathfrak{p}_\lambda$,*

$$\phi(\mathfrak{p}) = (\pi_0, \cdots, \pi_{mt-1}, u^{(0)}_{c_{g_0}}, \ldots, u^{(t-1)}_{c_{g_{t-1}}}),$$

where for all $k \in \{0, \cdots, mt-1\}$, $c(\pi_k) = c_{p_k}$,

$$\pi_k = -\frac{1}{t} \sum_{j=0}^{t-1} (j+1) DH_\lambda(g_{j+1} \otimes g_j) + \sum_{j=k}^{mt-1} DH_\lambda(p_{j+1} \otimes p_j),$$

and m is the unique positive integer such that $(p_{(m-1)t}, \ldots, p_{mt-1}) \neq (g_0, \ldots, g_{t-1})$ and $(p_{m't}, \ldots, p_{(m'+1)t-1}) = (g_0, \ldots, g_{t-1})$ for all $m' \geq m$.

Then ϕ defines a bijection between $\mathcal{P}(\lambda)$ and the set ${}_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ of partitions of $\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ whose number of parts is divisible by t.

Proposition 2.9. *Let ${}^d\mathcal{P}$ be the set of classical partitions where all parts are divisible by d. There is a bijection Φ_d between ${}^d_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ and ${}_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}} \times {}^d\mathcal{P}$, such that if $\Phi_d(\pi) = (\mu, \nu)$, then $|\pi| = |\mu| + |\nu|$, and by setting $c_{g_0} \cdots c_{g_{t-1}} = 1$, we have $C(\pi) = C(\mu)$.*

The additional parameter d allows us to have a refined equality and to simplify some calculations. It is particularly useful when $DH_\lambda(\mathcal{B} \otimes \mathcal{B}) \in d\mathbb{Z}$, in which case the parts of our partitions all belong to the same congruence class modulo d.

3 Example of application: character formulas for the standard level 1 modules of the Lie algebra $A_{2n-1}^{(2)} (n \geq 3)$

The crystal $\mathcal{B}$ of the vector representation of $A_{2n-1}^{(2)} (n \geq 3)$ is given by the crystal graph in Figure 1 with, for all $u \in \{1, \ldots, n\}$, the weights $\overline{\mathrm{wt}}(u) = -\overline{\mathrm{wt}}(\overline{u}) = \frac{1}{2}\alpha_n + \sum_{i=u}^{n-1} \alpha_i$. Here, the null root is $\delta = \alpha_0 + \alpha_1 + \alpha_n + 2\sum_{i=2}^{n-1} \alpha_i$.

$b^{\Lambda_0} = b_{\Lambda_1} = 1 \qquad b^{\Lambda_1} = b_{\Lambda_0} = \overline{1}$

$\mathfrak{p}_{\Lambda_0} = (\cdots \overline{1}1\overline{1}1\overline{1}) \qquad \mathfrak{p}_{\Lambda_1} = (\cdots 1\overline{1}1\overline{1}1)$

$\mathcal{B}$:

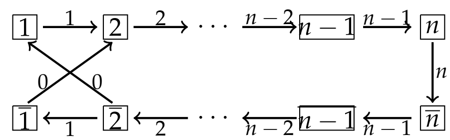

Figure 1: Crystal graph $\mathcal{B}$ of the vector representation for the Lie algebra $A_{2n-1}^{(2)} (n \geq 3)$.

The energy function such that $H(1 \otimes \overline{1}) = -1$ is given by the following matrix, where $H(b_1 \otimes b_2)$ is the entry in column b_1 and row b_2:

$$H = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ \overline{n} \\ \vdots \\ \overline{2} \\ \overline{1} \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \cdots & n & \overline{n} & \cdots & \overline{2} & \overline{1} \end{array} \\ \left(\begin{array}{cccccccc} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & & & 1^* & & \vdots \\ \vdots & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & & 0^* & & \ddots & \ddots & & \vdots \\ 0 & 0 & & & & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{array}\right) \end{array}.$$

We start by computing the character of $L(\Lambda_0)$. Recall that the ground state path of Λ_0 is $\mathfrak{p}_{\Lambda_0} = (g_k)_{k=0}^{\infty}$ with $g_{2k} = \overline{1}$ and $g_{2k+1} = 1$ for all $k \geq 0$. Here, the period of the ground state path is $t = 2$, and our choice of particular value $H(1 \otimes \overline{1}) = -1$ for the energy function gives $H(g_{2k+2} \otimes g_{2k+1}) = -H(g_{2k+1} \otimes g_{2k}) = 1$. Thus we have $H(1 \otimes \overline{1}) + H(\overline{1} \otimes 1) = 0$, and by (2.3), $H_{\Lambda_0} = H$. By (2.4), we obtain that $u^{(0)} = -1$ and $u^{(1)} = 1$.

We apply Theorem 2.7 with $d = 2$ and $D = 2$, which is allowed because $H(g_1 \otimes g_0) + 2H(g_2 \otimes g_1) = 1$. We obtain

$$\sum_{\pi \in {}^{2}_{2}\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}} C(\pi) q^{|\pi|} = \frac{e^{-\Lambda_0}\mathrm{ch}(L(\Lambda_0))}{(q^2;q^2)_\infty}, \tag{3.1}$$

where $q = e^{-\delta/2}$ and $c_b = e^{\overline{\mathrm{wt}}b}$ for all $b \in \mathcal{B}$. Recall that ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$ is the set of multi-grounded partitions $\pi = (\pi_0, \dots, \pi_{2s-1}, -1_{c_{\overline{1}}}, 1_{c_1})$ with relation $\gg$ and ground $c_{\overline{1}}, c_1$, **having an even number of parts**, such that for all $k \in \{0, \dots, 2s-1\}$,

$$\pi_k - \pi_{k+1} - 2H(p_{k+1} \otimes p_k) \in 2\mathbb{Z}_{\geq 0}, \tag{3.2}$$

where $c(\pi_k) = c_{p_k}$ and $\pi_{2s} = -1_{c_{\overline{1}}}$.

We observe that, by (3.2) and the fact that $u^{(0)} = -1$, the multi-grounded partitions of ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$ have parts with odd sizes, as the differences between consecutive parts are even and the grounds' sizes are odd (indeed, we always have the fixed tail $((-1)_{c_{\overline{1}}}, 1_{c_1})$). Besides, computing the generating function for partitions in ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$ is not too difficult. It suffices to notice that, combined with (3.2), $\gg$ is the following partial order on the set of coloured odd integers:

$$\begin{matrix} (-1)_{c_{\overline{1}}} \\ 1_{c_1} \end{matrix} \ll 1_{c_2} \ll \cdots \ll 1_{c_n} \ll 1_{c_{\overline{n}}} \ll \cdots \ll 1_{c_{\overline{2}}} \ll \begin{matrix} 1_{c_{\overline{1}}} \\ 3_{c_1} \end{matrix} \ll 3_{c_2} \ll \cdots .$$

We also note that, since $H(b \otimes b) = 1$ for all $b \in \mathcal{B}$, only parts coloured c_1 and $c_{\overline{1}}$ can appear several times, in sequences of the form

$$\cdots \ll (2k-1)_{c_{\overline{1}}} \ll (2k+1)_{c_1} \ll (2k-1)_{c_{\overline{1}}} \ll \cdots \ll (2k-1)_{c_{\overline{1}}} \ll (2k+1)_{c_1} \ll \cdots .$$

The generating function of these sequences for a fixed integer $k \geq 1$ is given by

$$\frac{(1 + c_{\overline{1}}q^{2k-1})(1 + c_1 q^{2k+1})}{(1 - c_{\overline{1}}c_1 q^{4k})},$$

where the denominator generates pairs $((2k-1)_{c_{\overline{1}}}, (2k+1)_{c_1})$ that can repeat arbitrarily many times, and the numerator accounts for the possibility of having an isolated $(2k+1)_{c_1}$ on the left end of the sequence, or an isolated $(2k-1)_{c_{\overline{1}}}$ on the right end of the sequence.

Note that for $k = 0$, only the sequence $(1_{c_1}, (-1)_{c_{\overline{1}}}, 1_{c_1})$ can occur at the tail of the partitions grounded in $c_{\overline{1}}, c_1$, but not the sequence $((-1)_{c_{\overline{1}}}, 1_{c_1}, (-1)_{c_{\overline{1}}}, 1_{c_1})$, as this would violate the definition of multi-grounded partitions. So, if we temporarily forget the condition on the even number of parts in ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$, the generation function would be

$$(1 + c_1 q) \cdot \frac{(-c_1 q^3, -c_{\overline{1}}q, \dots, -c_n q, -c_{\overline{n}}q; q^2)_\infty}{(c_{\overline{1}}c_1 q^4; q^4)_\infty} = \frac{(-c_1 q, -c_{\overline{1}}q, \dots, -c_n q, -c_{\overline{n}}q; q^2)_\infty}{(c_{\overline{1}}c_1 q^4; q^4)_\infty}.$$

Now to take into account the fact that there are an even number of parts, we use (1.2). Then the final expression (1.3) follows by using (3.1).

We now turn to $L(\Lambda_1)$ with a similar reasoning. Recall that the ground state path of Λ_1 is $(g_k)_{k=0}^{\infty}$ with $g_{2k+1} = \overline{1}$ and $g_{2k} = 1$ for all $k \geq 0$. We still have $H_{\Lambda_1} = H$, and by setting $D = 2$, we have by (2.4) that $u^0 = 1$ and $u^{(1)} = -1$. Theorem 2.7 gives

$$\sum_{\pi \in {}^2_2\mathcal{P}^{\gg}_{c_1 c_{\overline{1}}}} C(\pi) q^{|\pi|} = \frac{e^{-\Lambda_1}\mathrm{ch}(L(\Lambda_1))}{(q^2;q^2)_\infty},$$

where $q = e^{-\delta/2}$ and $c_b = e^{\overline{\mathrm{wt}} b}$ for all $b \in \mathcal{B}$.

So we need to study the set of multi-grounded partitions with ground $c_1, c_{\overline{1}}$ corresponding to ${}^2_2\mathcal{P}^{\gg}_{c_1 c_{\overline{1}}}$. We have almost the same set of partitions as in ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}} c_1}$, except that now the tail is always $(1_{c_1}, (-1)_{\overline{1}})$, and we can end with the sequence $((-1)_{\overline{1}}, 1_{c_1}, (-1)_{\overline{1}})$, but not with $(1_{c_1}, (-1)_{\overline{1}}, 1_{c_1}, (-1)_{\overline{1}})$.

Thus the generating function where we temporarily omit the condition on the parity of the number of parts is given by

$$(1 + c_{\overline{1}} q^{-1}) \cdot \frac{(-c_1 q^3, -c_{\overline{1}} q, \ldots, -c_n q, -c_{\overline{n}} q; q^2)_\infty}{(c_{\overline{1}} c_1 q^4; q^4)_\infty} = \frac{(-c_1 q^3, -c_{\overline{1}} q^{-1}, \ldots, -c_n q, -c_{\overline{n}} q; q^2)_\infty}{(c_{\overline{1}} c_1 q^4; q^4)_\infty},$$

and the proof ends similarly as for $L(\Lambda_0)$.

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References

[1] N. Bartlett and S. O. Warnaar. "Hall–Littlewood polynomials and characters of affine Lie algebras". *Adv. Math.* **285** (Nov. 2015), pp. 1066–1105. DOI.

[2] S. Capparelli. "On some representations of twisted affine Lie algebras and combinatorial identities". *J. Algebra* **154** (1993), pp. 335–355.

[3] J. Dousse and I. Konan. "Generalisations of Capparelli's and Primc's identities, I: Coloured Frobenius partitions and combinatorial proofs". 2019. arXiv:1911.13191.

[4] J. Dousse and I. Konan. "Generalisations of Capparelli's and Primc's identities, II: Perfect $A_{n-1}^{(1)}$ crystals and explicit character formulas". 2019. arXiv:1911.13189.

[5] J. Dousse and I. Konan. "Multi-grounded partitions and character formulas". 2021. arXiv:2103.04983.

[6] M. J. Griffin, K. Ono, and S. O. Warnaar. "A framework of Rogers–Ramanujan identities and their arithmetic properties". *Duke Math. J.* **8** (2016), pp. 1475–1527.

[7] J. Hong and S. Kang. *Introduction to Quantum Groups and Crystal Bases*. Vol. 42. Graduate Studies in Mathematics. American Mathematical Society, Feb. 2002.

[8] V. Kac. *Infinite Dimensional Lie Algebras*. 3rd edition. Cambridge University Press, 1990.

[9] S. Kang, M. Kashiwara, K. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki. "Affine Crystals and Vertex Models". *Int. Journ. Mod. Phys. A* **7** (Apr. 1992), pp. 449–484. DOI.

[10] S. Kang, M. Kashiwara, K. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki. "Perfect crystals of quantum affine Lie algebras". *Duke Math. J.* **68**.3 (Dec. 1992), pp. 499–607. DOI.

[11] I. Konan. "Weighted words at degree two, II: flat partitions, regular partitions, and application to level one perfect crystals". 2020. arXiv:2002.00300.

[12] J. Lepowsky and S. Milne. "Lie algebraic approaches to classical partition identities". *Adv. Math.* **29**.1 (1978), pp. 15–59.

[13] J. Lepowsky and R. L. Wilson. "The structure of standard modules, I: Universal algebras and the Rogers–Ramanujan identities". *Invent. Math.* **77** (1984), pp. 199–290.

[14] A. Meurman and M. Primc. "Annihilating ideals of standard modules of $sl(2,\mathbb{C})^{\sim}$ and combinatorial identities". *Adv. Math.* **64** (1987), pp. 177–240.

[15] D. Nandi. "Partition Identities Arising from Standard $A_2^{(2)}$-modules of Level 4". PhD thesis. Rutgers University, 2014.

[16] M. Primc. "Some crystal Rogers–Ramanujan type identities". *Glas. Math. Ser. III* **34** (1999), pp. 73–86.

[17] M. Primc and T. Šikić. "Combinatorial bases of basic modules for affine Lie algebras $C_n^{(1)}$". *J. Math. Phys.* **57**.9 (2016), p. 091701.

[18] E. M. Rains and S. O. Warnaar. "Bounded Littlewood identities". *Mem. Amer. Math. Soc.* (2015). To appear. arXiv:1506.02755.

[19] I. Siladić. "Twisted $sl(3,\mathbb{C})^{\sim}$-modules and combinatorial identities". *Glasnik Matematicki,* **52**.1 (2017), pp. 53–77.

[20] M. Wakimoto. *Lectures on Infinite-Dimensional Lie Algebra*. World Scientific, 2001. DOI.