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Flag Hilbert–Poincaré Series and Igusa Zeta Functions of Hyperplane Arrangements

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Abstract. We introduce and study a class of multivariate rational functions associated with hyperplane arrangements, called flag Hilbert–Poincaré series. These series are intimately connected with Igusa local zeta functions of products of linear polynomials, and their motivic and topological relatives. Our main results include a self-reciprocity result for central arrangements defined over fields of characteristic zero. We also prove combinatorial formulae for a specialization of the flag Hilbert–Poincaré series for irreducible Coxeter arrangements of types A, B, and D in terms of total partitions of the respective types. We show that a different specialization of the flag Hilbert–Poincaré series, which we call the coarse flag Hilbert–Poincaré series, exhibits intriguing non-negativity features and — in the case of Coxeter arrangements — connections with Eulerian polynomials. For numerous classes and examples of hyperplane arrangements, we determine their (coarse) flag Hilbert–Poincaré series. Some computations were aided by a SageMath package we developed.

Keywords: Hyperplane arrangements, Igusa's local zeta function, Eulerian polynomials, Stirling numbers of the second kind, Hilbert series, Hadamard products, topological zeta functions, total partitions, representable matroids, Coxeter arrangements, braid arrangements

1 Introduction

We present some of the main results of [7], an abridged version of [6]. A hyperplane arrangement over a field $\mathbb{K}$ is a finite set $\mathcal{A}$ of affine hyperplanes in $\mathbb{K}^d$ for some integer $d := \dim(\mathcal{A})$. We introduce and study a multivariate rational function $\mathsf{fHP}_{\mathcal{A}}(Y, \boldsymbol{T}) \in \mathbb{Q}(\boldsymbol{T})[Y]$, called the *flag Hilbert–Poincaré series* of $\mathcal{A}$, encompassing much of the topology and combinatorics of $\mathcal{A}$. As we shall explain, various substitutions yield connections to several enumeration problems associated with hyperplane arrangements.

In order to define $\mathsf{fHP}_{\mathcal{A}}$, we introduce some further notation. Let $\mathcal{L}(\mathcal{A})$ be the intersection poset of $\mathcal{A}$, ordered by reverse-inclusion. Two hyperplane arrangements are

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equivalent if their intersection posets are isomorphic. We denote by $\widehat{0}$ (resp. $\widehat{1}$) the bottom (resp. top) element of a poset (provided $\widehat{1}$ exists). Observe that $\widehat{1} \in \mathcal{L}(\mathcal{A})$ if and only if $\mathcal{A}$ is central, *i.e.* $\bigcap_{H\in\mathcal{A}} H \neq \varnothing$. Define $\widetilde{\mathcal{L}}(\mathcal{A}) = \mathcal{L}(\mathcal{A}) \setminus \{\widehat{0}\}$ and $\overline{\mathcal{L}}(\mathcal{A}) = \mathcal{L}(\mathcal{A}) \setminus \{\widehat{0}, \widehat{1}\}$. For $x \in \mathcal{L}(\mathcal{A})$, we write $\mathrm{rk}(x) := \mathrm{rk}_{\mathcal{L}(\mathcal{A})}(x)$ for the *rank* of x, viz. the supremum over the lengths of all chains from $\widehat{0}$ to x. For a poset P, the *order complex* $\Delta(P)$ associated with P is the simplicial complex with vertex set P, whose simplices are the flags of P. For $x \in \mathcal{L}(\mathcal{A})$, define hyperplane arrangements

$$\begin{aligned} \mathcal{A}_x &= \{H \in \mathcal{A} \mid x \subseteq H\} && \text{(subarrangement)},\\ \mathcal{A}^x &= \{x \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_x,\ x \cap H \neq \varnothing\} && \text{(restriction)}. \end{aligned}$$

Set $\mathcal{A}_\varnothing := \mathcal{A}$. Interlacing these constructions we obtain, for $x, y \in \mathcal{L}(\mathcal{A})$, the arrangement $\mathcal{A}_y^x := (\mathcal{A}^x)_y = (\mathcal{A}_y)^x$. Recall further the *Poincaré polynomial* [8, Definition 2.48]

$$\pi_{\mathcal{A}}(Y) = \sum_{x\in\mathcal{L}(\mathcal{A})} \mu(\widehat{0}, x)(-Y)^{\mathrm{rk}(x)} \in \mathbb{Z}[Y]$$

associated with $\mathcal{A}$, where μ is the Möbius function on $\mathcal{L}(\mathcal{A})$; *cf.* [15, Definition 3.15]. The Poincaré polynomial is closely related to the characteristic polynomial $\chi_{\mathcal{A}}(Y)$ of $\mathcal{A}$ via the identity (see [8, Definition 2.52])

$$\chi_{\mathcal{A}}(Y) = Y^d \pi_{\mathcal{A}}(-Y^{-1}).$$

We require the following flag generalization: for $F = (x_1 < x_2 < \cdots < x_\ell) \in \Delta(\mathcal{L}(\mathcal{A}))$ (possibly empty), set $x_0 = \widehat{0}$ and $x_{\ell+1} = \varnothing$, and define

$$\pi_F(Y) = \prod_{k=0}^{\ell} \pi_{\mathcal{A}_{x_{k+1}}^{x_k}}(Y) \in \mathbb{Z}[Y].$$

The following function is the main protagonist of [6].

Definition 1.1. Let $\boldsymbol{T} := (T_x)_{x\in\widetilde{\mathcal{L}}(\mathcal{A})}$ be indeterminates. The *flag Hilbert–Poincaré series* associated with $\mathcal{A}$ is

$$\mathsf{fHP}_{\mathcal{A}}(Y, \boldsymbol{T}) = \sum_{F\in\Delta(\widetilde{\mathcal{L}}(\mathcal{A}))} \pi_F(Y) \prod_{x\in F} \frac{T_x}{1 - T_x} \in \mathbb{Q}(\boldsymbol{T})[Y].$$

If $\mathcal{A}$ is central, then

$$\mathsf{fHP}_{\mathcal{A}}(Y, \boldsymbol{T}) = \frac{1}{1 - T_{\widehat{1}}} \sum_{F\in\Delta(\overline{\mathcal{L}}(\mathcal{A}))} \pi_F(Y) \prod_{x\in F} \frac{T_x}{1 - T_x}.$$

We remark that $\mathsf{fHP}_{\mathcal{A}}(0, \boldsymbol{T})$ is the (fine) Hilbert series of the Stanley–Reisner ring of the order complex of $\widetilde{\mathcal{L}}(\mathcal{A})$; see Proposition 4.2.

Example 1.2. Suppose $\mathsf{A}_2 := \{H_{ij} \mid 1 \leqslant i < j \leqslant 3\} \subset \mathbb{Q}^3$ is the arrangement, where H_{ij} is the set of zeros to $X_i - X_j$. This is also known as the Coxeter arrangement of type A_2. Then, writing T_{ij} for $T_{H_{ij}}$, we have

$$\mathsf{fHP}_{\mathsf{A}_2}(Y, \boldsymbol{T}) = \frac{1}{1 - T_{\hat{1}}}\left((1+Y)(1+2Y) + (1+Y)\left(\frac{T_{12}}{1-T_{12}} + \frac{T_{13}}{1-T_{13}} + \frac{T_{23}}{1-T_{23}}\right)\right).$$

The following self-reciprocity result for central arrangements over fields of characteristic zero is our first main theorem. The *rank* of $\mathcal{A}$, denoted by $\mathrm{rk}(\mathcal{A})$, is the rank of a maximal element of $\mathcal{L}(\mathcal{A})$.

Theorem A (Self-reciprocity)**.** *Let $\mathcal{A}$ be a central hyperplane arrangement over a field of characteristic zero. Then*

$$\mathsf{fHP}_{\mathcal{A}}\left(Y^{-1}, (T_x^{-1})_{x \in \widetilde{\mathcal{L}}(\mathcal{A})}\right) = (-Y)^{-\mathrm{rk}(\mathcal{A})} T_{\hat{1}} \cdot \mathsf{fHP}_{\mathcal{A}}(Y, \boldsymbol{T}).$$

The restriction to fields of characteristic zero reflects our method of proof rather than any known counterexamples in positive characteristic.

Various substitutions of the variables of the flag Hilbert–Poincaré series yield connections to seemingly different enumeration problems:

First, we explain in Section 2 that flag Hilbert–Poincaré series encode the same information as certain $\mathfrak{p}$-adic integrals associated with hyperplane arrangements (see Theorem B). In Section 3 we explicate the specific connections to the well-studied class of (both uni- and multivariate) *Igusa local zeta functions* associated with products of linear polynomials. In [6, Section 2.2], we also consider topological zeta functions, which are closely related to the Igusa zeta functions. Another bivariate substitution relates the flag Hilbert–Poincaré series of a hyperplane arrangement $\mathcal{A}$ with the motivic zeta function $Z_{M(\mathcal{A})}(Y, T)$ introduced in [3, Definition 1.1] associated with the (representable) matroid $M(\mathcal{A})$ determined by $\mathcal{A}$.

Second, we discuss in Section 2 an alternative combinatorial formula (see Theorem C) for specific multivariate substitutions, viz. *atom zeta functions*, associated with *classical* Coxeter arrangements — viz. irreducible Coxeter arrangements of types A, B, or D — in terms of total partitions and rooted trees.

Third, we focus in Section 4 on *coarse flag Hilbert–Poincaré series*, viz. the bivariate "coarsening" of the flag Hilbert–Poincaré series $\mathsf{fHP}_{\mathcal{A}}(Y, \boldsymbol{T})$ obtained by setting $T_x = T$ for all x. Our Theorem D presents coarse flag Hilbert–Poincaré series associated with Coxeter arrangements as "Y-analogs" of Hilbert series of the Stanley–Reisner rings of the first barycentric subdivisions of standard simplices. On the level of rational generating functions, this is reflected by an intriguing connection with Eulerian polynomials.

We record some examples of the rational functions $\mathsf{fHP}_{\mathcal{A}}$ in Section 5. The case of Boolean arrangements $\mathcal{A} = \mathsf{A}_1^n$ (see Section 5.1) is of particular interest: specific substitutions of the functions $\mathsf{fHP}_{\mathsf{A}_1^n}$ arise in the study [11] of the average sizes of kernels of generic matrices with support constraints over finite quotients of compact discrete valuation rings; see [6, Section 4.8].

2 Flag Hilbert–Poincaré series and $\mathfrak{p}$-adic integrals

For general arrangements $\mathcal{A}$ over fields of characteristic zero, the functions $\mathsf{fHP}_{\mathcal{A}}$ are universal objects from which various $\mathfrak{p}$-adic integrals associated with $\mathcal{A}$ may be obtained via specializations. To discuss this connection, we first recall some representability properties of hyperplane arrangements.

If K is a field and $\mathcal{A}_K$ is a hyperplane arrangement defined over K such that $\mathcal{L}(\mathcal{A}) \cong \mathcal{L}(\mathcal{A}_K)$ as posets, then we say that $\mathcal{A}$ is *K-representable* and call $\mathcal{A}_K$ a *K-representation* of $\mathcal{A}$. If, as we now assume, $\mathcal{A}$ is a hyperplane arrangement defined over a field $\mathbb{K}$ of characteristic zero, there exists a finite extension K of $\mathbb{Q}$ such that $\mathcal{A}$ is K-representable; *cf.* [9, Proposition 6.8.11]. Having fixed such a K-representation of $\mathcal{A}$, we may further assume, without loss of generality, that each $H \in \mathcal{A}$ is of the form $H = V(L)$, where $L(\boldsymbol{X}) = c_L + \sum_{j=1}^{d} \alpha_{L,j} X_j \in \mathcal{O}_K[\boldsymbol{X}]$ is an affine linear polynomial over $\mathcal{O}_K$, the ring of integers of the number field K. These choices allow us, in fact, to identify the arrangement $\mathcal{A}$ with the collection of polynomials L arising in this way. We will use this freedom frequently.

In the sequel we denote by $\mathfrak{o}$ a compact discrete valuation ring (cDVR) with an $\mathcal{O}_K$-module structure. This could be a finite extension of the completion $\mathcal{O}_{K,\mathfrak{p}}$ of $\mathcal{O}_K$ at a nonzero prime ideal $\mathfrak{p}$ (in characteristic zero) or a power series ring of the form $\mathbb{F}_q[\![X]\!]$, where $\mathbb{F}_q$ is the residue field of such a ring (in positive characteristic).

Denoting by $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{o}$, we write $\mathcal{A}(\mathfrak{o}/\mathfrak{p})$ for the reduction of $\mathcal{A}$ modulo $\mathfrak{p}$. If $\mathcal{L}(\mathcal{A}) \cong \mathcal{L}(\mathcal{A}(\mathfrak{o}/\mathfrak{p}))$, then $\mathcal{A}$ is said to have *good reduction over* $\mathbb{F}_q$, provided $\mathfrak{o}/\mathfrak{p}$ has cardinality q. It is well-known that $\mathcal{A}$ has good reduction over $\mathbb{F}_q$ for all such q not divisible by finitely many ("bad") primes; *cf.* [14, Chap. 5.1].

We now explain the connection between the flag Hilbert–Poincaré series $\mathsf{fHP}_{\mathcal{A}}$ and various (multi- and univariate) $\mathfrak{p}$-adic integrals associated with the hyperplane arrangement $\mathcal{A}$.

Definition 2.1. The *analytic zeta function* of $\mathcal{A}$ over $\mathfrak{o}$ is

$$\zeta_{\mathcal{A}(\mathfrak{o})}(\boldsymbol{s}) = \int_{\mathfrak{o}^{\dim(\mathcal{A})}} \prod_{x \in \widetilde{\mathcal{L}}(\mathcal{A})} \|\mathcal{A}_x\|^{s_x} \, |\mathrm{d}\boldsymbol{X}|,$$

where s_x is a complex variable for each $x \in \widetilde{\mathcal{L}}(\mathcal{A})$, further $\|\mathcal{X}\| := \max\{|f| \mid f \in \mathcal{X}\}$ for a finite set $\mathcal{X} \subset \mathfrak{o}$, where $|f|$ is the $\mathfrak{p}$-adic absolute value on $\mathfrak{o}$, and $|\mathrm{d}\boldsymbol{X}|$ is the additive Haar measure on $\mathfrak{o}^{\dim(\mathcal{A})}$, normalized so that $\mathfrak{o}^{\dim(\mathcal{A})}$ has measure 1.

Example 2.2. The analytic zeta function of A_2, *cf.* Example 1.2, over $\mathfrak{o}$ is

$$\begin{aligned}\zeta_{\mathsf{A}_2(\mathfrak{o})}(\boldsymbol{s}) &= \int_{\mathfrak{o}^3} |X_1 - X_2|^{s_{12}}|X_1 - X_3|^{s_{13}}|X_2 - X_3|^{s_{23}} \,\|X_1 - X_2, X_1 - X_3, X_2 - X_3\|^{s_{\hat{1}}}\, |\mathrm{d}\boldsymbol{X}| \\ &= \frac{1}{1 - q^{-2-s_{12}-s_{13}-s_{23}-s_{\hat{1}}}}\Bigg((1-q^{-1})(1-2q^{-1}) \\ &\qquad +(1-q^{-1})^2\left(\frac{q^{-1-s_{12}}}{1-q^{-1-s_{12}}} + \frac{q^{-1-s_{13}}}{1-q^{-1-s_{13}}} + \frac{q^{-1-s_{23}}}{1-q^{-1-s_{23}}}\right)\Bigg).\end{aligned}$$

One sees the striking similarity between the rational functions in Example 1.2 and Example 2.2. Our next main result establishes that the functions $\zeta_{\mathcal{A}(\mathfrak{o})}(\boldsymbol{s})$ and $\mathsf{fHP}_{\mathcal{A}}(Y, \boldsymbol{T})$ determine each other, in the following precise sense.

Theorem B. *Let $\mathcal{A}$ be a hyperplane arrangement over a number field K. For indeterminates $\boldsymbol{s} := (s_x)_{x\in\widetilde{\mathcal{L}}(\mathcal{A})}$ and $\boldsymbol{r} := (r_x)_{x\in\widetilde{\mathcal{L}}(\mathcal{A})}$ and $x \in \widetilde{\mathcal{L}}(\mathcal{A})$, let*

$$\begin{aligned} g_x(\boldsymbol{s}) &= \mathrm{rk}(x) + \sum_{y\in\widetilde{\mathcal{L}}(\mathcal{A}_x)} s_y, \\ h_x(\boldsymbol{r}) &= \sum_{y\in\widetilde{\mathcal{L}}(\mathcal{A}_x)} (r_y - \mathrm{rk}(y))\mu(y,x). \end{aligned}$$

If $\mathfrak{o}$ is a cDVR and an $\mathcal{O}_K$-module with residue field cardinality q such that $\mathcal{A}$ has good reduction over $\mathbb{F}_q$, then

$$\begin{aligned} \zeta_{\mathcal{A}(\mathfrak{o})}(\boldsymbol{s}) &= \mathsf{fHP}_{\mathcal{A}}\left(-q^{-1}, \left(q^{-g_x(\boldsymbol{s})}\right)_{x\in\widetilde{\mathcal{L}}(\mathcal{A})}\right), \\ \mathsf{fHP}_{\mathcal{A}}\left(-q^{-1}, (q^{-r_x})_{x\in\widetilde{\mathcal{L}}(\mathcal{A})}\right) &= \zeta_{\mathcal{A}(\mathfrak{o})}\left((h_x(\boldsymbol{r}))_{x\in\widetilde{\mathcal{L}}(\mathcal{A})}\right). \end{aligned}$$

The interpretation of $\mathsf{fHP}_{\mathcal{A}}$ in terms of the $\mathfrak{p}$-adic integrals $\zeta_{\mathcal{A}(\mathfrak{o})}$ expressed by Theorem B is key to our proof of Theorem A.

3 Igusa and atom zeta functions

Assume now, as in Section 2, that $\mathcal{A}$ is a K-representation and $\mathfrak{o}$ is a cDVR and an $\mathcal{O}_K$-module. An important specialization of the multivariate function $\zeta_{\mathcal{A}(\mathfrak{o})}(\boldsymbol{s})$ yields the (univariate) *Igusa local zeta function* (over $\mathfrak{o}$) associated with the product $f_{\mathcal{A}}(\boldsymbol{X}) := \prod_{L\in\mathcal{A}} L(\boldsymbol{X})$ of linear polynomials $L \in \mathcal{O}_K[\boldsymbol{X}]$ (see [2]):

$$\mathsf{Z}_{f_{\mathcal{A}},\mathfrak{o}}(s) := \zeta_{\mathcal{A}(\mathfrak{o})}\left(\left(s\cdot\delta_{|\mathcal{A}_x|=1}\right)_{x\in\widetilde{\mathcal{L}}(\mathcal{A})}\right) = \int_{\mathfrak{o}^{\dim(\mathcal{A})}} |f_{\mathcal{A}}|^s |\mathrm{d}\boldsymbol{X}|; \tag{3.1}$$

here s is a complex variable. Motivic zeta functions related with such integrals have been studied and can be used to understand the topological zeta function associated with $f_{\mathcal{A}}(\boldsymbol{X})$. This is executed, for example, in [1] for generic central arrangements, among others; *cf.* Section 5.2. See Section 5.3 for a formula for Igusa's local zeta function associated with the braid arrangement A_3.

The specialization $\mathsf{Z}_{f_{\mathcal{A}},\mathfrak{o}}(s)$ defined in (3.1) loses sight of all variables not corresponding to atoms (*i.e.* minimal elements in $\widetilde{\mathcal{L}}(\mathcal{A})$) and cannot distinguish atoms. We consider the following, slightly more distinguishing $\mathfrak{p}$-adic specialization of $\zeta_{\mathcal{A}(\mathfrak{o})}(\boldsymbol{s})$.

Definition 3.1. The *atom zeta function* of $\mathcal{A}$ is

$$\zeta^{\mathrm{at}}_{\mathcal{A}(\mathfrak{o})}\left((s_L)_{L\in\mathcal{A}}\right) = \zeta_{\mathcal{A}(\mathfrak{o})}\left((s_x \cdot \delta_{|\mathcal{A}_x|=1})_{x\in\widetilde{\mathcal{L}}(\mathcal{A})}\right) = \int_{\mathfrak{o}^{\dim(\mathcal{A})}} \prod_{L\in\mathcal{A}} |L|^{s_L}\, |\mathrm{d}\boldsymbol{X}|.$$

Here we identified atoms with elements $L \in \mathcal{A}$. We note that the independent variables $(s_L)_{L\in\mathcal{A}}$ allow for the treatment of multi-arrangements in the sense of [1].

We remark that the atom zeta function is the finest coarsening of the multivariate zeta function $\zeta_{\mathcal{A}(\mathfrak{o})}(\boldsymbol{s})$ that is, in general, multiplicative with respect to direct products of hyperplane arrangements. Namely, if $\mathcal{A}$ and $\mathcal{A}'$ are arrangements of hyperplanes in (disjoint vector spaces) K^d and $K^{d'}$ and $\mathfrak{o}$ is as above, then, by Fubini's theorem,

$$\zeta^{\mathrm{at}}_{(\mathcal{A}\times\mathcal{A}')(\mathfrak{o})}((\boldsymbol{s},\boldsymbol{s}')) = \zeta^{\mathrm{at}}_{\mathcal{A}(\mathfrak{o})}(\boldsymbol{s})\zeta^{\mathrm{at}}_{\mathcal{A}'(\mathfrak{o})}(\boldsymbol{s}').$$

Our next result paraphrases an explicit combinatorial formula for atom zeta functions associated with classical Coxeter arrangements; *cf.* [6, Theorem 5.6]. There we define, in particular, for $n \in \mathbb{N}$, the sets $\mathrm{TP}_{\mathsf{X},n}$ of total partitions of type X_n; for type $\mathsf{X} = \mathsf{A}$, these are also defined in [13, Example 5.2.5] and related to Schröder's fourth problem.

Theorem C. *Let $\mathsf{X} \in \{\mathsf{A},\mathsf{B},\mathsf{D}\}$ and $n \in \mathbb{N}$, with $n \geqslant 2$ if $\mathsf{X} = \mathsf{D}$. Then there exist, for all $\tau \in \mathrm{TP}_{\mathsf{X},n}$, explicitly determined polynomials $\pi_{\mathsf{X},\tau}(Y) \in \mathbb{Z}[Y]$ and products of geometric progressions $\mathsf{Cgp}_{\mathsf{X},\tau}(Z,(T_L)_{L\in\mathsf{X}_n})$ such that the following holds: for all cDVR $\mathfrak{o}$ with residue field cardinality q, assumed to be odd unless $\mathsf{X} = \mathsf{A}$,*

$$\zeta^{\mathrm{at}}_{\mathsf{X}_n(\mathfrak{o})}\left((s_L)_{L\in\mathsf{X}_n}\right) = \frac{1}{1-q^{-n-\sum_{L\in\mathsf{X}_n} s_L}} \sum_{\tau\in\mathrm{TP}_{\mathsf{X},n}} \pi_{\mathsf{X},\tau}(-q^{-1})\mathsf{Cgp}_{\mathsf{X},\tau}\left(q^{-1},(q^{-s_L})_{L\in\mathsf{X}_n}\right).$$

As a consequence of Theorem C, we obtain an explicit formula for Igusa's local zeta function $\mathsf{Z}_{f_{\mathsf{A}_n},\mathfrak{o}}(s)$ in terms of (unlabeled) rooted trees with $n+1$ leaves; see Section 5.3.

4 Coarse flag Hilbert–Poincaré series

Consider now the bivariate specialization of the flag Hilbert–Poincaré series $\mathsf{fHP}_{\mathcal{A}}$ obtained by setting $T_x = T$ for each $x \in \widetilde{\mathcal{L}}(\mathcal{A})$:

Definition 4.1. The *coarse flag Hilbert–Poincaré series* of $\mathcal{A}$ is

$$\mathsf{cfHP}_{\mathcal{A}}(Y,T) = \sum_{F\in\Delta(\widetilde{\mathcal{L}}(\mathcal{A}))} \pi_F(Y)\left(\frac{T}{1-T}\right)^{|F|} \in \mathbb{Q}(T)[Y].$$

We define the polynomial $\mathcal{N}_{\mathcal{A}}(Y,T) \in \mathbb{Q}[Y,T]$ by the formula

$$\mathsf{cfHP}_{\mathcal{A}}(Y,T) = \frac{\mathcal{N}_{\mathcal{A}}(Y,T)}{(1-T)^{\mathrm{rk}(\mathcal{A})}}.$$

We explore a number of remarkable properties of these rational functions, including nonnegativity features of $\mathcal{N}_{\mathcal{A}}(Y,T)$ and — in the case of Coxeter arrangements — connections with Eulerian and Stirling numbers.

In Proposition 4.2, we observe that $\mathcal{N}_{\mathcal{A}}(0,T)$ has nonnegative coefficients. Its proof is based on the fact that $\mathsf{cfHP}_{\mathcal{A}}(0,T)$ is the coarse Hilbert series of the Stanley–Reisner ring of the order complex of $\widetilde{\mathcal{L}}(\mathcal{A})$. The Cohen–Macaulayness of this complex implies the nonnegativity of the associated h-vector, *i.e.* the coefficients of $\mathcal{N}_{\mathcal{A}}(0,T)$.

Recall that the nth *Eulerian polynomial* $E_n(T)$ is defined via

$$E_n(T) = \sum_{w\in S_n} T^{\mathrm{des}(w)} \in \mathbb{Z}[T],$$

where $\mathrm{des}(w) := |\{i \in [n-1] \mid w(i) > w(i+1)\}|$. Let $S(n,k)$ be the Stirling number of the second kind; see [15, Section 1.9]. It is well-known that

$$\frac{E_n(T)}{(1-T)^n} = \sum_{k=1}^{n} k!\; S(n,k)\left(\frac{T}{1-T}\right)^{k-1} \tag{4.1}$$

is the (coarse) Hilbert series of the Stanley–Reisner ring $\mathbb{F}[\mathrm{sd}(\partial\Delta_{n-1})]$ associated with the first barycentric subdivision of the boundary of the $(n-1)$-dimensional simplex Δ_{n-1} over a field $\mathbb{F}$; *cf.* [10, Theorem 9.1].

A real hyperplane arrangement $\mathcal{A}$ is a *Coxeter arrangement* if the set of reflections across its hyperplanes fixes $\mathcal{A}$ and forms a finite Coxeter group under composition. We call $\mathcal{A}$ *irreducible* if it is not a direct product of two nontrivial arrangements. Finite Coxeter arrangements may be decomposed as direct products of irreducible Coxeter arrangements. The latter come in two classes: *classical* Coxeter arrangements of types A, B, or D and *exceptional* Coxeter arrangements of types E_6, E_7, E_8, F_4, G_2, H_2, H_3, H_4, or $\mathsf{I}_2(m)$ for $m \geqslant 7$.

The following result shows that the coarse flag Hilbert–Poincaré series of (most) Coxeter arrangements may be viewed as "Y-analogs" of the Hilbert series (4.1).

Theorem D. *Let $\mathcal{A}$ be a Coxeter arrangement with no irreducible factor equivalent to* E_8 *and* $\mathbb{F}$ *be a field. Then*

$$\frac{\mathsf{cfHP}_{\mathcal{A}}(1,T)}{\pi_{\mathcal{A}}(1)} = \frac{E_{\mathrm{rk}(\mathcal{A})}(T)}{(1-T)^{\mathrm{rk}(\mathcal{A})}} = \mathrm{Hilb}(\mathbb{F}[\mathrm{sd}(\partial\Delta_{\mathrm{rk}(\mathcal{A})-1})], T). \tag{4.2}$$

In other words,

$$\mathcal{N}_{\mathcal{A}}(1,T) = \pi_{\mathcal{A}}(1)E_{\mathrm{rk}(\mathcal{A})}(T)$$

and, equivalently, for $1 \leqslant k \leqslant \mathrm{rk}(\mathcal{A})$,

$$\sum_{\substack{F\in\Delta(\overline{\mathcal{L}}(\mathcal{A}))\\ |F|=k-1}} \frac{\pi_F(1)}{\pi_{\mathcal{A}}(1)} = k!\; S(\mathrm{rk}(\mathcal{A}),k).$$

The Stirling numbers of the second kind enter our proof of Theorem D via a simple formula, essentially due to Cayley, for the numbers of plane trees of a given length and number of leaves.

Simple examples show that the conclusion of Theorem D does not hold for general, non-Coxeter hyperplane arrangements, even when they are central: the coefficients of $\mathcal{N}_{\mathcal{A}}(1,T)$ are typically not multiples of $\pi_{\mathcal{A}}(1)$. However, equation (4.2) of Theorem D holds for small-rank non-Coxeter restrictions of type-D arrangements. For a recent strengthening of Theorem D see [4], where they show that a real central hyperplane arrangement satisfies (4.2) if and only if all of its chambers (*i.e.* the open regions in $\mathbb{R}^d$ not contained in any hyperplane) are simplicial cones.

To prove Theorem D, we first reduce to the irreducible case by showing that coarse flag Hilbert–Poincaré series are, essentially, Hadamard multiplicative; see [6, Proposition 6.3]. For $\mathsf{X} \in \{\mathsf{A},\mathsf{B},\mathsf{D}\}$, the result is proven, type-by-type, by constructing a bijection between flags of $\mathcal{L}(\mathsf{X}_n)$ and certain labeled rooted trees. We computed the coarse flag Hilbert–Poincaré series of the other irreducible Coxeter arrangement with the help of HypIgu [5], a SageMath [16] package developed by the first author to compute (coarse) flag Hilbert–Poincaré series and other rational functions associated with hyperplane arrangements. The results of these computations, along with many other examples, are recorded in the appendix of [6]; in each case, the validity of Theorem D follows by inspection. The type E_8 is excluded from Theorem D only because we do not supply a proof nor an explicit computation.

All our computations support the following general nonnegativity conjecture.

Conjecture E. *For all hyperplane arrangements $\mathcal{A}$, the polynomial $\mathcal{N}_{\mathcal{A}}(Y,T) \in \mathbb{Z}[Y,T]$ has nonnegative coefficients.*

Indeed, the polynomial $\mathcal{N}_{\mathcal{A}}(Y,T)$ has nonnegative coefficients for all of the arrangements in the appendix of [6]; see Section 5.4 for the arrangement E_6. It is shown in [4] that Conjecture E holds for all rank 3 central hyperplane arrangements. We furthermore view Conjecture E as an extension of the following observation, which uses deep results from algebraic combinatorics. We note that $\pi_{\mathcal{A}}(Y)$ is the Poincaré polynomial of a quotient of an exterior algebra, known as the Orlik–Solomon algebra [8, Theorem 3.68].

Proposition 4.2. *For all hyperplane arrangements $\mathcal{A}$ we have*

$$\mathcal{N}_{\mathcal{A}}(Y, 0) = \pi_{\mathcal{A}}(Y)$$

and, for all fields $\mathbb{F}$,

$$\frac{\mathsf{cfHP}_{\mathcal{A}}(0, T)}{1 - T} = \mathrm{Hilb}(\mathbb{F}[\Delta(\mathcal{L}(\mathcal{A}))], T).$$

In particular, the coefficients of both $\mathcal{N}_{\mathcal{A}}(Y, 0)$ and $\mathcal{N}_{\mathcal{A}}(0, T)$ are nonnegative.

5 Examples

We discuss various examples of flag Hilbert–Poincaré series.

5.1 Boolean arrangements

Consider the arrangement $\mathcal{A}$ comprising all the coordinate hyperplanes in $\mathbb{K}^n$, also known as the Boolean arrangement and equivalent to A_1^n. Since $\mathcal{A}_x^y$ is Boolean for all $x, y \in \mathcal{L}(\mathcal{A})$ with $y < x$, it follows that $\pi_F(Y) = (1 + Y)^n$ for all $F \in \Delta(\widetilde{\mathcal{L}}(\mathcal{A}))$. Thus,

$$\mathsf{fHP}_{\mathsf{A}_1^n}(Y, \boldsymbol{T}) = (1 + Y)^n \sum_{F \in \Delta(\widetilde{\mathcal{L}}(\mathcal{A}))} \prod_{x \in F} \frac{T_x}{1 - T_x}.$$

We identify $\Delta(\widetilde{\mathcal{L}}(\mathcal{A}))$ with $\widetilde{\mathrm{WO}}([n])$, the poset of chains of nonempty subsets of $[n]$ and rewrite $\mathsf{fHP}_{\mathcal{A}}(Y, (T_I)_{I \in \mathcal{P}(n) \setminus \{\varnothing\}})$ in terms of the *weak order zeta function* (cf. [12, Definition 2.9])

$$\mathcal{I}_n^{\mathrm{WO}}\left((T_I)_{I \in \mathcal{P}(n) \setminus \{\varnothing\}}\right) := \sum_{F \in \widetilde{\mathrm{WO}}([n])} \prod_{I \in F} \frac{T_I}{1 - T_I};$$

Proposition 5.1. *For $n \geqslant 1$,*

$$\mathsf{fHP}_{\mathsf{A}_1^n}(Y, \boldsymbol{T}) = (1 + Y)^n \mathcal{I}_n^{\mathrm{WO}}(\boldsymbol{T}).$$

The neat factorization of $\mathsf{fHP}_{\mathsf{A}_1^n}(Y, \boldsymbol{T})$ as a product of a polynomial in Y and a rational function in $\boldsymbol{T}$ seems to be atypical for these series. We have not observed such a factorization anywhere outside the family of Boolean arrangements.

5.2 Generic central arrangements

Let $m, n \in \mathbb{N}$ with $n \leqslant m$. We consider the arrangement $\mathcal{U}_{n,m}$ of m hyperplanes through the origin in $\mathbb{K}^n$ in general position. That is, for each $k \leqslant n$, every k-set of hyperplanes intersects in a codimension-k subspace. This is also known as the m-element uniform matroid of rank n. Observe that $\mathcal{U}_{n,n}$ is Boolean, seen in Section 5.1.

For a set I, denote by $\mathcal{P}(I)$ the power set of I and by $\mathcal{P}(I;k)$ the set of subsets of I of cardinality k. For $I, J \subset \mathbb{N}_0$, we define $\mathcal{P}(I;J) = \bigcup_{j\in J} \mathcal{P}(I;j) \subseteq \mathcal{P}(I)$. Observe that for $n \leqslant m$ we have $\overline{\mathcal{L}}(\mathcal{U}_{n,m}) \cong \mathcal{P}([m];[n-1])$ and $\widetilde{\mathcal{L}}(\mathcal{U}_{n,n}) \cong \mathcal{P}([n];[n])$.

The next proposition generalizes Proposition 5.1, as $\mathsf{A}_1^n \cong \mathcal{U}_{n,n}$.

Proposition 5.2. *Let $m, n \in \mathbb{N}$ with $n \leqslant m$. For $\boldsymbol{T} = (T_{\hat{1}}, (T_I)_{I\in\mathcal{P}([m];[n-1])})$,*

$$\mathsf{fHP}_{\mathcal{U}_{n,m}}(Y, \boldsymbol{T}) = \frac{1+Y}{1-T_{\hat{1}}} \sum_{k=0}^{n-1} \binom{m-1}{k} Y^k + \sum_{\ell=1}^{n-1} \sum_{k=0}^{n-\ell-1} \frac{(1+Y)^{\ell+1}}{1-T_{\hat{1}}} \binom{m-\ell-1}{k} Y^k \times \sum_{I\in\mathcal{P}([m];\ell)} T_I \mathcal{I}_{\ell}^{\mathrm{WO}}\left((T_J)_{J\in\mathcal{P}(I;[\ell])} \right).$$

5.3 Braid arrangement A_3

Our work allows us to obtain explicit formulae for the Igusa zeta functions associated with the braid arrangements of type A in terms of rooted trees. Consider, for example, the braid arrangement A_3, with $f_{\mathsf{A}_3}(\boldsymbol{X}) = \prod_{1\leqslant i<j\leqslant 4}(X_i - X_j)$.

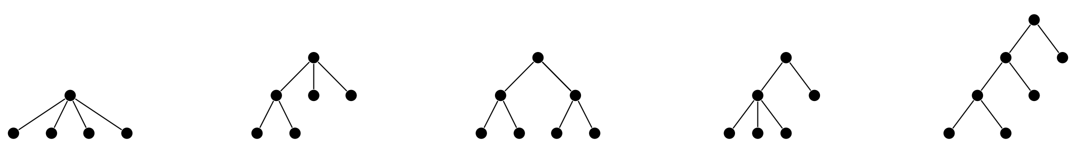

Figure 5.1: Five rooted trees with four leaves.

By [6, Corollary 5.7], for all cDVR $\mathfrak{o}$ with residue field cardinality q, Igusa's zeta function associated with $f := f_{\mathsf{A}_3}$ is

$$\mathsf{Z}_{f,\mathfrak{o}}(s) = \frac{1-q^{-1}}{1-q^{-3-6s}}\Bigg((1-2q^{-1})(1-3q^{-1}) + \frac{6q^{-1-s}(1-q^{-1})(1-2q^{-1})}{1-q^{-1-s}} + \frac{3q^{-2-2s}(1-q^{-1})^2}{(1-q^{-1-s})^2} + \frac{4q^{-2-3s}(1-q^{-1})(1-2q^{-1})}{1-q^{-2-3s}} + \frac{12q^{-3-4s}(1-q^{-1})^2}{(1-q^{-1-s})(1-q^{-2-3s})} \Bigg).$$

The five summands correspond to the five rooted trees in Figure 5.1. Each summand is determined from combinatorial data readily read off from the corresponding tree.

5.4 Coarse flag Hilbert–Poincaré series

In the appendix of [6] we record the coarse flag Hilbert–Poincaré series of various hyperplane arrangements, including the irreducible Coxeter arrangements of rank oat most seven. Using our package HypIgu we computed, for instance, the numerator of the coarse flag Hilbert–Poincaré series of the arrangement E_6:

$$\begin{aligned}\mathcal{N}_{\mathsf{E}_6}(Y,T) = {} & 1 + 36Y + 510Y^2 + 3600Y^3 + 13089Y^4 + 22284Y^5 + 12320Y^6 \\ & + (4591 + 57420Y + 289824Y^2 + 748080Y^3 + 1020819Y^4 \\ & + 671940Y^5 + 162206Y^6)T + (103681 + 888840Y + 3011919Y^2 \\ & + 5080320Y^3 + 4411839Y^4 + 1858680Y^5 + 300401Y^6)T^2 \\ & + (300401 + 1858680Y + 4411839Y^2 + 5080320Y^3 + 3011919Y^4 \\ & + 888840Y^5 + 103681Y^6)T^3 + (162206 + 671940Y + 1020819Y^2 \\ & + 748080Y^3 + 289824Y^4 + 57420Y^5 + 4591Y^6)T^4 + (12320 \\ & + 22284Y + 13089Y^2 + 3600Y^3 + 510Y^4 + 36Y^5 + Y^6)T^5.\end{aligned}$$

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