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Inequality of a Class of Near-Ribbon Skew Schur Q-Functions

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Abstract. The problem of determining when two skew Schur Q-functions are equal is still largely open. It has been studied in the case of ribbon shapes in 2008 by Barekat and van Willigenburg, and this paper approaches the problem for *near-ribbon* shapes, formed by adding one box to a ribbon skew shape. We particularly consider *frayed ribbons*, that is, the near-ribbons whose shifted skew shape is not an ordinary skew shape. We conjecture, with evidence, that all Schur Q-functions of frayed ribbon shape are distinct up to antipodal reflection. We prove this conjecture for several infinite families of frayed ribbons, using a new approach via the "lattice walks" version of the shifted Littlewood–Richardson rule discovered in 2018 by Gillespie, Levinson, and Purbhoo.

Keywords: Schur Q functions, shifted tableaux, Littlewood–Richardson rules

1 Introduction

In this extended abstract, following the results of [7], we provide new results on the open problem of determining when two skew Schur Q-functions are equal. The (non-skew) Schur Q-functions $Q_\lambda(x_1, x_2, \ldots)$, originally defined by Schur [22], are analogues of the classical Schur functions for shifted partitions λ, and are themselves symmetric functions. The Schur Q-functions naturally arise in the projective representation theory of the symmetric group [23], the crystal base theory of the quantum queer Lie superalgebra [4, 8, 9, 10], and the intersection theory of Schubert varieties in the odd orthogonal Grassmannian [5, 18].

The Schur Q-functions have several equivalent combinatorial definitions, including as the $t = -1$ evaluation of the Hall–Littlewood Q-polynomials [14]. Another is in terms of semistandard shifted tableaux, leading to many combinatorial developments analogous to those for ordinary Schur functions [12, 20, 26]. We recall the definition here via shifted tableaux as follows.

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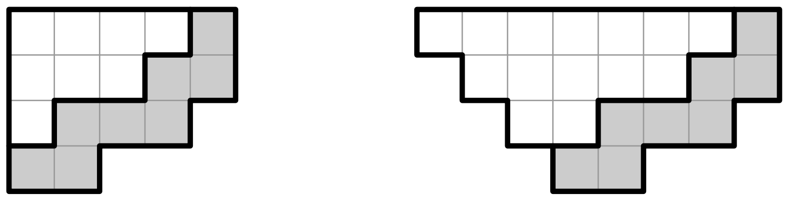


Figure 1: The ordinary skew shape $(5,5,4,2)/(4,3,1)$ at left, and the shifted skew shape $(8,7,5,2)/(7,5,2)$ at right. Both sets of shaded squares have the same underlying shape, but their indexing is different since the latter is shifted. In the latter, one box is on the staircase.

Recall that a **partition** of n is a tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\sum_i \lambda_i = n$. We say λ is **strict** if in fact $\lambda_1 > \lambda_2 > \cdots > \lambda_k$. The **size** of λ is $|\lambda| = \sum_i \lambda_i = n$. Its **Young diagram**, or simply **diagram**, is the left-justified array of boxes in which the i-th row from the top contains λ_i boxes (we use the 'English' convention for Young diagrams). If the Young diagram of μ is contained in that of λ, we write λ/μ for the **(ordinary) skew shape** formed by deleting the boxes of μ from λ. The diagram of a single partition λ is called a **straight shape**.

The **shifted Young diagram**, or simply **shifted diagram**, of a strict partition λ is formed by shifting the i-th row of the ordinary Young diagram of λ exactly $i-1$ steps to the right for all i. A **shifted skew shape** is the difference λ/μ of two nested shifted diagrams. We can think of shifted diagrams as fitting inside a triangular "staircase shape". A square in a skew shifted diagram is said to be **on the staircase** if it is in the leftmost possible position in its row (see Figure 1). We define shifted tableaux using the "doubled alphabet" of symbols:

$$1' < 1 < 2' < 2 < 3' < 3 < \cdots . \tag{1.1}$$

Definition 1.1. A **shifted semistandard Young tableau** (ShSSYT) is a filling of a shifted skew shape with primed and unprimed letters from (1.1) such that rows and columns are weakly increasing from left to right and top to bottom, primed letters can only be repeated in columns, and unprimed letters can only be repeated in rows.

We write ShSSYT(λ/μ) for the set of all shifted semistandard Young tableaux of shape λ/μ. The **reading word** of a shifted tableau is the word formed by concatenating the rows from bottom to top, and the **reading order** of the entries in a tableau is the total order given by the reading word. An example of a shifted semistandard Young tableau is shown below.

	$1'$	1	1	1
	$1'$	2		
$1'$	1			
1				

The **monomial** associated to a shifted semistandard Young tableau T is

$$x^T := x_1^{m_1} x_2^{m_2} \cdots,$$

where m_i is the total number of i or i' entries in T for each i. For instance, the monomial for the tableau shown above is $x_1^8 x_2$. The tuple $(m_1, m_2, m_3, \ldots)$ of exponents is called the **content** of T. We often write X for the set of variables $x_1, x_2, \ldots$.

Definition 1.2. The **skew Schur Q-function** for the skew shape λ/μ is the symmetric function

$$Q_{\lambda/\mu}(X) = \sum_{T \in \text{ShSSYT}(\lambda/\mu)} x^T.$$

In the study of skew Schur Q-functions, the following natural problem remains largely open.

Question 1.3. *When are two skew Schur Q-functions equal to each other?*

The natural analog of Question 1.3 has been studied more thoroughly for the unshifted case of ordinary Schur functions, which similarly arise in representation theory and geometry. In [25], van Willigenburg characterized the case when a skew Schur function is equal to a straight shape Schur function, finding that $s_{\lambda/\mu}$ and s_ν are equal only when λ/μ and ν are the same shape, or $180°$ rotations of each other. Billera, Thomas, and van Willigenburg [2] determined an exact condition for the equality of ribbon Schur functions. Reiner, Shaw, and van Willigenburg [19] expanded on this result, giving further conditions for equality for general shapes, and soon after McNamara and van Willigenburg [15] gave a single composition operation that maintains Schur equality. Similar results for the problem of determining when the difference of two skew Schur functions is Schur positive were given in, for instance, [11, 13, 16, 27].

In the case of Schur Q-functions, Salmasian [21] found exact criteria for when the Schur Q-function $Q_{\lambda/\mu}$ of a shifted skew shape was equal to that of a shifted straight shape Schur Q-function Q_ν. Barekat and van Willigenburg [1] investigated the problem of Schur Q-function equality in the case of ribbons, finding a compositional construction that gives families of shapes with equal Schur Q-function, and conjecturing that it is a necessary and sufficient condition for equality. However, the remaining results from the ordinary Schur function case have not yet been replicated for Schur Q-functions.

Building off of the results of Barekat and van Willigenburg [1], we examine shifted skew shapes that are near-ribbons, defined as follows.

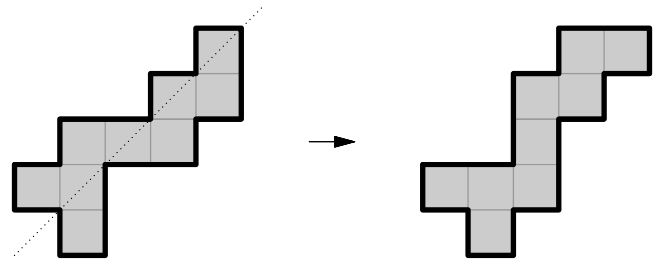


Figure 2: A frayed ribbon D and its antipodal reflection D^a.

Definition 1.4. A **near-ribbon** is a connected non-ribbon shape for which it is possible to remove one square to form a ribbon.

This class of skew Schur Q-functions is self contained, in the following sense.

Proposition 1.5. *If D and E are shifted skew shapes such that $Q_D = Q_E$, and D is a near-ribbon, then E is also a near-ribbon.*

In this direction, we have found ample computational and theoretical evidence that, remarkably, the subclass of shifted near-ribbons that are not themselves ordinary skew shapes have distinct Schur Q-functions up to antipodal reflection. We call these shapes frayed ribbons, and they can also be defined as follows.

Definition 1.6. A **frayed ribbon** is a shifted near-ribbon containing two squares on the staircase. (See Figure 2.)

We state our main conjecture precisely as follows. Define D^a to be the **antipodal reflection** of a shifted skew shape across the northeast-southwest diagonal (Figure 2). In [3], it was shown that the antipodal map preserves the Schur Q-functions.

Conjecture 1.7. *If D and E are frayed ribbons such that $Q_D = Q_E$, then either $D = E$ or $D = E^a$.*

This conjecture is in sharp contrast to the results for ribbons in [1], in which infinitely many pairs of non-antipodal ribbons, formed by "composing" previously equal pairs in different ways, were found to have equal Schur Q-functions. It is therefore surprising that adding the extra square on the staircase appears to distinguish all of the corresponding Schur Q-functions (up to antipodal reflection).

We have verified Conjecture 1.7 by computer for all frayed ribbons up to size 11, and prove it for several infinite families of frayed ribbons below. In the statement below, a **turn** of a frayed ribbon is a square in both a nontrivial row and nontrivial column of the ribbon structure (not including the square adjacent to the two squares on the staircase).

Theorem 1.8. *If D and E are frayed ribbons with $Q_D = Q_E$, then:*

- *D and E have the same number of turns;*
- *If D and E have no turn or one turn, then $D = E$ or $D = E^a$;*
- *If D has two turns and at most one square between the turns, then $D = E$ or $D = E^a$.*

It is worth noting that while Theorem 1.8 and computational evidence indicates that all non-antipodal pairs of distinct frayed ribbons have distinct Schur Q-functions, there are examples showing that several natural generalizations of Conjecture 1.7 do not hold. For instance, not all non-antipodal pairs of distinct connected shifted skew shapes having at least two boxes on the staircase have distinct Schur Q-functions. We provide counterexamples to this effect, and to other potential generalizations, in Section 4.

2 Walks and shifted Littlewood–Richardson coefficients

To prove Theorem 1.8, we expand the skew Schur Q-functions in terms of the straight shape Schur Q-functions, which form a basis of a the subring of symmetric functions generated by the odd-degree power sum symmetric functions [14]. In particular, the skew Schur Q-functions $Q_{\lambda/\mu}$, which also lie in this subring, expand positively as a sum of straight shape Schur Q-functions Q_ν:

$$Q_{\lambda/\mu} = \sum_{\nu} f^{\lambda}_{\mu\nu} Q_\nu.$$

Here the coefficients $f^{\lambda}_{\mu\nu}$ are nonnegative integers, and are called the **shifted Littlewood–Richardson coefficients**. They have several known combinatorial interpretations [6, 17, 23], and here we use the interpretation defined in [6] in terms of lattice walks.

Definition 2.1. Let w be a word in the alphabet $\{1', 1, 2', 2\}$. The $1/2$**-walk** of the word w is a lattice walk in the first quadrant using one of the four unit steps

$$\longrightarrow\; = \;(1,0) \qquad \longleftarrow\; = \;(-1,0) \qquad \uparrow\; = \;(0,1) \qquad \downarrow\; = \;(0,-1)$$

for each letter of the word as we read from left to right. The walk starts at the origin $(0,0)$, and at the i-th step we read w_i and draw the next step of the walk according to Figure 3, with two cases based on whether or not the step starts on one of the x or y axes. In particular, any 2 is an up arrow, any $1'$ is a right arrow, a 1 is either right if on an axis or down if not, and a $2'$ is either up if on an axis or left if not. We will generally write the label each step of the walk by the letter w_i, so as to represent both the word and its walk on the same diagram.

Definition 2.2 (Starred entries)**.** We write 1^* to denote a letter that is either $1'$ or 1, 2^* to denote a letter that is either $2'$ or 2, *etc.*

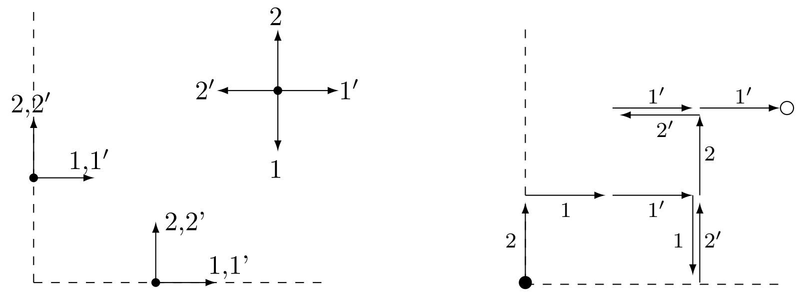


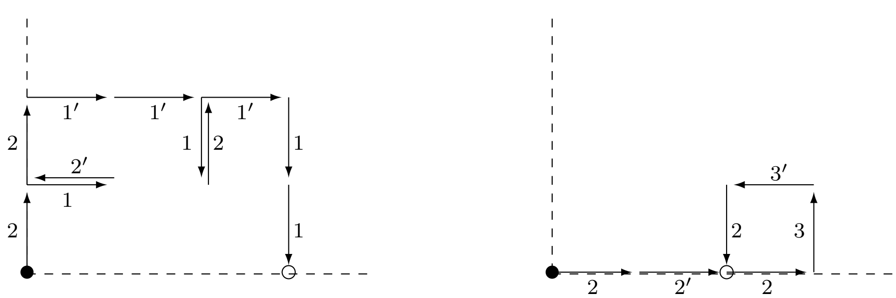
Figure 3: At left, the directions assigned to the letters $1', 1, 2', 2$ in the lattice walk of a word, depending on whether or not the step starts on an axis. At right, the walk for $w = 211'12'22'1'1'$.

If w is a word in the alphabet $\{i', i, (i+1)', i+1\}$ for any i, we similarly define the $i/(i+1)$**-walk** of w by replacing 1^* with i^* and 2^* with $(i+1)^*$ in the above definition. We can then define the $i/(i+1)$-walk of any word in the doubled alphabet as follows.

Definition 2.3. If w is a word in the alphabet $\{1', 1, 2', 2, 3', 3, \ldots\}$, the $i/(i+1)$**-walk** of w is the $i/(i+1)$-walk of the subword of w formed by its i^* and $(i+1)^*$ elements (which we often refer to as the $i/(i+1)$**-subword**).

Definition 2.4. A word w is **ballot** if, for every i, the $i/(i+1)$-walk of w ends at a point on the x axis.

Example 2.5. Suppose $w = 212'231'3'1'121'11$. Then its 1/2-subword is $212'21'1'121'11$, whose walk is shown below at left. Its 2/3-subword is $22'233'2$, whose walk is shown below at right. Both walks end on the x-axis, so the word w is ballot.



We now need to recall the definition of a tableau in canonical form.

Definition 2.6. A skew shifted semistandard Young tableau is in **canonical form** if the first i^* in reading order is unprimed for every i.

Note that if T is semistandard, its first i^* in reading order may be changed to being primed or unprimed and always still yield another semistandard tableau. Thus, the canonical form is simply enforcing this choice to be unprimed. The shifted jeu de taquin process [20, 26] is only well-defined for skew tableaux in canonical form, which is why it appears in the shifted Littlewood–Richardson rule.

Definition 2.7. A skew shifted Young tableau T is a **shifted ballot** tableau if it is semistandard, in canonical form, and its reading word is ballot.

Shifted ballot tableaux are also referred to as shifted Littlewood–Richardson tableaux, because they enumerate the shifted Littlewood–Richardson coefficients. We may now state the shifted Littlewood–Richardson rule.

Theorem 2.8 ([6, Theorem 1.5])**.** *The shifted Littlewood–Richardson coefficient $f^{\lambda}_{\mu\nu}$ is equal to the number of shifted ballot tableaux of shape λ/μ and content ν.*

3 Distinctness for frayed ribbon shapes

We now turn our attention to Theorem 1.8. We begin by proving the first statement of Theorem 1.8, below in Proposition 3.2.

Lemma 3.1. *The top row of any shifted ballot tableau has only 1^* entries, and in fact has at most one $1'$.*

Proof. By the ballot condition on the $1/2$-walk, the last 1^* or 2^* in the word must be a 1^*, since no 2^* arrow can ever end on the x axis. Similarly the last 2^* in any ballot reading word comes after the last 3^*, and so on, meaning that the last letter in reading order is 1^*. Thus, by the semistandard condition, the entire top row consists of 1^* entries, with at most one $1'$ at the start of the row. □

We define an **outer turn** to be a square with adjacent squares to the left and above, and an **inner turn** to be a square with adjacent squares the right and below, with neither adjacent square on the staircase.

Proposition 3.2. *Suppose D is a frayed ribbon shape of size n with k turns. Then the coefficient of $Q_{(n-2,2)}$ in the expansion of Q_D is $2k$.*

Proof. We assume without loss of generality, using the antipodal map, that the second-to-bottom row of D has more than two squares. First, note that in a shifted ballot tableau of shape D with content $(n-2,2)$, the entry in the bottom row must be a 2 by semistandardness and the canonical form condition. Moreover, the other 2^* must be in the corner of one of the outer turns by semistandardness and by Lemma 3.1.

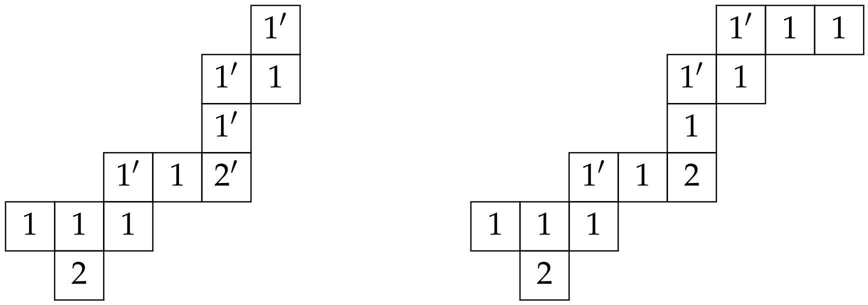


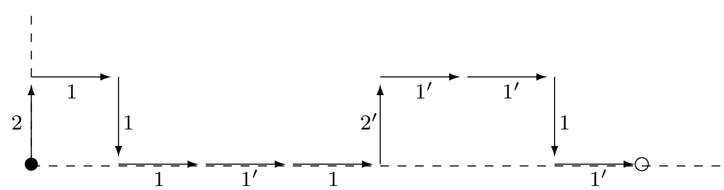
Figure 4: At left, a shifted ballot tableau of a frayed ribbon with three outer turns and two inner turns, that is, $t_0 = 3$ and $t_1 = 2$ in the notation of the proof of Proposition 3.2. At right, a shifted ballot tableau of a frayed ribbon with $t_0 = t_1 = 3$.

Now, if t_0 is the number of outer turns and t_1 is the number of inner turns, we have $t_0 + t_1 = k$ and either $t_0 = t_1$ (if the topmost turn is an inner turn) or $t_0 = t_1 + 1$ (if the topmost turn is an outer turn). See Figure 4 for each such case.

For each outer turn of D, there are exactly four semistandard fillings containing a 2^* in the corner of that turn; in particular, the 2^* may be either 2 or $2'$, and the square above it may be either 1 or $1'$. We now check which of these have ballot reading words. The reading word is

$$211(1^* \cdots 1^*)2^*1^*(1^* \cdots 1^*)$$

where the strings $(1^* \cdots 1^*)$ in parentheses have a mix of primed and unprimed 1 entries that are uniquely determined by the shape of D. Just before the second 2^*, the walk is on the x axis, and the 2^* is an up-step. So the walk returns to the x-axis if and only if an unprimed 1 appears after the 2^*, as shown in the walk below (which corresponds to the tableau at left in Figure 4).



This is guaranteed to happen by semistandardness if there is an inner turn after the outer turn which contains the 2^*, but if not, then the only way it is guaranteed is if the 1^* just following the 2^* is unprimed.

If $t_0 = t_1$, we have an inner turn after all outer turns, so each of the t_0 outer corners contributes four shifted ballot tableaux, and the coefficient is $4t_0 = 2(t_0 + t_1) = 2k$. If instead $t_0 = t_1 + 1$, then the $t_0 - 1$ lowest outer corners contribute four shifted ballot tableaux, but the topmost outer corner contributes only two since the 1^* above it must be unprimed. Thus we have a coefficient of $4(t_0 - 1) + 2 = 2t_0 + 2(t_1 + 1) - 2 = 2(t_0 + t_1) = 2k$. Therefore, in all cases, the coefficient is $2k$ as desired. $\square$

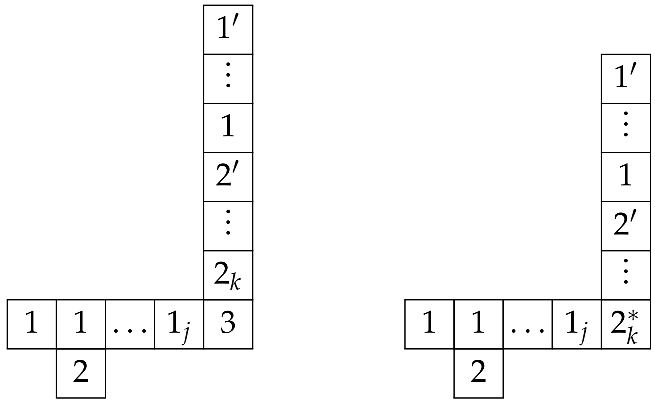


Figure 5: The forms that any shifted ballot tableau of a frayed ribbon with one turn must take, where if k is the number of 2^* entries in the long column and j is the number of 1 entries in the long row, we have $k < j$.

As a corollary we obtain the first statement of Theorem 1.8.

Corollary 3.3. *Let D and E be frayed ribbon shapes for which $Q_D = Q_E$. Then D and E have the same number of turns.*

3.1 Frayed ribbons with one turn

We now focus on the second statement of Theorem 1.8. Note that there are only two frayed ribbon shapes with no turns — the straight shifted shape $(n-1,1)$ and its antipodal reflection — so the statement holds when there are no turns.

We therefore consider the case of one turn. Since we are considering antipodal pairs, we also assume without loss of generality that the shape of T's frayed ribbon goes "to the right and then up" from the frayed part, that is, its second-to-bottom row has more than two squares. For full proofs, see [7]; as a short summary, one can show the following by analyzing lattice walks.

Proposition 3.4. *A shifted Young tableau T that is a frayed ribbon shape with one (outer) turn is a shifted ballot tableau if and only if it is of one of the two forms shown in Figure 5.*

Proposition 3.4 leads to the following explicit Schur Q-function expansion for one turn frayed ribbons.

Corollary 3.5. *Let D be a frayed ribbon with one turn of size n and final column height $h+1$. Then*

$$Q_D = Q_{(n-1,1)} + 2\sum_{i=2}^{m_1(h)} Q_{(n-i,i)} + \sum_{i=2}^{m_2(h)} Q_{(n-i,i,1)}$$

where $m_1(h) = \min(h+1, n-h-2)$ *and* $m_2(h) = \min(h, n-h-2)$.

Theorem 3.6. *Suppose D and E are frayed ribbons with one turn of size n, where D has column height h and E has column height* ℓ *for* $h \neq \ell$. *Then* $Q_D \neq Q_E$.

Proof. Suppose $Q_D = Q_E$. Then by Corollary 3.5, we must have $m_1(h) = m_1(\ell)$ and $m_2(h) = m_2(\ell)$. The former equation states that $\min(h+1, n-h-2) = \min(\ell+1, n-\ell-2)$. Since $h \neq \ell$, it follows that either $h+1 = n-\ell-2$ or $n-h-2 = \ell+1$, which are in fact equivalent statements, and so they both hold. From $m_2(h) = m_2(\ell)$, we have $\min(h, n-h-2) = \min(\ell, n-\ell-2)$, and so by the same reasoning we have $h = n-\ell-2$, which contradicts our equality above. Hence $Q_D \neq Q_E$. □

3.2 Frayed ribbons with two turns

For the third statement of Theorem 1.8, we use similar methods to the cases above to analyze the coefficients. In particular, when $D = \lambda/\mu$ is a frayed ribbon with two turns and *no* squares between the turns, we show that the pair of coefficients $f^{\lambda}_{\mu,(n-k,k)}$ and $f^{\lambda}_{\mu,(n-k-1,k,1)}$ distinguish the Schur Q-functions Q_D.

Similarly, when $D = \lambda/\mu$ is a frayed ribbon with two turns and *one* square between the turns, we show that the pair of coefficients $f^{\lambda}_{\mu,(n-k,k)}$ and $f^{\lambda}_{\mu,(n-k-2,k,2)}$ distinguish the Schur Q-functions Q_D. See [7] for details.

4 Further Observations

We conclude here with three examples pertaining to natural generalizations of Conjecture 1.7 and Theorem 1.8. The examples in this section were found using SageMath [24].

Example 4.1. It is natural to ask whether the "frayed" aspect of frayed ribbons is in fact enough to distinguish any Schur Q-functions that are not antipodal. More specifically, perhaps any two distinct non-antipodal connected skew shifted shapes having at least two boxes on the staircase diagonal (but are not necessarily frayed ribbons) have distinct Schur Q-functions. In fact, this is not the case - the shifted shapes $(6,5,4,2,1)/(5,4,1)$ and $(6,5,2,1)/(5,1)$ are a non-antipodal pair of shapes with at least two boxes on the staircase whose Schur Q-functions are equal:

$$Q_{(6,5,4,2,1)/(5,4,1)} = Q_{(6,5,2,1)/(5,1)} = Q_{(6,2)} + 2Q_{(5,3)} + 2Q_{(5,2,1)} + 2Q_{(4,3,1)}.$$

Example 4.2. The Schur Q-functions of frayed ribbons are not necessarily distinct from those of other near-ribbons that are not frayed. Indeed, we have

$$Q_{(4,3,1)/(3)} = Q_{(4,3)/(2)} = Q_{(4,1)} + Q_{(3,2)},$$

and $(4,3,1)/(3)$ and $(4,3)/(2)$ are a frayed ribbon and near-ribbon respectively.

Example 4.3. There exist pairs of near-ribbons that are *both* not frayed in which equality holds, and their Schur Q-functions are not trivially equal by being antipodal, transposed, or antipodal transposed shapes. As an example, we have

$$Q_{(7,6,5,3)/(6,5,2)} = Q_{(7,6,5,1)/(6,4,1)} = 3Q_{(4,3,1)} + 3Q_{(5,2,1)} + 5Q_{(5,3)} + 4Q_{(6,2)} + Q_{(7,1)},$$

and the shapes $(7,6,5,3)/(6,5,2)$ and $(7,6,5,1)/(6,4,1)$ are non-frayed near-ribbons that are not trivially equivalent.

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