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Classifying Levi-Spherical Schubert Varieties

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Abstract. A Schubert variety in the flag manifold GL_n/B is *Levi-spherical* if the action of a Borel subgroup in a Levi subgroup of a standard parabolic has an open dense orbit. We present some recent combinatorial developments on this topic, including a classification in terms of *spherical elements* of a symmetric group. We offer a new conjecture that extends the classification to other Lie types, along with supporting evidence.

Keywords: key polynomials, Schubert varieties, Levi subgroups, spherical varieties

1 Introduction

1.1 Schubert varieties and Levi-sphericality

Let $\text{Flags}(\mathbb{C}^n)$ be the variety of *complete flags* $\langle 0\rangle \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n$, where F_i is a subspace of dimension i. The group GL_n of invertible $n\times n$ matrices over $\mathbb{C}$ acts transitively on $\text{Flags}(\mathbb{C}^n)$ by change-of-basis. Define the *standard flag* by $F_i = \text{span}(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_i)$ where $\vec{e}_i$ is the i-th standard basis vector. The stabilizer of this flag is $B \subset GL_n$, the *Borel subgroup* of upper triangular invertible matrices. Thus, $\text{Flags}(\mathbb{C}^n) \cong GL_n/B$. The Borel B acts on GL_n/B with finitely many orbits. These orbits are the *Schubert cells* $X_w^\circ = BwB/B \cong \mathbb{C}^{\ell(w)}$ and are indexed by w in the symmetric group $\mathfrak{S}_n$. Their closures $X_w := \overline{X_w^\circ}$ are the *Schubert varieties* and are of interest in combinatorial algebraic geometry and Lie theory. We refer the reader to [7] for more background.

For $I \subseteq J(w) := \{j \in [n-1] : w^{-1}(j) > w^{-1}(j+1)\}$, let $L_I \subseteq GL_n$ be the Levi subgroup of invertible block diagonal matrices

$$L_I \cong GL_{d_1-d_0} \times GL_{d_2-d_1} \times \cdots \times GL_{d_k-d_{k-1}} \times GL_{d_{k+1}-d_k}.$$

L_I acts on X_w; see, *e.g.*, [10, Section 1.2]. This is the main concept of our interest:

Definition 1.1 ([10, Definition 1.8])**.** X_w is *L_I-spherical* if X_w has an open dense orbit of a Borel subgroup of L_I. If in addition, $I = J(w)$, X_w is *maximally spherical*.

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The purpose of this extended abstract is to review recent work [10, 11, 9, 3, 8] about Definition 1.1. We also describe new (as yet, unpublished) progress for other Lie types.

1.2 Levi spherical permutations and the classification theorem

Let $G = GL_n$. Its Weyl group $W \cong \mathfrak{S}_n$ consists of permutations of $[n] := \{1, 2, \ldots, n\}$. Thus W is generated, as a Coxeter group, by the simple transpositions $S = \{s_i = (i\ i+1) : 1 \leq i \leq n-1\}$. The set of *left descents* is $J(w) = \{j \in [n-1] : w^{-1}(j) > w^{-1}(j+1)\}$ ($j \in J(w)$ if $j+1$ appears to the left of j in w's one-line notation). Let $\ell(w)$ be the *Coxeter length* of w. For $w \in \mathfrak{S}_n$,

$$\ell(w) = \#\{1 \leq i < j \leq n : w(i) > w(j)\}$$

is the number of *inversions* of w.

A *parabolic subgroup* W_I of W is the subgroup generated by a subset $I \subset S$. Furthermore, a *standard Coxeter element* $c \in W_I$ is the product of the elements of I listed in some order. Let $w_0(I)$ denote the longest element of W_I.

Definition 1.2 ([9, Definition 1.1])**.** Let $w \in W$ and fix $I \subseteq J(w)$. Then w is *I-spherical* if $w_0(I)w$ is a standard Coxeter element for some parabolic subgroup $W_{I'}$ of W.

In [10, Conjecture 3.2], a conjectural combinatorial classification of L_I-spherical Schubert varieties was stated. In [9] (see Section 3) it was proved that said conjecture is equivalent to the following theorem of *ibid*.

Theorem 1.3 ([9, Theorem 1.5])**.** *Let $w \in \mathfrak{S}_n$ and $I \subseteq J(w)$. $X_w \subseteq GL_n/B$ is L_I-spherical if and only if w is I-spherical.*

The proof uses the theory of *Demazure characters* and their manifestation in algebraic combinatorics, the *key polynomials*. One of the results used is a classification of *multiplicity-free* key polynomials [11, Theorem 1.1]. This is explained in Section 2.

Let us also mention some other related results. Theorem 1.3 is used in C. Gaetz's [8], which proves [10, Conjecture 3.8]. Consequently, this gives a pattern avoidance criterion for maximally spherical Schubert varieties [8, Theorem 1.4, Corollary 1.5]. Earlier work of D. Brewster–R. Hodges–A. Yong [3] proved a weaker numerical assertion, that

$$\lim_{n\to\infty} \Pr[w \in S_n, w \text{ is } I\text{-spherical for some } I \subseteq J(w)] = 0,$$

as well as its geometric counterpart

$$\lim_{n\to\infty} \Pr[w \in S_n, X_w \text{ is } L_I\text{-spherical for some } I \subseteq J(w)] = 0.$$

However, the proofs in *ibid.* did not depend on [11] but rather a definition of *proper permutations*. Work in preparation of J. Balogh, D. Brewster, and the second author extend the results of *ibid.* to other Lie types.

Our focus now turns to extending Theorem 1.3 to other Lie types. In Section 4 we report on our ongoing project in that direction after [10, 11, 3, 9].

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2 Key polynomials and sphericality

The problem of deciding if a Schubert variety is Levi spherical is closely connected to the algebraic combinatorics of key polynomials.

2.1 Key polynomials

Let $\mathsf{Pol} := \mathbb{Z}[x_1, x_2, \ldots, x_n]$ be the polynomial ring in the indeterminates $x_1, x_2, \ldots, x_n$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathsf{Comp}_n$, the *key polynomial* κ_α is defined as follows. If α is weakly decreasing, then $\kappa_\alpha := \prod_i x_i^{\alpha_i}$. Otherwise, suppose $\alpha_i > \alpha_{i+1}$. Let

$$\pi_i \colon \mathsf{Pol} \to \mathsf{Pol}, \quad f \mapsto \frac{x_i f(\ldots, x_i, x_{i+1}, \ldots) - x_{i+1} f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}},$$

and $\kappa_\alpha = \pi_i(\kappa_{\widehat{\alpha}})$ where $\widehat{\alpha} := (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots)$.

The operators π_i satisfy the relations

$$\begin{aligned} \pi_i \pi_j &= \pi_j \pi_i \ \text{ (for } |i-j| > 1) \\ \pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1} \\ \pi_i^2 &= \pi_i; \end{aligned}$$

see [14]. Recall that the *Demazure product* on $\mathfrak{S}_n$ is defined by

$$w * s_i = \begin{cases} ws_i & \text{if } \ell(ws_i) = \ell(w) + 1 \\ 0 & \text{otherwise.} \end{cases}.$$

This product is associative. Then $R = (s_{i_1}, \cdots, s_{i_\ell})$ is a *Hecke word* of w if $w = s_{i_1} * s_{i_2} * \cdots * s_{i_\ell}$. For any $w \in \mathfrak{S}_n$ one unambiguously defines

$$\pi_w := \pi_{i_1} \pi_{i_2} \cdots \pi_{i_\ell},$$

where $R = (s_{i_1}, \ldots, s_{i_\ell})$ is a Hecke word of w.

Next, suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$ is a partition, and $w \in \mathfrak{S}_n$. Define

$$\kappa_{w\lambda} := \kappa_{\lambda_{w^{-1}(1)}, \ldots, \lambda_{w^{-1}(n)}}.$$

Therefore, $\kappa_{w\lambda} = \pi_w \kappa_\lambda$.

2.2 Split-symmetry and multiplicity-freeness

We recall some notions from [10, Section 4]. Suppose

$$d_0 := 0 < d_1 < d_2 < \cdots < d_k < d_{k+1} := n$$

and $D = \{d_1, \ldots, d_k\}$. Let Π_D be the subring of Pol consisting of the polynomials that are separately symmetric in $X_i := \{x_{d_{i-1}+1}, \ldots, x_{d_i}\}$ for $1 \leq i \leq k+1$. If $f \in \Pi_D$, f is *D-split-symmetric*.

The ring Π_D has a basis of *D-Schur polynomials*

$$s_{\lambda^1,\ldots,\lambda^k} := s_{\lambda^1}(X_1)s_{\lambda^2}(X_2)\cdots s_{\lambda^k}(X_k),$$

where

$$(\lambda^1, \ldots, \lambda^k) \in \mathsf{Par}_D := \mathsf{Par}_{d_1-d_0} \times \cdots \times \mathsf{Par}_{d_{k+1}-d_k},$$

and Par_t is the set of partitions with at most t nonzero-parts. See [10, Definition 4.3, Corollary 4.4]. Thus, for any $f \in \Pi_D$ there is a unique expression

$$f = \sum_{(\lambda^1,\ldots,\lambda^k)\in \mathsf{Par}_D} c_{\lambda^1,\ldots,\lambda^k} s_{\lambda^1,\ldots,\lambda^k}. \tag{2.1}$$

Definition 2.1 **([10, Definition 4.7]).** If $c_{\lambda^1,\ldots,\lambda^k} \in \{0,1\}$ for all $(\lambda^1, \ldots, \lambda^k) \in \mathsf{Par}_D$, f is *D-multiplicity-free*.

Example 2.2 (Vieta's formulas, a reinterpretation). Let $f = \prod_{i=2}^{n}(x_1 + x_i)$. This polynomial is D-split symmetric for $D = \{1\}$, *i.e.*, it is separately symmetric in $\{x_1\}$ and $\{x_2, \ldots, x_n\}$. Then (2.1) is the D-multiplicity-free expansion

$$f = s_{n-1}(x_1)s_{\emptyset}(x_2, \ldots, x_n) + s_{n-2}(x_1)s_1(x_2, \ldots, x_n) + \cdots + s_{\emptyset}(x_1)s_{1^{n-1}}(x_2, \ldots, x_n). \tag{2.2}$$

Thinking of f as a monic polynomial in x_1 with roots $-x_2, -x_3, \ldots, -x_n$, (2.2) is just stating Vieta's formulas.

Definition 2.1 unifies two disparate concepts of multiplicity-freeness:

(MF1) Suppose $f = f(x_1, \ldots, x_n)$ is symmetric and

$$f = \sum_{\lambda \in \mathsf{Par}_n} c_\lambda s_\lambda.$$

Then f is *multiplicity-free* if $c_\lambda \in \{0,1\}$ for all λ. This is the case $D = \emptyset$. For example, J. Stembridge [17] classified multiplicity-freeness when $f = s_\mu s_\nu$. See [10] for additional references.

(MF2) Now let
$$f = \sum_{\alpha \in \mathsf{Comp}_n} c_\alpha x^\alpha \in \mathsf{Pol}.$$
f is *multiplicity-free* if $c_\alpha \in \{0, 1\}$ for all α. This corresponds to $D = [n-1]$. For instance, recent work of A. Fink-K. Mészáros-A. St. Dizier [6] characterizes multiplicity-free Schubert polynomials.

In [11, Theorem 1.1], an analogue, for key polynomials, of the aforementioned result [6] was proved. That key polynomial result plays a role in the proof of Theorem 1.3.

Definition 2.3 (Composition patterns [11, Definition 4.8])**.** Let
$$\mathsf{Comp} := \bigcup_{n=1}^{\infty} \mathsf{Comp}_n.$$
For $\alpha = (\alpha_1, \ldots, \alpha_\ell), \beta = (\beta_1, \ldots, \beta_k) \in \mathsf{Comp}$, α *contains the composition pattern* β if there exist integers $j_1 < j_2 < \cdots < j_k$ that satisfy:

- $(\alpha_{j_1}, \ldots, \alpha_{j_k})$ is order isomorphic to β ($\alpha_{j_s} \leq \alpha_{j_t}$ if and only if $\beta_s \leq \beta_t$),
- $|\alpha_{j_s} - \alpha_{j_t}| \geq |\beta_s - \beta_t|$.

The first condition is the naïve notion of pattern containment, while the second allows for minimum relative differences. If α does not contain β, then α *avoids* β. For $S \subset \mathsf{Comp}$, α avoids S if α avoids all the compositions in S.

Example 2.4. The composition $(3, \underline{1}, 4, \underline{2}, \underline{2})$ contains $(0, 1, 1)$. It avoids $(0, 2, 2)$.

Define
$$\mathsf{KM} = \{(0,1,2), (0,0,2,2), (0,0,2,1), (1,0,3,2), (1,0,2,2)\}.$$
Let $\overline{\mathsf{KM}}_n$ be those $\alpha \in \mathsf{Comp}_n$ that avoid KM.

Theorem 2.5 ([11, Theorem 1.1])**.** *κ_α is $[n-1]$-multiplicity-free if and only if $\alpha \in \overline{\mathsf{KM}}_n$.*

It is an open problem to classify when $\kappa_\alpha \in \Pi_D$ is D-multiplicity-free. (The analogous question for Schubert polynomials, whose solution would generalize [6] is also open.)

2.3 Geometry to combinatorics connection

This fact from [10] allows us to turn the geometric question of Levi-sphericality into D-multiplicity-freeness of key polynomials:

Theorem 2.6 ([10, Theorem 4.13])**.** *Let $\lambda \in \mathsf{Par}_n$, and $w \in \mathfrak{S}_n$. Suppose $I \subseteq J(w)$ and $D = [n-1] - I$. X_w is L_I-spherical if and only if $\kappa_{w\lambda}$ is D-multiplicity-free for all $\lambda \in \mathsf{Par}_n$.*

In view of Theorem 2.6, the following is clearly equivalent to Theorem 1.3.

Theorem 2.7. *Let $D = [n-1] - I$. w is I-spherical if and only if $\kappa_{w\lambda}$ is D-multiplicity-free for all $\lambda \in \mathsf{Par}_n$.*

2.4 Proof sketch for Theorem 2.7 (and Theorem 1.3)

We outline the argument from [9]. The "$\Rightarrow$" proof starts with two simple observations:

Lemma 2.8. *If $w = w_0(I)c$ where c is a standard Coxeter element, then $\kappa_{w\lambda} = \pi_{w_0(I)}\kappa_{c\lambda}$.*

For any $\alpha \in \mathsf{Comp}_n$, let

$$a_{\alpha_1+n-1,\alpha_2+n-2,\ldots,\alpha_n} := \det(x_j^{\lambda_i+n-i})_{1\leq i,j\leq n}.$$

In particular, $\Delta_n := a_{n-1,n-2,\ldots,0} = \prod_{1\leq j<k\leq n}(x_j - x_k)$ is the *Vandermonde determinant*.

Define a *generalized Schur polynomial s_α* by

$$s_\alpha(x_1,\ldots,x_n) := a_{\alpha_1+n-1,\alpha_2+n-2,\ldots,\alpha_n}/a_{n-1,n-2,\ldots,1,0}. \tag{2.3}$$

Definition 2.9 ([9, Definition 3.4])**.** If $\beta = (\beta_1,\ldots,\beta_n) \in \mathsf{Comp}_n$ and $i < j \in [n-1]$, define $t_{ij}\colon \mathsf{Comp}_n \to \mathsf{Comp}_n$ by

$$t_{ij}(\ldots,\beta_i,\ldots,\beta_j,\ldots) = (\ldots,\beta_j-(j-i),\ldots,\beta_i+(j-i),\ldots). \tag{2.4}$$

Also let $t_i := t_{i\ i+1}$.

This is well-known, and clear from (2.3) and the row-swap property of determinants:

Lemma 2.10. $s_{t_i\alpha}(x_1,\ldots,x_n) = -s_\alpha(x_1,\ldots,x_n)$. *If $\alpha_{i+1} = \alpha_i + 1$ then $s_\alpha(x_1,\ldots,x_n) = 0$.*

It follows that:

Lemma 2.11. *Let $\beta \in \mathsf{Comp}_n$, then*

$$\pi_{w_0(I)}(x_1^{\beta_1}\cdots x_n^{\beta_n}) \in \{0, \pm s_{\alpha^1,\ldots,\alpha^k}\},$$

where $(\alpha^1,\ldots,\alpha^k) \in \mathsf{Par}_D$.

Fix $\gamma \in \mathsf{Par}_D$. We argue [9, Proposition 5.7] that the set

$$\mathcal{P}_{c\lambda,\gamma} := \{\beta \in \mathsf{Comp}_n : [x^\beta]\kappa_{c\lambda} \neq 0 \text{ and } \pi_{w_0(I)}x^\beta = \pm s_\gamma\}$$

has the structure of a poset isomorphic to an interval in (strong) Bruhat order of the Young subgroup $\mathfrak{S}_{d_1-d_0} \times \cdots \times \mathfrak{S}_{d_{k+1}-d_k}$ of $\mathfrak{S}_n$. This poset isomorphism is deduced in part by using combinatorial properties of key polynomials from [1, 5, 12]. The technical core is to establish a "diamond property" (in the sense of [15]) for $\mathcal{P}_{c\lambda,\gamma}$; this is [9, Theorem 5.3]. The upshot is that if

$$\Phi\colon \mathcal{P}_{c\lambda,\gamma} \to \mathfrak{S}_{d_1-d_0} \times \cdots \times \mathfrak{S}_{d_{k+1}-d_k}$$

is the aforementioned poset isomorphism, then in fact

$$\pi_{w_0(I)}x^\beta = (-1)^{\ell(\Phi(\beta))}s_\gamma.$$

Multiplicity-freeness of $\kappa_{w\lambda}$ then follows from this (mild extension of a) result of V. Deodhar [4], thus completing the (sketch) proof of $\Rightarrow$:

Lemma 2.12 ([9, Lemma 5.6])**.** *Let $\mathfrak{S} := \mathfrak{S}_{d_1-d_0} \times \cdots \times \mathfrak{S}_{d_{k+1}-d_k}$ be a Young subgroup of $\mathfrak{S}_n$. Suppose $[u,v] \subset \mathfrak{S}$ is an interval. Then*

$$\sum_{u\leq w\leq v} (-1)^{\ell(uw)} = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

Sketch proof of Theorem 2.7 "⇐": Now suppose w is not I-spherical. By Proposition 3.4, $u := w_0(I)w$ contains either a 321 pattern or a 3412 pattern. We select a suitable λ depending on which pattern u contains. Then we show that $\kappa_{w\lambda}$ has multiplicity. This is achieved using Kohnert's rule for key polynomials [12] combined with some further analysis of the poset $\mathcal{P}_{u\lambda}$ (defined similarly to $\mathcal{P}_{c\lambda}$ above). □

The following example, illustrates the ⇒ argument.

Example 2.13. Let $w = 265439871$ and $\lambda = 987654321$. Then $J(w) = \{1,3,4,5,7,8\}$ and let $I = J(w)$. Thus $w_0(I) = 216543987$ and w factors as $w_0(I)c$ with c the standard Coxeter element $c = 134567892 = s_2s_3s_4s_5s_6s_7s_8$. Additionally, $c^{-1} = 192345678$ and $w^{-1} = 915432876$. This yields $\alpha = c\lambda = 918765432$, and $w\lambda = 195678234$.

Since $D = [9] - I = \{2,6\}$, the key polynomial $\kappa_{w\lambda} = \kappa_{195678234} \in \Pi_D$ is separately symmetric in the sets of indeterminates $\{x_1,x_2\}, \{x_3,x_4,x_5,x_6\}, \{x_7,x_8,x_9\}$.

By [10, Theorem 4.13(II)], the fact that c is a standard Coxeter element implies that $\kappa_{c\lambda}$ is $[n-1]$-multiplicity-free. Now we consider the term $x^{981765432}$ that appears in $\kappa_{c\lambda}$.

Observe $\pi_{w_0(I)}(x^{981765432}) = s_{98,\underline{17}65,432} = -s_{98,6\underline{26}5,432} = s_{9,65\underline{35},432} = -s_{98,6544,432}$, where in each step we have underlined the swaps from applying Lemma 2.10.

The $\beta \in \mathsf{Comp}_n$ such that the monomial x^β of $\kappa_{c\lambda}$ satisfies $\pi_{w_0(I)}(x^\beta) = \pm s_{98,6544,432}$, along with the signs they contribute, are:

$$[9,8,1,7,6,5,4,3,2] : -1,\ [9,8,2,7,6,4,4,3,2] : 1,\ [9,8,6,2,6,5,4,3,2] : 1,$$
$$[9,8,4,7,3,5,4,3,2] : 1,\ [9,8,6,3,6,4,4,3,2] : -1,\ [9,8,6,5,3,5,4,3,2] : -1,$$
$$[9,8,4,7,4,4,4,3,2] : -1,\ [9,8,6,5,4,4,4,3,2] : 1.$$

These elements form the poset $\mathcal{P}_{c\lambda,\gamma=98,6544,432}$ which is shown in Figure 1 and is isomorphic to the interval $[\mathrm{id}, s_3s_4s_5]$ in Bruhat order. The associated coefficients sum to zero, agreeing with the preceding discussion on the Möbius function. □

3 Another definition of I-spherical elements

Let Φ be a finite crystallographic root system, with positive roots Φ^+, and simple roots $\Delta = \{\alpha_1,\ldots,\alpha_r\}$. Let W be its finite Weyl group with corresponding simple generators $S = \{s_1,s_2,\ldots,s_r\}$, where we have fixed a bijection of $[r] := \{1,2,\ldots,r\}$ with the nodes

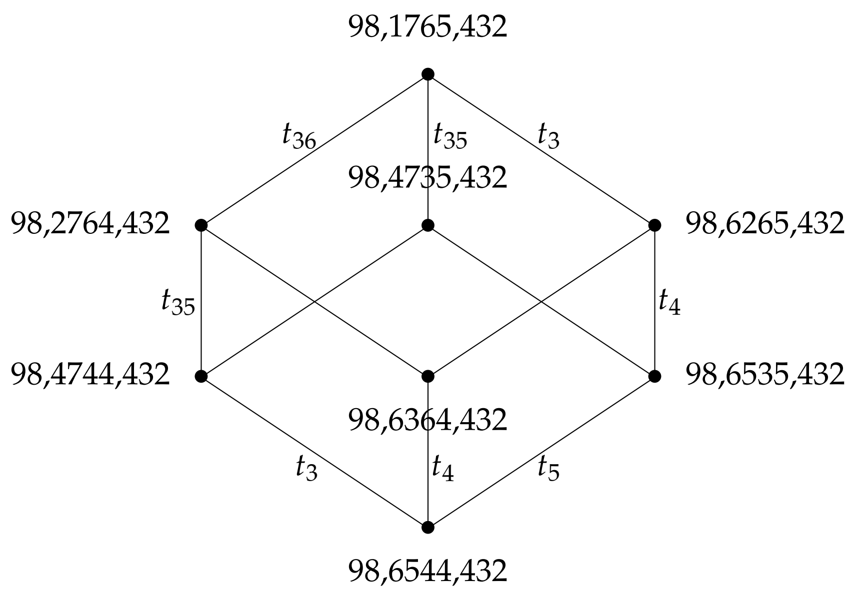

Figure 1: The poset $\mathcal{P}_{c\lambda,\gamma}$ for $c = 234567918$, $\lambda = 987654321$, $\gamma = 986544432$, $I = \{1,3,4,5,7,8\}$ with some edges labeled.

of the Dynkin diagram $\mathcal{G}$. Let $\mathrm{Red}(w)$ be the set of the *reduced expressions* $w = s_{i_1}\cdots s_{i_k}$, where $k = \ell(w)$ is the *Coxeter length* of w. The *left descents* of w are

$$J(w) = \{j \in [r] : \ell(s_j w) < \ell(w)\}.$$

For $I \in 2^{[r]}$, let $\mathcal{G}_I$ be the induced subdiagram of $\mathcal{G}$. Write $\mathcal{G}_I = \bigcup_{z=1}^{m} \mathcal{C}^{(z)}$ as its decomposition into connected components. Let $w_0^{(z)}$ be the longest element of the parabolic subgroup $W_{I^{(z)}}$ generated by $I^{(z)} = \{s_j : j \in \mathcal{C}^{(z)}\}$. This general-type definition of I-spherical was proposed in [10]:

Definition 3.1 ([10, Definition 1.1])**.** Let $w \in W$ and fix $I \subset J(w)$. Then w is *I-spherical* if there exists $R = s_{i_1}\cdots s_{i_{\ell(w)}} \in \mathrm{Red}(w)$ such that

- $\#\{t \mid i_t = j\} \leq 1$ for all $j \in [r] - I$, and
- $\#\{t \mid i_t \in \mathcal{C}^{(z)}\} \leq \ell(w_0^{(z)}) + \#\mathrm{vertices}(\mathcal{C}^{(z)})$ for $1 \leq z \leq m$.

Such an R is called an *I-witness*.

Definition 1.2 makes sense in the general context as well. However, that notion differs from Definition 3.1 in type D_4 and F_4 (although we suspect they are equivalent for B_n and C_n types). Nevertheless, this next proposition says that Definition 3.1 is, in general, "close" to Definition 1.2.

Proposition 3.2 ([9, Proposition 2.6])**.** *If $w \in W$ is I-spherical (in the sense of Definition 3.1), then there exists an I-witness R of w of the form $R = R'R''$ where $R' \in \mathrm{Red}(w_0(I))$ and $R'' \in \mathrm{Red}(w_0(I)w)$.*

Moreover, in type A, the two notions are indeed equivalent:

Theorem 3.3 ([9, Theorem 1.3])**.** *Definitions 1.2 and 3.1 are equivalent for $W = \mathfrak{S}_n$.*

Proof sketch: The $\Rightarrow$ direction is clear.

For the converse recall that $w \in \mathfrak{S}_n$ *contains the pattern* $u \in \mathfrak{S}_k$ if there exists $i_1 < i_2 < \cdots < i_k$ such that $w(i_1), w(i_2), \ldots, w(i_k)$ is in the same relative order as $u(1), u(2), \ldots, u(k)$. Furthermore w *avoids* u if no such indices exist. $\square$

Proposition 3.4 ([18])**.** *A permutation $w \in \mathfrak{S}_n$ is a product of distinct generators,* i.e.*, a standard Coxeter element in some parabolic subgroup, if and only if w avoids 321 and 3412.*

Assume w is I-spherical with some I-witness. By Proposition 3.2 and Definition 3.1, we write $w = w_0(I)u$ such that there is a reduced word $R'' = s_{i_1} \cdots s_{i_{\ell(u)}}$ of u such that

- s_{d_i} appears at most once in R''; and
- $\#\{m \mid d_{t-1} < i_m < d_t\} < \binom{d_t - d_{t-1}+1}{2} - \binom{d_t - d_{t-1}}{2} = d_t - d_{t-1}$ for $1 \le t \le k+1$.

By Proposition 3.4, it remains to show that $u = w_0(I) \cdot w$ avoids 321 and 3412. This is established by direct considerations. $\square$

4 A (new) classification conjecture for all Lie types

Let G be a complex, connected, semisimple Lie group. Fix a choice B Borel subgroup and its maximal torus T. The *generalized flag variety* is G/B. Its Weyl group is $W \cong N(T)/T$; it is generated by simple reflections $S = \{s_1, s_2, \ldots, s_r\}$ as in Section 3. The *Schubert varieties* $\overline{BwB/B}$ are indexed by $w \in W$. For $I \subseteq J(w)$, there is a parabolic subgroup $P_I \supset B$. Let L_I be the standard Levi subgroup of P_I. As explained in [10, Section 1.2], Definition 1.1 extends *verbatim* to this more general setting. This is the main conjecture of this report:

Conjecture 4.1. *Let $I \subseteq J(w)$. X_w is L_I-spherical if and only if $w \in W$ is I-spherical (in the sense of Definition 1.2).*

We claim (details omitted here) that Theorem 2.6 generalizes to this context, with the exception that the key polynomial is replaced by the more general notion of *Demazure character* $D_{w,\lambda}$ where $\lambda \in \mathbb{Q}[\Lambda]$ is a weight, that is $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}$. $D_{w,\lambda}$ is an element of the *weight ring*, *i.e.* the Laurent polynomial ring generated by formal exponentials $e^{\pm\omega}$ where ω is a fundamental weight associated to G.

Using SageMath we are able to check in the classical types B_n, C_n, D_n ($n \leq 6$) that for a fixed dominant integral weight $\lambda(n)$ (that depends only on n), w is not I-spherical if and only if $D_{w,\lambda(n)}$ is not multiplicity-free as an L_I-character. This gives a complete verification of the "$\Rightarrow$" direction of Conjecture 4.1 for these low-rank cases; it also gives nontrivial evidence for the converse.

Since we have already remarked that Definition 3.1 and Definition 1.2 disagree in type D_4, it follows that [10, Conjecture 1.9] is false for $G = SO_8$. This disproves the general version of the general-type conjecture of [10].

Now, we have further evidence for "$\Leftarrow$":

Theorem 4.2. *Conjecture 4.1 "$\Leftarrow$" holds for $G = Sp_{2n}$ (type C_n).*

The proof also should extend to type D_n. We now sketch the type C_n argument. The main idea is to use the fact that $G = Sp_{2n}$ may be realized as the fixed point locus of an involution σ on $H = SL_{2n}$. We recall this construction and refer the reader to [13, Section 6] for additional details. Define the block matrix

$$E = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix},$$

where J is the $n \times n$ matrix with 1's on the antidiagonal and 0's elsewhere. Let $\sigma\colon H \to H$ be the map that sends A to $E(A^T)^{-1}E^{-1}$. Then

$$G = \{A \in H | A^T E A = E\} = \{A \in H | E(A^T)^{-1}E^{-1} = A\} = H^{\sigma}.$$

More is true. Let B_H be the Borel subgroup of upper triangular matrices in H, and T_H the subgroup of diagonal matrices. Setting $B_G = B_H^{\sigma}$ and $T_G = T_H^{\sigma}$, B_G and T_G are, respectively, a Borel subgroup and maximal torus in G.

Let $W_H = N_H(T_H)/T_H$ be the Weyl group of H and $W_G = N_G(T_G)/T_G$ be the Weyl group of G. Then $N_G(T_G) = N_H(T_H)^{\sigma}$, and hence there is a canonical injection $\iota\colon W_G \hookrightarrow W_H$. Identifying W_G with its image under ι gives

$$W_G = \{(a_1, \ldots, a_{2n}) \in S_{2n} | a_i = 2n + 1 - a_{2n+1-i} \text{ for } i \in [2n]\}.$$

For $w = (a_1, \ldots, a_{2n}) \in W_G$, let $\mathsf{ex}(w) := |\{i \in [n] | a_i > n\}|$.

Proposition 4.3 ([13, Proposition 6.1.0.1])**.** *For $w = (a_1, \ldots, a_{2n}) \in W_G$, we have $\ell_G(w) = \frac{1}{2}(\ell_H(\iota(w)) + \mathsf{ex}(w))$, where $\ell_G(w)$ is the Coxeter length of $w \in W_G$ (and similarly for $\ell_H(w)$).*

Corollary 4.4. *If w_0^G and w_0^H are the long elements in W_G and W_H (resp.) then $\iota(w_0^G) = w_0^H$.*

Let $\overline{\sigma}\colon [2n] \to [2n]$ be the map which sends i to $2n - i$. The canonical injection $\iota\colon W_G \hookrightarrow W_H$ is the group homomorphism [2, Section 8.1] with

$$\iota(s_i) = \begin{cases} s_i s_{\overline{\sigma}(i)} & \text{if } i < n \\ s_i & \text{if } i = n \end{cases}. \tag{4.1}$$

For $w \in W_G$, denote the set of left descents of w as $J_G(w)$. Denote the set of left descents of $\iota(w) \in W_H$, as $J_H(w)$. The map $\overline{\sigma}$ induces a map, which we also denote $\overline{\sigma}$, from $\mathcal{P}([2n])$, the power set of $[2n]$, to itself. Let $\bar{\imath}\colon \mathcal{P}([n]) \mapsto \mathcal{P}([2n])^{\overline{\sigma}}$ be the map that sends $S \in \mathcal{P}([n])$ to $T \subseteq \mathcal{P}([2n])^{\overline{\sigma}}$ where $i \in T$ if and only if $i \in S$.

This is proved using the *exchange property* of Bruhat order and Proposition 4.3:

Lemma 4.5. *Let $w \in W_G$. Then $\bar{\imath}(J_G(w)) = J_H(w) \in \mathcal{P}([2n])^{\overline{\sigma}}$.*

Using Corollary 4.4 and Lemma 4.5 one shows:

Proposition 4.6. *Let $w \in W_G$ and let $I_G \subseteq J_G(w)$ with $I_H \subseteq J_H(w)$ such that $\bar{\imath}(I_G) = I_H \in \mathcal{P}([2n])^{\overline{\sigma}}$. Then w is I_G-spherical implies $\iota(w)$ is I_H-spherical.*

Proof of Theorem 4.2: Let $I_H \subseteq J_H(w)$ such that $\iota(I_G) = I_H \in \mathcal{P}([2n])^{\overline{\sigma}}$. If w is I_G-spherical, then $\iota(w)$ is I_H-spherical by Proposition 4.6. By Theorem 1.3, $X_{\iota(w)}$ is L_{I_H}-spherical. By [16, Theorem 2.1.2], this is equivalent to the existence of a Borel subgroup B_{L_H} in L_{I_H} such that B_{L_H} has finitely many orbits in $X_{\iota(w)}$. Then, as a set, $X_{\iota(w)} = \bigcup_{1\leq k\leq z} B_{L_H} \cdot x_k$ for some $z \in \mathbb{Z}_{>0}$ and $x_1, \ldots, x_z \in X_{\iota(w)}$. Now, $X_w = X_{\iota(w)} \cap G/B_G$ [13, Proposition 6.1.1.2], and therefore, set-theoretically,

$$X_w = \left(\bigcup_{1\leq k\leq z} B_{L_H} \cdot x_k \right) \cap G/B_G \tag{4.2}$$

Suppose that $B_{L_H} \cdot x_k \cap G/B_G \neq \emptyset$. Modifying x_k if necessary, we may assume without loss that $x_k \in G/B_G$. The parabolic subgroup $P_{I_G} = P^{\sigma}_{I_H}$ and its Levi $L_{I_G} = L^{\sigma}_{I_H}$. Further, $B_{L_G} := B^{\sigma}_{L_H}$ is a Borel in L_{I_G}. We claim that $B_{L_H} \cdot x_k \cap G/B_G = B_{L_G} \cdot x_k$. Proving this claim completes our proof since then (4.2) implies B_{L_G} has finitely many orbits in X_w, which by [16, Theorem 2.1.2] is equivalent to X_w being L_{I_G}-spherical.
($\subseteq$) We have $B_{L_G} \cdot x_k \subseteq B_{L_H} \cdot x_k \cap G/B_G$ since $B_{L_G} \subseteq B_{L_H}$ and $B_{L_G} \subseteq \mathsf{stab}_G(X_w)$.
($\supseteq$) Let $b \in B_{L_H}$. Suppose that $bx_k \in G/B_G$. Let $\overline{x}_k$ be a coset representative of x_k in G. Then $bx_k \in G/B_G$ implies $b\overline{x}_k \in G$. This implies $\overline{x}_k^T b^T E b \overline{x}_k = E$ which further implies $b^T E b = (\overline{x}_k^T)^{-1} E (\overline{x}_k)^{-1} = E \overline{x}_k E^{-1} E (\overline{x}_k)^{-1} = E$. Thus $b \in B^{\sigma}_{L_H} = B_{L_G}$. □

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