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Mockingbird Lattices

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Abstract. We study combinatorial and order theoretic structures arising from the fragment of combinatory logic spanned by the basic combinator $\mathbf{M}$. This basic combinator, named as the Mockingbird by Smullyan, is defined by the rewrite rule $\mathbf{M}\mathsf{x}_1 \to \mathsf{x}_1\mathsf{x}_1$. We prove that the reflexive and transitive closure of this rewrite relation is a partial order on terms on $\mathbf{M}$ and that all connected components of its rewrite graph are Hasse diagrams of lattices. This last result is based on the introduction of lattices on some forests. We enumerate the elements, the edges of the Hasse diagrams, and the intervals of these lattices with the help of formal power series on terms and on forests.

Keywords: partial orders, lattices, combinatory logic, rewrite systems, treelike structures, formal power series

Introduction

Combinatory logic [5] is a model of computation introduced by Schönfinkel [7] and developed by Curry [3] with the objective to abstain from the need of bound variables specific to the λ-calculus. Its combinatorial heart is formed by terms, which are binary trees with labeled leaves, and rules to compute a result from a term, which are rewrite relations on trees [2]. An important instance is the system containing the basic combinators $\mathbf{K}$ and $\mathbf{S}$ with the rewrite rules $\mathbf{K}\mathsf{x}_1\mathsf{x}_2 \to \mathsf{x}_1$ and $\mathbf{S}\mathsf{x}_1\mathsf{x}_2\mathsf{x}_3 \to \mathsf{x}_1\mathsf{x}_3(\mathsf{x}_2\mathsf{x}_3)$. This system is important because it is combinatorially complete: each λ-term can be translated, by bracket abstraction algorithms [7] into a term over $\mathbf{K}$ and $\mathbf{S}$ emulating it.

A lot of other basic combinators with their own rewrite rules have been introduced by Smullyan in [9] after —now widely used— bird names, forming the enchanted forest of combinator birds. For instance, $\mathbf{K}$ is the Kestrel and $\mathbf{S}$ is the Starling. Usual computer science questions consist in considering a fragment of combinatory logic, that is a finite set of basic combinators with their rewrite rules, and to ask whether **(a)** Given two terms t and t', can we decide if t and t' can be rewritten eventually in a same term? This is known as the word problem [2, 12]. It admits a positive answer for some basic combinators like the Lark [10, 11] and the Warbler [10] but is still open for the Starling [1]; **(b)** Given a

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term $\mathfrak{t}$, can we decide if all rewrite sequences starting from $\mathfrak{t}$ are finite? This is known as the strong normalization problem. It admits a positive answer for the Starling [15] and the Jay [6].

Here, we pursue this study in a different direction by adopting a combinatorial, order theoretic, and enumerative point of view. In particular, by denoting by $\preccurlyeq$ (resp. $\equiv$) the reflexive and transitive (resp. reflexive, symmetric, and transitive) closure of the rewrite relation, we try to **(a′)** Determine if $\preccurlyeq$ is a partial order relation; **(b′)** Determine in this case if each interval of this poset is a lattice; **(c′)** Enumerate the $\equiv$-equivalence classes of terms with respect to some size notions. This work fits in this general project consisting in mixing combinatory logic with combinatorics.

We start this project by studying the system made of the basic combinator $\mathbf{M}$, known as the Mockingbird [9, 13]. By drawing some portions of its rewrite graph, the first properties that stand out are that the graph does not contain any nontrivial loops and that its connected components are finite and have exactly one minimal and one maximal element. At this stage, driven by computer exploration, we conjecture that $\preccurlyeq$ is a partial order relation and that each $\equiv$-equivalence class is a lattice with respect to $\preccurlyeq$. This lattice property is for us a good clue for the fact that this system contains rather rich combinatorial properties. To prove this, we introduce a new lattice on duplicative forests, that are kinds of treelike structures, and construct a poset isomorphism between this last poset and the poset on terms on $\mathbf{M}$. The Mockingbird lattice of order $d \geqslant 0$ is the lattice $\mathrm{M}(d)$ consisting in the combinators on $\mathbf{M}$ greater than or equal to the right comb combinator on $\mathbf{M}$ of degree d. Since any combinator on $\mathbf{M}$ can be seen as a binary tree, this provides a new lattice structure on these objects. Many similar lattices have been studied on binary trees such as among others the Tamari lattice [14]. However, unlike these lattices having for any $d \geqslant 0$ a cardinality equal to the d-th Catalan number, the elements of $\mathrm{M}(d)$ are enumerated by a different integer sequence. To obtain enumerative results about the Mockingbird lattices and all the posets of terms on $\mathbf{M}$ in general, we use formal power series on terms and on duplicative forests. In this way, we enumerate the maximal and minimal elements of the poset of all terms on $\mathbf{M}$, and the cardinality, the number of edges of the Hasse diagram, and the number of intervals of $\mathrm{M}(d)$.

This paper is organized as follows. Section 1 contains definitions about terms, rewrite relations, and combinatory logic systems. In Section 2, we study the combinatory logic system on $\mathbf{M}$ and the Mockingbird lattices. Section 3 contains enumerative results. This text ends with the presentation of some open questions.

General notations and conventions. For any integers i and j, $[i, j]$ denotes the set $\{i, i+1, \dots, j\}$. For any integer i, $[i]$ denotes the set $[1, i]$ and $[\![i]\!]$ denotes the set $[0, i]$. For any set A, A^* is the set of words on A. For any $w \in A^*$ and $a \in A$, $|w|_a$ is the number of occurrences of a in w. The only word of length 0 is the empty word ϵ. If P is a statement, we denote by $\mathbb{1}_P$ the indicator function (equals to 1 if P holds and 0 otherwise).

1 Combinatory logic systems

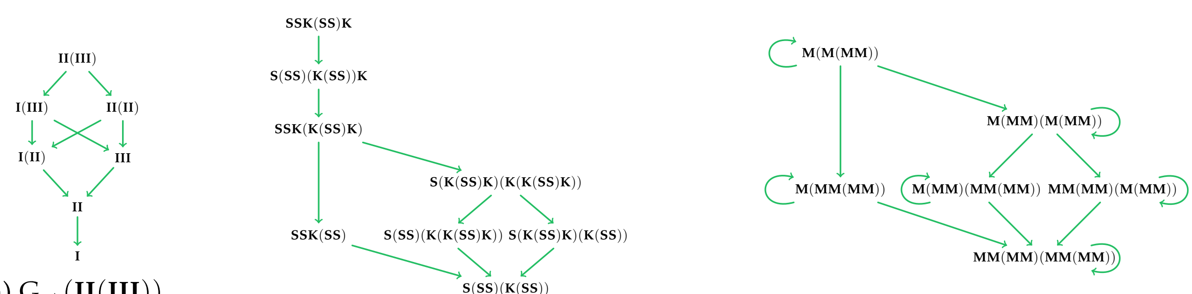
An *alphabet* is a finite set $\mathfrak{G}$. Its elements are called *basic combinators*. Any element of the set $\mathbb{X} := \bigcup_{n \geqslant 1} \mathbb{X}_n$, where $\mathbb{X}_n := \{x_1, \dots, x_n\}$, is a *variable*. The set $\mathfrak{T}(\mathfrak{G})$ of *$\mathfrak{G}$-terms* is so that any variable of $\mathbb{X}$ is a $\mathfrak{G}$-term, any basic combinator of $\mathfrak{G}$ is a $\mathfrak{G}$-term, and if $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two $\mathfrak{G}$-terms, then $(\mathfrak{t}_1 \star \mathfrak{t}_2)$ is a $\mathfrak{G}$-term. Any term is thus a rooted planar binary tree where leaves are decorated by variables or by basic combinators. We shall express terms concisely by removing superfluous parentheses by considering that $\star$ associates to the left and also by removing the symbols $\star$. For instance, if $\mathfrak{G} = \{\mathbf{A}, \mathbf{B}\}$, the $\mathfrak{G}$-term $\mathfrak{t} := (((\mathbf{A} \star \mathbf{B}) \star (x_1 \star x_2)) \star \mathbf{A})$ writes concisely as $\mathbf{AB}(x_1 x_2)\mathbf{A}$. Let $\mathfrak{t}$ be a $\mathfrak{G}$-term. The *degree* $\deg(\mathfrak{t})$ of $\mathfrak{t}$ is the number of internal nodes of $\mathfrak{t}$ seen as a binary tree. The *depth* of a node u of $\mathfrak{t}$ is the number of internal nodes in the path connecting the root of $\mathfrak{t}$ and u. The *height* $\mathrm{ht}(\mathfrak{t})$ of $\mathfrak{t}$ is the maximal depth among all the nodes of $\mathfrak{t}$. A *combinator* is a term having no occurrence of any variable.

Let $\mathfrak{t}$ and $\mathfrak{t}'_1, \dots, \mathfrak{t}'_n$, $n \geqslant 0$, be $\mathfrak{G}$-terms. The *composition* of $\mathfrak{t}$ with $\mathfrak{t}'_1, \dots, \mathfrak{t}'_n$ is the $\mathfrak{G}$-term $\mathfrak{t}[\mathfrak{t}'_1, \dots, \mathfrak{t}'_n]$ obtained by simultaneously replacing for all $i \in [n]$ all occurrences of the variables x_i in $\mathfrak{t}$ by $\mathfrak{t}'_i$. For instance $x_1(\mathbf{A}x_1)(x_4 x_2)[\mathbf{B}, x_1 x_3] = \mathbf{B}(\mathbf{AB})(x_4(x_1 x_3))$. Given two $\mathfrak{G}$-terms $\mathfrak{t}$ and $\mathfrak{s}$, $\mathfrak{s}$ is a *factor* of $\mathfrak{t}$ if $\mathfrak{t} = \mathfrak{t}'[\mathfrak{s}_1, \dots, \mathfrak{s}_{i-1}, \mathfrak{s}, \mathfrak{s}_{i+1}, \dots, \mathfrak{s}_n][\mathfrak{r}_1, \dots, \mathfrak{r}_m]$ for some integers $n, m \geqslant 0$ and $\mathfrak{G}$-terms $\mathfrak{t}'$, $\mathfrak{s}_1, \dots, \mathfrak{s}_{i-1}$, $\mathfrak{s}_{i+1}, \dots, \mathfrak{s}_n$, $\mathfrak{r}_1, \dots, \mathfrak{r}_m$, where x_i appears in $\mathfrak{t}'$. When this property does not hold, $\mathfrak{t}$ *avoids* $\mathfrak{s}$.

A *rewrite relation* on $\mathfrak{T}(\mathfrak{G})$ is a binary relation $\to$ on $\mathfrak{T}(\mathfrak{G})$. A *combinatory logic system* (or *CLS* for short) is a pair $\mathcal{C} := (\mathfrak{G}, \to)$ where $\mathfrak{G}$ is an alphabet and $\to$ is a rewrite relation on $\mathfrak{T}(\mathfrak{G})$ such that for each basic combinator $\mathbf{C}$ of $\mathfrak{G}$, there is exactly one rule of the form $\mathbf{C}x_1 \dots x_n \to \mathfrak{t}_{\mathbf{C}}$ where $n \geqslant 1$ and $\mathfrak{t}_{\mathbf{C}}$ is a term having no basic combinators and having all variables in $\mathbb{X}_n$. The integer n is the *order* of $\mathbf{C}$ in $\mathcal{C}$. Some well-known combinators $\mathbf{C}$ together with the terms $\mathfrak{t}_{\mathbf{C}}$ appearing among other in [9] are the Identity bird $\mathbf{I}$ of order 1 with $\mathfrak{t}_{\mathbf{I}} = x_1$, the Mockingbird $\mathbf{M}$ of order 1 with $\mathfrak{t}_{\mathbf{M}} = x_1 x_1$, the Kestrel $\mathbf{K}$ of order 2 with $\mathfrak{t}_{\mathbf{K}} = x_1$, and the Starling $\mathbf{S}$ of order 3 with $\mathfrak{t}_{\mathbf{S}} = x_1 x_3 (x_2 x_3)$. The *context closure* of $\to$ is the binary relation $\Rightarrow$ on $\mathfrak{T}(\mathfrak{G})$ defined as follows. For any $\mathbf{C} \in \mathfrak{G}$, by denoting by n the order of $\mathbf{C}$, we have $\mathbf{C}x_1 \dots x_n[\mathfrak{s}_1, \dots, \mathfrak{s}_n] \Rightarrow \mathfrak{t}_{\mathbf{C}}[\mathfrak{s}_1, \dots, \mathfrak{s}_n]$ for any $\mathfrak{s}_1, \dots, \mathfrak{s}_n \in \mathfrak{T}(\mathfrak{G})$, and $\mathfrak{t}_1 \mathfrak{t}_2 \Rightarrow \mathfrak{t}'_1 \mathfrak{t}_2$ for any $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}(\mathfrak{G})$ whenever $\mathfrak{t}_1 \Rightarrow \mathfrak{t}'_1$, and $\mathfrak{t}_1 \mathfrak{t}_2 \Rightarrow \mathfrak{t}_1 \mathfrak{t}'_2$ for any $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}(\mathfrak{G})$ whenever $\mathfrak{t}_2 \Rightarrow \mathfrak{t}'_2$. For instance, if $\mathcal{C}$ is the CLS containing the basic combinators $\mathbf{K}$ and $\mathbf{S}$, we have $\mathbf{S}(\mathbf{KKS})\mathbf{K}(\mathbf{SS}) \Rightarrow \mathbf{SKK}(\mathbf{SS}) \Rightarrow \mathbf{K}(\mathbf{SS})(\mathbf{K}(\mathbf{SS})) \Rightarrow \mathbf{SS}$.

Given a CLS $\mathcal{C} := (\mathfrak{G}, \to)$, we denote by $\preccurlyeq$ the preorder defined as the reflexive and transitive closure of $\Rightarrow$. The *rewrite graph* $\mathrm{G}_{\mathcal{C}}$ of $\mathcal{C}$ is the digraph of the binary relation $\Rightarrow$ on $\mathfrak{T}(\mathfrak{G})$. For any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, $\mathrm{G}_{\mathcal{C}}(\mathfrak{t})$ is the subgraph of $\mathrm{G}_{\mathcal{C}}$ restrained on $\{\mathfrak{t}' \in \mathfrak{T}(\mathfrak{G}) : \mathfrak{t} \preccurlyeq \mathfrak{t}'\}$. When $\preccurlyeq$ is antisymmetric, $\mathcal{C}$ has the *poset property* and we denote by $\mathcal{P}_{\mathcal{C}}$ the poset $(\mathfrak{T}(\mathfrak{G}), \preccurlyeq)$. For any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, $\mathcal{P}_{\mathcal{C}}(\mathfrak{t})$ is the subposet of $\mathcal{P}_{\mathcal{C}}$ having $\mathfrak{t}$ as smallest element. When $\mathcal{C}$ has the poset property and, for any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, $\mathcal{P}_{\mathcal{C}}(\mathfrak{t})$ is a lattice, $\mathcal{C}$ has the *lattice property*. We denote by $\equiv$ the equivalence relation defined as the reflexive, symmetric, and transitive closure of $\Rightarrow$. If for any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, the $\equiv$-equivalence class $[\mathfrak{t}]_\equiv$ of $\mathfrak{t}$ is finite, then $\mathcal{C}$ is *locally finite*. When $\mathcal{C}$ has the poset property and, for any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, $[\mathfrak{t}]_\equiv$ has a unique minimal element, $\mathcal{C}$ is *rooted*. If for any $\mathfrak{t}, \mathfrak{s}_1, \mathfrak{s}_2 \in \mathfrak{T}(\mathfrak{G})$, $\mathfrak{t} \preccurlyeq \mathfrak{s}_1$ and $\mathfrak{t} \preccurlyeq \mathfrak{s}_2$ implies the existence of $\mathfrak{t}' \in \mathfrak{T}(\mathfrak{G})$ such that $\mathfrak{s}_1 \preccurlyeq \mathfrak{t}'$ and $\mathfrak{s}_2 \preccurlyeq \mathfrak{t}'$, then $\mathcal{C}$ is *confluent*.

Consider for instance the CLS $\mathcal{C}$ containing the combinator $\mathbf{I}$. It is straightforward to show that $\mathcal{C}$ has the poset property. Nevertheless, $\mathcal{C}$ has not the lattice property, as suggested by the Hasse diagram shown in Figure 1a. It is known that the CLS containing the combinators $\mathbf{K}$ and $\mathbf{S}$ has not the poset property. Figure 1b shows a subgraph of the rewrite graph of this CLS. Figure 1c shows a subgraph of the rewrite graph of the CLS containing the combinator $\mathbf{M}$. We shall study in details this CLS in the next sections.



(a) $\mathrm{G}_\rightarrow(\mathbf{II}(\mathbf{III}))$.

(b) $\mathrm{G}_\rightarrow(\mathbf{SSK}(\mathbf{SS})\mathbf{K})$.

(c) $\mathrm{G}_\rightarrow(\mathbf{M}(\mathbf{M}(\mathbf{MM})))$.

Figure 1: Some subgraphs of rewrite graphs of some CLS.

A basic combinator $\mathbf{C}$ is *hierarchical* if for any $i \in [n]$, x_i appears in $\mathfrak{t}_\mathbf{C}$ at depth $n+1-i$. For instance, all the terms $\mathfrak{t}_\mathbf{C}$ such that $\mathbf{C}$ are hierarchical and of order 3 or less are $\mathsf{x}_1\mathsf{x}_1$, $\mathsf{x}_1\mathsf{x}_1\mathsf{x}_2$, $\mathsf{x}_2(\mathsf{x}_1\mathsf{x}_1)$, $\mathsf{x}_1\mathsf{x}_1\mathsf{x}_2\mathsf{x}_3$, $\mathsf{x}_2(\mathsf{x}_1\mathsf{x}_1)\mathsf{x}_3$, $\mathsf{x}_3(\mathsf{x}_1\mathsf{x}_1\mathsf{x}_2)$, and $\mathsf{x}_3(\mathsf{x}_2(\mathsf{x}_1\mathsf{x}_1))$.

Proposition 1.1. *Any CLS is confluent.*

Proposition 1.2. *If all basic combinators of a CLS $\mathcal{C}$ are hierarchical, then $\mathcal{C}$ is locally finite and all the $\mathfrak{G}$-terms of a same connected component of $\mathrm{G}_\mathcal{C}$ have the same height.*

Observe that by Propositions 1.1 and 1.2, if $\mathcal{C}$ has the poset property and all its basic combinators are hierarchical, then for any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, the subposet $[\mathfrak{t}]_\equiv$ of $\mathcal{P}_\mathcal{C}$ has exactly one maximal element. If additionally $\mathcal{C}$ is rooted, then for any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, the subposet $[\mathfrak{t}]_\equiv$ of $\mathcal{P}_\mathcal{C}$ has exactly one minimal element.

2 The Mockingbird combinatory logic system

Let $\mathcal{C} := (\mathfrak{G}, \rightarrow)$ be the CLS such that $\mathfrak{G} := \{\mathbf{M}\}$. We call $\mathcal{C}$ the *Mockingbird CLS*. From now, we shall simply write G instead of $\mathrm{G}_\mathcal{C}$.

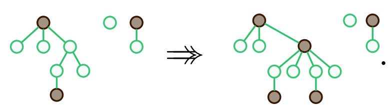
Proposition 2.1. *The Mockingbird CLS is locally finite, has the poset property, and is rooted.*

Proposition 2.1 is a consequence of the fact that $\mathbf{M}$ is hierarchical and of Proposition 1.2. By Proposition 2.1, $\mathcal{P}_{\mathcal{C}}$ is a well-defined poset. From now, we shall simply write $\mathcal{P}$ instead of $\mathcal{P}_{\mathcal{C}}$.

Proposition 2.2. *A combinator* $\mathfrak{t}$ *is a maximal (resp. minimal) element of* $\mathcal{P}$ *if and only if* $\mathfrak{t}$ *avoids* $\mathbf{M}(\mathsf{x}_1\mathsf{x}_2)$ *(resp.* $(\mathsf{x}_1\mathsf{x}_2)(\mathsf{x}_1\mathsf{x}_2)$*).*

A *duplicative tree* is a planar rooted tree such that each node is either a *black node* $\bullet$ or a *white node* $\circ$. A *duplicative forest* is a word $\mathfrak{f}$ of duplicative trees. We denote by $\mathcal{D}$ (resp. $\mathcal{D}^*$) the set of such trees (resp. forests). The *height* $\mathrm{ht}(\mathfrak{f})$ of $\mathfrak{f}$ is the number of internal nodes in a longest path following edges connecting a node to one of its child. Each expression using some occurrences of $\square$ denotes the two expressions obtained by replacing simultaneously all $\square$ either by $\circ$ or by $\bullet$. The *grafting product* is the operation $\square$ on $\mathcal{D}^*$ such that for any $\mathfrak{f} \in \mathcal{D}^*$, $\square(\mathfrak{f})$ is the duplicative tree obtained by grafting the roots of the duplicative trees of $\mathfrak{f}$ on a common root node $\square$. The *concatenation product* is the binary operation $\centerdot$ on $\mathcal{D}^*$ such that for any $\mathfrak{f}_1, \mathfrak{f}_2 \in \mathcal{D}^*$, $\mathfrak{f}_1 \centerdot \mathfrak{f}_2$ is the duplicative forest made of the trees of $\mathfrak{f}_1$ and then of the trees of $\mathfrak{f}_2$.

Let $\Rrightarrow$ be the binary relation on $\mathcal{D}^*$ such that for any $\mathfrak{f}, \mathfrak{f}' \in \mathcal{D}^*$, we have $\mathfrak{f} \Rrightarrow \mathfrak{f}'$ if $\mathfrak{f}'$ can be obtained from $\mathfrak{f}$ by selecting a white node of $\mathfrak{f}$, by turning it into black, and by duplicating its sequence of descendants. For instance, we have



(2.1)

Observe that in this case, there are more black nodes in $\mathfrak{f}'$ than in $\mathfrak{f}$. Hence, the reflexive and transitive closure $\ll$ of $\Rrightarrow$ is antisymmetric so that $(\mathcal{D}^*, \ll)$ is a poset. For any $\mathfrak{f} \in \mathcal{D}^*$, we denote by $\mathcal{D}^*(\mathfrak{f})$ the subposet of $(\mathcal{D}^*, \ll)$ on the set $\{\mathfrak{f}' \in \mathcal{D}^* : \mathfrak{f} \ll \mathfrak{f}'\}$. Figure 2 shows the Hasse diagram of the poset $\mathcal{D}^*(\mathfrak{f})$ for an $\mathfrak{f} \in \mathcal{D}^*$. According to this Hasse

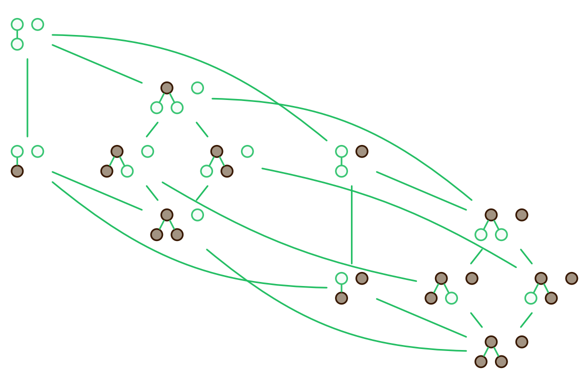


Figure 2: The Hasse diagram of a maximal interval of the poset of duplicative forests.

diagram, the poset $\mathcal{D}^*$ is not graded.

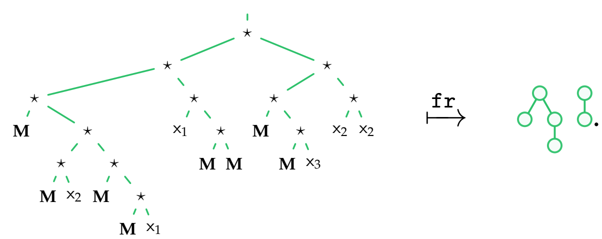
Let $\wedge$ and $\vee$ be the two binary, commutative, and associative partial operations on $\mathcal{D}^*$ defined, for any $\ell \geqslant 0$, $\mathfrak{f}_1, \dots, \mathfrak{f}_\ell \in \mathcal{D}$, $\mathfrak{f}'_1, \dots, \mathfrak{f}'_\ell \in \mathcal{D}$, and $\mathfrak{f}, \mathfrak{f}', \mathfrak{f}'' \in \mathcal{D}^*$, by

$$\begin{aligned} \mathfrak{f}_1 \dots \mathfrak{f}_\ell \wedge \mathfrak{f}'_1 \dots \mathfrak{f}'_\ell &:= (\mathfrak{f}_1 \wedge \mathfrak{f}'_1) \dots (\mathfrak{f}_\ell \wedge \mathfrak{f}'_\ell), \\ \square(\mathfrak{f}) \wedge \square(\mathfrak{f}') &:= \square(\mathfrak{f} \wedge \mathfrak{f}'), \\ \circ(\mathfrak{f}) \wedge \bullet(\mathfrak{f}'\mathfrak{f}'') &:= \circ(\mathfrak{f} \wedge \mathfrak{f}' \wedge \mathfrak{f}'') \end{aligned} \tag{2.2}$$

$$\begin{aligned} \mathfrak{f}_1 \dots \mathfrak{f}_\ell \vee \mathfrak{f}'_1 \dots \mathfrak{f}'_\ell &:= (\mathfrak{f}_1 \vee \mathfrak{f}'_1) \dots (\mathfrak{f}_\ell \vee \mathfrak{f}'_\ell), \\ \square(\mathfrak{f}) \vee \square(\mathfrak{f}') &:= \square(\mathfrak{f} \vee \mathfrak{f}'), \\ \circ(\mathfrak{f}) \vee \bullet(\mathfrak{f}'\mathfrak{f}'') &:= \bullet((\mathfrak{f} \vee \mathfrak{f}')(\mathfrak{f} \vee \mathfrak{f}'')). \end{aligned} \tag{2.3}$$

Proposition 2.3. *For any $\mathfrak{f} \in \mathcal{D}^*$, the poset $\mathcal{D}^*(\mathfrak{f})$ is a lattice for the operations $\wedge$ and $\vee$.*

We call *duplicative forest lattice* of $\mathfrak{f} \in \mathcal{D}^*$ the lattice $\mathcal{D}^*(\mathfrak{f})$. To show that each subposet $\mathcal{P}(\mathfrak{t})$, $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, of $\mathcal{P}$ is a lattice, we introduce a poset isomorphism between $\mathcal{P}(\mathfrak{t})$ and an interval of a lattice of duplicative forests. For this, let $\mathtt{fr} colon \mathfrak{T}(\mathfrak{G}) \to \mathcal{D}^*$ be the map defined, for any $\mathsf{x}_i \in \mathbb{X}$ and $\mathfrak{t}, \mathfrak{t}', \mathfrak{t}'' \in \mathfrak{T}(\mathfrak{G})$, by $\mathtt{fr}(\mathsf{x}_i) := \mathtt{fr}(\mathbf{M}) := \mathtt{fr}(\mathbf{MM}) := \epsilon$, $\mathtt{fr}(\mathbf{M}\mathsf{x}_i) := \circ$, $\mathtt{fr}(\mathbf{M}(\mathfrak{t}\mathfrak{t}')) := \circ(\mathtt{fr}(\mathfrak{t}\mathfrak{t}'))$, $\mathtt{fr}(\mathsf{x}_i\mathfrak{t}) := \mathtt{fr}(\mathfrak{t})$, and $\mathtt{fr}((\mathfrak{t}\mathfrak{t}')\mathfrak{t}'') := \mathtt{fr}(\mathfrak{t}\mathfrak{t}') \centerdot \mathtt{fr}(\mathfrak{t}'')$. For instance,



(2.4)

Immediately from the definition, we observe that this map is not injective. It can be shown by structural induction on duplicative forests that the image of $\mathtt{fr}$ is the set of all duplicative forests with no black nodes.

Proposition 2.4. *For any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, the posets $\mathcal{P}(\mathfrak{t})$ and $\mathcal{D}^*(fr(\mathfrak{t}))$ are isomorphic.*

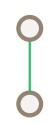
Theorem 2.5. *For any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$, the poset $\mathcal{P}(\mathfrak{t})$ is a finite lattice.*

The *Mockingbird lattice* of order $d \geqslant 0$ is the lattice $\mathrm{M}(d) := \mathcal{P}(\mathfrak{r}_d)$ where $\mathfrak{r}_d$ is the combinator defined by $\mathfrak{r}_0 := \mathbf{M}$ and, for any $d \geqslant 1$, by $\mathfrak{r}_d := \mathbf{M}\mathfrak{r}_{d-1}$. Figure 3 shows the Hasse diagrams of the first Mockingbird lattices. These lattices are not graded, not self-dual, and not semidistributive.

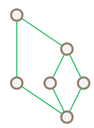
Theorem 2.6. *For any $\mathfrak{f} \in \mathcal{D}^*$, the poset $\mathcal{D}^*(\mathfrak{f})$ is isomorphic to a maximal interval of a Mockingbird lattice.*

Theorem 2.6 justifies the fact that the study of the Mockingbird lattices is universal enough because these lattices contain as maximal interval all duplicative forests lattices.

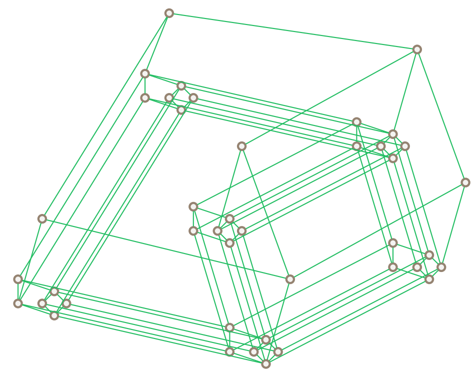
(a) $\mathrm{M}(0) \simeq \mathrm{M}(1)$.



(b) $\mathrm{M}(2)$.



(c) $\mathrm{M}(3)$.



(d) $\mathrm{M}(4)$.

Figure 3: The Hasse diagrams of the Mockingbird lattices $\mathrm{M}(d)$ for $d \in [\![4]\!]$.

3 Enumerative properties

Let $\mathbb{K}$ be any field of characteristic zero. For any set X, let $\mathbb{K}\langle X\rangle$ be the linear span of X. The dual space of $\mathbb{K}\langle X\rangle$ is denoted by $\mathbb{K}\langle\langle X\rangle\rangle$ and is by definition the space of the maps $\mathbf{f}\colon X \to \mathbb{K}$, called *$X$-series*. The coefficient $\mathbf{f}(x)$ of any $x \in X$ is denoted by $\langle x, \mathbf{f}\rangle$. The *support* of $\mathbf{f}$ is the set $\mathrm{Supp}(\mathbf{f}) := \{x \in X : \langle x, \mathbf{f}\rangle \neq 0\}$. The *characteristic series* of any subset X' of X is the series $\mathbf{c}(X')$ having X' as support and such that the coefficient of each $x \in X'$ is 1. For any $k \geqslant 0$, $\mathrm{T}^k\mathbb{K}\langle\langle X\rangle\rangle$ is the k-th tensor power of $\mathbb{K}\langle\langle X\rangle\rangle$. Elements of this space are possibly infinite linear combinations of tensors $x_1 \otimes \cdots \otimes x_k$, where for any $i \in [k]$, $x_i \in X$. The *tensor algebra* of $\mathbb{K}\langle\langle X\rangle\rangle$ is the space $\mathrm{T}^*\mathbb{K}\langle\langle X\rangle\rangle := \bigoplus_{k \geqslant 0} \mathrm{T}^k\mathbb{K}\langle\langle X\rangle\rangle$.

A linear map $\phi\colon \mathrm{T}^{k_1}\mathbb{K}\langle\langle X\rangle\rangle \to \mathrm{T}^{k_2}\mathbb{K}\langle\langle X\rangle\rangle$, $k_1, k_2 \geqslant 0$, is a *(k_1, k_2)-operation* on $\mathbb{K}\langle\langle X\rangle\rangle$. The *diagonal coproduct* is the $(1,2)$-operation Δ on $\mathbb{K}\langle\langle X\rangle\rangle$ satisfying $\Delta(x) = x \otimes x$ for any $x \in X$. When X is endowed with an associative operation $\star\colon X^2 \to X$, the *$\star$-flattening map* is for any $k \geqslant 1$ the $(k,1)$-operation $\mathrm{P}^k_\star$ on $\mathbb{K}\langle\langle X\rangle\rangle$ satisfying $\mathrm{P}^k_\star(x_1 \otimes \cdots \otimes x_k) = x_1 \star \cdots \star x_k$ for any $x_1, \dots, x_k \in X$. When X is endowed with an n-ary operation $\star : X^n \to X$, $n \geqslant 0$, the *linearization* of $\star$ is the $(n,1)$-operation $\bar{\star}$ on $\mathbb{K}\langle\langle X\rangle\rangle$ satisfying $\bar{\star}(x_1 \otimes \cdots \otimes x_n) = \star(x_1, \dots, x_n)$ for any $x_1, \dots, x_n \in X$. When $n = 1$, by a slight abuse of notation, for any $k \geqslant 1$ and $x_1, \dots, x_k \in X$, we set $\bar{\star}(x_1 \otimes \cdots \otimes x_k) := \bar{\star}(x_1) \otimes \cdots \otimes \bar{\star}(x_k)$. To lighten the notation, when $\star$ is a $(2,1)$-operation on $\mathbb{K}\langle\langle X\rangle\rangle$, we will use $\star$ as an infix operation by writing $\mathbf{f}_1 \star \mathbf{f}_2$ for $\star(\mathbf{f}_1 \otimes \mathbf{f}_2)$ for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{K}\langle\langle X\rangle\rangle$.

The space of the usual power series on the formal parameter z is denoted by $\mathbb{K}\langle\langle \mathrm{z}\rangle\rangle$. For any $F, F' \in \mathbb{K}\langle\langle \mathrm{z}\rangle\rangle$, $F[\mathrm{z} := F']$ is the series of $\mathbb{K}\langle\langle \mathrm{z}\rangle\rangle$ obtained by substituting F' for z in F. The *Hadamard product* is the binary operation $\boxtimes$ on $\mathbb{K}\langle\langle \mathrm{z}\rangle\rangle$ defined linearly for any $n_1, n_2 \geqslant 0$ by $\mathrm{z}^{n_1} \boxtimes \mathrm{z}^{n_2} := \mathbb{1}_{n_1 = n_2}\, \mathrm{z}^{n_1}$. The *max product* is the binary operation $\uparrow$ on $\mathbb{K}\langle\langle \mathrm{z}\rangle\rangle$ defined linearly for any $n_1, n_2 \geqslant 0$ by $\mathrm{z}^{n_1} \uparrow \mathrm{z}^{n_2} := \mathrm{z}^{\max\{n_1, n_2\}}$. If X is endowed with a map $\omega\colon X \to \mathbb{N}$, the *$\omega$-enumeration map* is the partial map $\mathrm{en}_\omega\colon \mathrm{T}^*\mathbb{K}\langle\langle X\rangle\rangle \to \mathbb{K}\langle\langle \mathrm{z}\rangle\rangle$ defined linearly for any $k \geqslant 1$ and $x_1, \dots, x_k \in X$ by $\mathrm{en}_\omega(x_1 \otimes \cdots \otimes x_k) := \mathrm{z}^{\omega(x_1)} \uparrow \dots \uparrow \mathrm{z}^{\omega(x_k)}$. For any $\mathbf{f} \in \mathrm{T}^*\mathbb{K}\langle\langle X \rangle\rangle$, the generating series $\mathrm{en}_\omega(\mathbf{f})$ is the *ω-enumeration* of $\mathbf{f}$. In the sequel, we shall use the following strategy to enumerate a set X with respect to such a map ω: we shall provide a description of $\mathbf{c}(X)$, then deduce a description of $\mathrm{en}_\omega(\mathbf{c}(X))$, and finally deduce from this a formula to compute the coefficients $\langle \mathrm{z}^n, \mathrm{en}_\omega(\mathbf{c}(X)) \rangle$, $n \geqslant 0$.

Recall now that $\star$ is the binary operation on $\mathfrak{T}(\mathfrak{G})$ satisfying, for any $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}(\mathfrak{G})$, $\mathfrak{t}_1 \star \mathfrak{t}_2 = \mathfrak{t}_1 \mathfrak{t}_2$. Proposition 2.2 leads to the following result.

Proposition 3.1. *The characteristic series $\mathbf{f}_{\max}$ of the maximal combinators of $\mathcal{P}$ satisfies*

$$\mathbf{f}_{\max} = \mathbf{M} + \mathbf{M}\mathbf{M} + \mathbf{f}_{\max} \,\overline{\star}\, \mathbf{f}_{\max} - \mathbf{M} \,\overline{\star}\, \mathbf{f}_{\max} \tag{3.1}$$

and the characteristic series $\mathbf{f}_{\min}$ of the minimal combinators of $\mathcal{P}$ satisfies

$$\mathbf{f}_{\min} = \mathbf{M} + \mathbf{M}\mathbf{M} + \mathbf{f}_{\min} \,\overline{\star}\, \mathbf{f}_{\min} - \overline{\star}(\Delta(\mathbf{f}_{\min})). \tag{3.2}$$

A consequence of Proposition 3.1 is that the deg-enumeration $F_{\max}$ of $\mathbf{f}_{\max}$ satisfies $F_{\max} = 1 + \mathrm{z} + \mathrm{z}F_{\max}^2 - \mathrm{z}F_{\max}$. The first coefficients are 1, 1, 1, 2, 4, 9, 21, 51 and form Sequence **A001006** (Motzkin numbers) of [8]. Another consequence of Proposition 3.1 is that the deg-enumeration $F_{\min}$ of $\mathbf{f}_{\min}$ satisfies $F_{\min} = 1 + \mathrm{z} + \mathrm{z}F_{\min}^2 - \mathrm{z}\, F_{\min}\left[\mathrm{z} := \mathrm{z}^2\right]$. We deduce from this that the number of these terms of degree $d \geqslant 0$ is $\mathbf{a}(d)$ where $\mathbf{a}$ is the integer sequence satisfying $\mathbf{a}(0) = \mathbf{a}(1) = 1$ and, for any $d \geqslant 2$,

$$\mathbf{a}(d) = \mathbf{b}(d-1) - \mathbb{1}_{d \text{ is odd}}\ \mathbf{a}((d-1)/2) \qquad \text{where} \qquad \mathbf{b}(d) := \sum_{i \in [\![d]\!]} \mathbf{a}(i)\ \mathbf{a}(d-i). \tag{3.3}$$

The first numbers are 1, 1, 2, 4, 12, 34, 108, 344 and form Sequence **A343663** of [8].

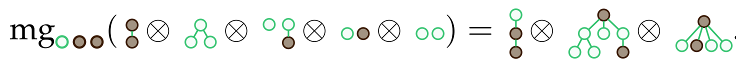
By Proposition 2.1, $\mathcal{C}$ has the properties described at the very end of Section 1. Therefore, the ht-enumerations of $\mathbf{f}_{\max}$ and $\mathbf{f}_{\min}$ are equal and is the generating series of the $\equiv$-equivalence classes of terms with respect to the height of their terms. By Proposition 3.1, by denoting it by F, it satisfies $F = 1 + \mathrm{z} + \mathrm{z}(F \uparrow F) - \mathrm{z}F$. Therefore, the number of these $\equiv$-equivalence classes of terms of height $h \geqslant 0$ is $\mathbf{a}(h)$ where $\mathbf{a}$ is the integer sequence satisfying $\mathbf{a}(0) = \mathbf{a}(1) = 1$ and, for any $h \geqslant 2$,

$$\mathbf{a}(h) = \mathbf{a}(h-1)^2 - \mathbf{a}(h-1) + 2\mathbf{a}(h-1) \sum_{i \in [h-1]} \mathbf{a}(i-1). \tag{3.4}$$

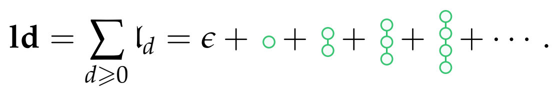
The first numbers are 1, 1, 2, 10, 170, 33490, 1133870930, 1285739648704587610 and form Sequence **A063573** of [8].

Let us now consider series on duplicative forests in order to obtain enumerative results on the Mockingbird lattices by using Proposition 2.4. For any $k \geqslant 1$ and $u \in \{\circ, \bullet\}^k$, the *merging product* is the $(k + |u|_\bullet, k)$-operation on $\mathbb{K}\langle\langle \mathcal{D}^* \rangle\rangle$ satisfying, for any $\mathfrak{f}_1, \dots, \mathfrak{f}_{k+|u|_\bullet} \in \mathcal{D}^*$, $\mathrm{mg}_\circ(\mathfrak{f}_1) = \circ(\mathfrak{f}_1)$, $\mathrm{mg}_{\circ u'}(\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_k) = \mathrm{mg}_\circ(\mathfrak{f}_1) \otimes \mathrm{mg}_{u'}(\mathfrak{f}_2 \otimes \cdots \otimes \mathfrak{f}_k)$,

$\mathrm{mg}_{\bullet}(\mathfrak{f}_1 \otimes \mathfrak{f}_2) = \bullet(\mathfrak{f}_1\mathfrak{f}_2)$, and $\mathrm{mg}_{\bullet u'}(\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_k) = \mathrm{mg}_{\bullet}(\mathfrak{f}_1 \otimes \mathfrak{f}_2) \otimes \mathrm{mg}_{u'}(\mathfrak{f}_3 \otimes \cdots \otimes \mathfrak{f}_k)$, where $u' \in \{\circ, \bullet\}^*$. For instance,


(3.5)

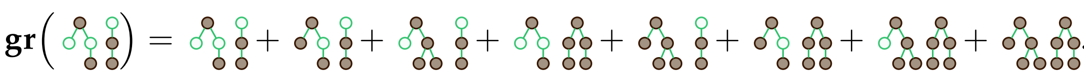
For any $d \geqslant 0$, the *d-ladder* is the duplicative forest $\mathfrak{l}_d$ defined recursively by $\mathfrak{l}_0 := \epsilon$ and, for any $d \geqslant 1$, by $\mathfrak{l}_d := \circ(\mathfrak{l}_{d-1})$. Let us denote by $\mathcal{L}$ the set $\bigcup_{d \geqslant 0} \mathcal{D}^*(\mathfrak{l}_d)$. The *series of ladders* is the unique $\mathcal{D}^*$-series $\mathbf{ld}$ satisfying $\mathbf{ld} = \epsilon + \bar{\circ}(\mathbf{ld})$. Hence,


(3.6)

Let $\mathbf{gr}$ be the $(1,1)$-operation on $\mathbb{K}\langle\langle \mathcal{D}^* \rangle\rangle$ satisfying, for any $\mathfrak{f} \in \mathcal{D}^*$,

$$\mathbf{gr}(\mathfrak{f}) = \sum_{\mathfrak{f}' \in \mathcal{D}^*(\mathfrak{f})} \mathfrak{f}'. \tag{3.7}$$

By definition, $\mathbf{gr}(\mathfrak{f})$ is the characteristic series of $\mathcal{D}^*(\mathfrak{f})$. For instance,


(3.8)

Observe that $\mathbf{gr}(\mathbf{ld})$ is the characteristic series of $\mathcal{L}$ and that $\mathrm{en}_{\mathrm{ht}}(\mathbf{gr}(\mathbf{ld}))$ is the generating series of the cardinalities of the lattices $\mathcal{D}^*(\mathfrak{l}_d)$, enumerated with respect to $d \geqslant 0$.

Theorem 3.2. *The series* $\mathbf{gr}(\mathbf{ld})$ *satisfies*

$$\mathbf{gr}(\mathbf{ld}) = \epsilon + \bar{\circ}(\mathbf{gr}(\mathbf{ld})) + \bar{\bullet}\Big(\mathbf{gr}\Big(\mathrm{P}^2_{\bullet}(\Delta(\mathbf{ld}))\Big)\Big). \tag{3.9}$$

We deduce from Theorem 3.2 that the ht-enumeration F of $\mathbf{gr}(\mathbf{ld})$ satisfies $F = 1 + \mathrm{z}F + \mathrm{z}(F \boxtimes F)$ so that for any $d \geqslant 1$, the number of elements in $\mathrm{M}(d)$ is $\mathbf{a}(d-1)$ where $\mathbf{a}$ is the integer sequence satisfying $\mathbf{a}(0) = 1$ and, for any $d \geqslant 1$,

$$\mathbf{a}(d) = \mathbf{a}(d-1) + \mathbf{a}(d-1)^2. \tag{3.10}$$

The sequence of the cardinalities of $\mathrm{M}(d)$, $d \geqslant 0$, starts by 1, 1, 2, 6, 42, 1806, 3263442, 10650056950806 and forms Sequence **A007018** of [8].

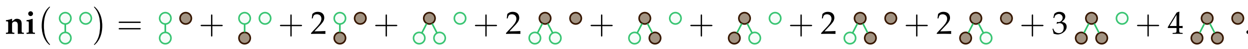
Let $\mathbf{cv}$ be the $(1,1)$-operation on $\mathbb{K}\langle\langle \mathcal{D}^* \rangle\rangle$ satisfying, for any $\mathfrak{f} \in \mathcal{D}^*$,

$$\mathbf{cv}(\mathfrak{f}) = \sum_{\substack{\mathfrak{f}' \in \mathcal{D}^* \\ \mathfrak{f} \Rrightarrow \mathfrak{f}'}} \mathfrak{f}'. \tag{3.11}$$

Let also $\mathbf{ni}$ be the $(1,1)$-operation on $\mathbb{K}\langle\langle \mathcal{D}^* \rangle\rangle$ satisfying $\mathbf{ni}(\mathfrak{f}) = \mathbf{cv}(\mathbf{gr}(\mathfrak{f}))$ for any $\mathfrak{f} \in \mathcal{D}^*$. By a straightforward computation, we obtain

$$\mathbf{ni}(\mathfrak{f}) = \sum_{\mathfrak{f}' \in \mathcal{D}^*(\mathfrak{f})} \#\{\mathfrak{f}'' \in \mathcal{D}^*(\mathfrak{f}) : \mathfrak{f}'' \Rrightarrow \mathfrak{f}'\}\, \mathfrak{f}', \tag{3.12}$$

so that the coefficient of each $\mathfrak{f}' \in \mathcal{D}^*(\mathfrak{f})$ in $\mathbf{ni}(\mathfrak{f})$ is the number of duplicative forests admitting $\mathfrak{f}'$ as covering in $\mathcal{D}^*(\mathfrak{f})$. For instance (see at the same time Figure 2),



(3.13)

Observe that $\mathrm{Supp}(\mathbf{ni}(\mathbf{ld})) = \mathcal{L} \setminus \{\mathfrak{l}_d : d \geqslant 0\}$ and that $\mathrm{en}_{\mathrm{ht}}(\mathbf{ni}(\mathbf{ld}))$ is the generating series of the number of edges of the Hasse diagrams of the lattices $\mathcal{D}^*(\mathfrak{l}_d)$, enumerated with respect to $d \geqslant 0$.

Theorem 3.3. *The series* $\mathbf{ni}(\mathbf{ld})$ *satisfies*

$$\mathbf{ni}(\mathbf{ld}) = \bar{\circ}(\mathbf{ni}(\mathbf{ld})) + \bar{\bullet}\Big(\mathbf{ni}\Big(\mathrm{P}^2_{\bullet}(\Delta(\mathbf{ld}))\Big)\Big) + \bar{\bullet}\Big(\mathrm{P}^2_{\bullet}(\Delta(\mathbf{gr}(\mathbf{ld})))\Big). \tag{3.14}$$

We deduce from Theorem 3.3 that the ht-enumeration F of $\mathbf{ni}(\mathbf{ld})$ satisfies $F = zF + zG + 2z(F \boxtimes G)$ where G is the ht-enumeration of $\mathbf{gr}(\mathbf{ld})$. Therefore, for any $d \geqslant 1$, the number of edges in the Hasse diagram of $\mathrm{M}(d)$ is $\mathbf{a}(d-1)$ where $\mathbf{a}$ is the integer sequence satisfying $\mathbf{a}(0) = 0$ and, for any $d \geqslant 1$,

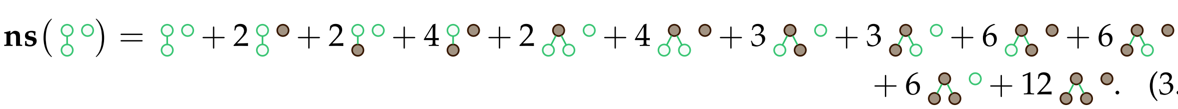
$$\mathbf{a}(d) = \mathbf{a}(d-1) + \mathbf{b}(d-1) + 2\mathbf{a}(d-1)\mathbf{b}(d-1), \tag{3.15}$$

where $\mathbf{b}$ is the integer sequence such that for any $d \geqslant 0$, $\mathbf{b}(d)$ is the number of elements of $\mathcal{D}^*(\mathfrak{l}_d)$, satisfying therefore (3.10). The sequence of the number of edges of the Hasse diagram of $\mathrm{M}(d)$, $d \geqslant 0$, starts by 0, 0, 1, 7, 97, 8287, 29942737, 195432804247687. This sequence does not appear in [8] for the time being.

Now, let $\mathbf{ns}$ be the $(1,1)$-operation on $\mathbb{K}\langle\langle\mathcal{D}^*\rangle\rangle$ satisfying $\mathbf{ns}(\mathfrak{f}) = \mathbf{gr}(\mathbf{gr}(\mathfrak{f}))$ for any $\mathfrak{f} \in \mathcal{D}^*$. By a straightforward computation, we obtain

$$\mathbf{ns}(\mathfrak{f}) = \sum_{\mathfrak{f}' \in \mathcal{D}^*(\mathfrak{f})} \#[\mathfrak{f}, \mathfrak{f}']\,\mathfrak{f}', \tag{3.16}$$

so that the coefficient of each $\mathfrak{f}' \in \mathcal{D}^*(\mathfrak{f})$ in $\mathbf{ns}(\mathfrak{f})$ is the number of duplicative forests smaller than or equal to $\mathfrak{f}'$ in $\mathcal{D}^*(\mathfrak{f})$. For instance (see at the same time Figure 2),



(3.17)

Contrary to what we have undertaken previously to express $\mathbf{gr}(\mathbf{ld})$ and $\mathbf{ni}(\mathbf{ld})$, we fail to directly express $\mathbf{ns}(\mathbf{ld})$. The trick here consists in considering first a slightly different series depending on a parameter $k \geqslant 1$ which can be seen as a catalytic parameter. For any $k \geqslant 1$, let $\mathbf{md}_k$ be the $(1,k)$-operation on $\mathbb{K}\langle\langle\mathcal{D}^*\rangle\rangle$ satisfying, for any $\mathfrak{f} \in \mathcal{D}^*$,

$$\mathbf{md}_k(\mathfrak{f}) = \sum_{\substack{\mathfrak{g}_1, \dots, \mathfrak{g}_k \in \mathcal{D}^*(\mathfrak{f}) \\ \mathfrak{g}_1 \wedge \dots \wedge \mathfrak{g}_k = \mathfrak{f}}} \mathfrak{g}_1 \otimes \dots \otimes \mathfrak{g}_k. \tag{3.18}$$

We call $\mathbf{md}_k(\mathfrak{f})$ the *meet k-decomposition* of $\mathfrak{f}$. Observe that $\mathbf{md}_1$ is the identity map.

Observe that $\mathrm{Supp}(\mathbf{ns}(\mathbf{ld})) = \mathcal{L}$ and that $\mathrm{en}_{\mathrm{ht}}(\mathbf{md}_1(\mathbf{ns}(\mathbf{ld})))$ is the generating series of the number of intervals of the lattices $\mathcal{D}^*(\mathfrak{l}_d)$, enumerated with respect to $d \geqslant 0$.

Theorem 3.4. *The series* $\mathbf{ns}(\mathbf{ld})$ *satisfies* $\mathbf{ns}(\mathbf{ld}) = \mathbf{md}_1(\mathbf{ns}(\mathbf{ld}))$ *where, for any* $k \geqslant 1$, *the series* $\mathbf{md}_k(\mathbf{ns}(\mathbf{ld}))$ *satisfies*

$$\mathbf{md}_k(\mathbf{ns}(\mathbf{ld})) = \epsilon^{\otimes k} + \sum_{u \in \{\circ, \bullet\}^k} \mathrm{mg}_u\Big(\mathbf{md}_{k+|u|_\bullet}(\mathbf{ns}(\mathbf{ld}))\Big) + \bar{\bullet}\Big(\mathbf{md}_k\Big(\mathbf{ns}\Big(\mathrm{P}^k_{\bullet}(\Delta(\mathbf{ld}))\Big)\Big)\Big). \tag{3.19}$$

We deduce from Theorem 3.4 that the ht-enumeration F of $\mathbf{ns}(\mathbf{ld})$ satisfies $F = F_1$ where, for any $k \geqslant 1$, F_k is the ht-enumeration of $\mathbf{md}_k(\mathbf{ns}(\mathbf{ld}))$ which satisfies $F_k = 1 + \mathsf{z}(F_k \boxtimes F_k) + \mathsf{z} \sum_{i \in [\![k]\!]} \binom{k}{i} F_{k+i}$. Therefore, for any $d \geqslant 1$, the number of intervals in $\mathrm{M}(d)$ is $\mathbf{a}_1(d-1)$ where for any $k \geqslant 1$, $\mathbf{a}_k$ is the integer sequence satisfying $\mathbf{a}_k(0) = 1$ and, for any $d \geqslant 1$,

$$\mathbf{a}_k(d) = \mathbf{a}_k(d-1)^2 + \sum_{i \in [\![k]\!]} \binom{k}{i} \mathbf{a}_{k+i}(d-1). \tag{3.20}$$

The sequence of the number of intervals of $\mathrm{M}(d)$, $d \geqslant 0$, starts by 1, 1, 3, 17, 371, 144513, 20932611523, 438176621806663544657. This sequence does not appear in [8] for the time being.

Open questions and future work

We have studied a CLS having many rich combinatorial properties despite its simplicity. This can be considered as the prototypical example for this kind of investigation. We expect to discover similar properties for more complex CLS. Additionally, here are three open questions raised by this work. **(1)** The description of minimal and maximal elements of $\mathcal{P}$ uses a notion of pattern avoidance in terms. This is a general fact: when a CLS $(\mathfrak{G}, \to)$ has the poset property, its minimal (resp. maximal) elements are the terms avoiding terms deduced from the ones appearing as right-hand (resp. left-hand) members of $\to$. Such an enumerative problem has been considered in [4] for the particular case of terms without repeating variables. We ask here for the general enumeration of terms avoiding a set of terms wherein multiple occurrences of a same variable are allowed. **(2)** We have shown that the Mockingbird CLS has the poset property, is rooted, and has the lattice property by employing some specific reasoning from the definition of the basic combinator **M**. A question here concerns the existence of a general criterion to decide if a CLS has the poset (resp. lattice) property and if it is rooted. **(3)** Finally, we have seen from Proposition 1.2 that being hierarchical is a sufficient condition for a CLS $\mathcal{C}$ to be locally finite. The question in this context consists in strengthening this result in order to obtain a necessary and sufficient condition for this last property.

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