Séminaire Lotharingien de Combinatoire **86B** (2022)
Article #33, 10 pp.

Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)

Affine Semigroups of Maximal Projective Dimension

Om Prakash Bhardwaj*[1], Kriti Goel†[2], and Indranath Sengupta‡[3]

[1]*Discipline of Mathematics, IIT Gandhinagar, India*
[2]*Discipline of Mathematics, IIT Gandhinagar, India*
[3]*Discipline of Mathematics, IIT Gandhinagar, India*

Abstract. A submonoid of $\mathbb{N}^d$ is of maximal projective dimension (MPD) if the associated affine semigroup k-algebra has the maximum possible projective dimension. Such submonoids have a nontrivial set of pseudo-Frobenius elements. We generalize the notion of symmetric semigroups, pseudo-symmetric semigroups, and row-factorization matrices for pseudo-Frobenius elements of numerical semigroups to the case of MPD-semigroups in $\mathbb{N}^d$. We prove that under suitable conditions these semigroups satisfy the generalized Wilf's conjecture. We prove that the generic nature of the defining ideal of the associated semigroup algebra of an MPD-semigroup implies the uniqueness of the row-factorization matrix for each pseudo-Frobenius element. Further, we give a description of pseudo-Frobenius elements and row-factorization matrices of gluing of MPD-semigroups. We prove that the defining ideal of gluing of MPD-semigroups is never generic.

Keywords: MPD-semigroup, pseudo-Frobenius elements, row-factorization matrix, generic toric ideals

1 Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and non-negative integers respectively. An affine semigroup S is a finitely generated submonoid of $\mathbb{N}^d$ for some positive integer d. When $d = 1$, affine semigroups correspond to numerical semigroups. Equivalently, a submonoid S of $\mathbb{N}$ is called a numerical semigroup if it has a finite complement in $\mathbb{N}$. If $S \neq \mathbb{N}$, then the largest integer not belonging to S is known as the Frobenius number of S, denoted by $\mathrm{F}(S)$. Also, the finiteness of $\mathbb{N} \setminus S$ implies that there exists at least one element $f \in \mathbb{N} \setminus S$ such that $f + (S \setminus \{0\}) \subset S$. These elements are called pseudo-Frobenius elements of the numerical semigroup S. The set of pseudo-Frobenius elements is denoted by $\mathrm{PF}(S)$. In particular, $\mathrm{F}(S)$ is a pseudo-Frobenius number. But, in general, this

*om.prakash@iitgn.ac.in
†kritigoel.maths@gmail.com
‡indranathsg@iitgn.ac.in

does not hold for affine semigroups in $\mathbb{N}^d$. The study of pseudo-Frobenius elements in affine semigroups over $\mathbb{N}^d$ is studied in [8], where the authors consider the complement of the affine semigroup in its rational polyhedral cone. They give a necessary and a sufficient condition for the existence of pseudo-Frobenius elements using properties of the associated semigroup ring. Let S be an affine semigroup and let k be a field. The semigroup ring $k[S] = \bigoplus_{s \in S} k\, \mathbf{t}^s$ of S is a k-subalgebra of the polynomial ring $k[t_1, \ldots, t_d]$, where $t_1, \ldots, t_d$ are indeterminates and $\mathbf{t}^s = \prod_{i=1}^{d} t_i^{s_i}$ for all $s = (s_1, \ldots, s_d) \in S$. In [8], the authors prove that an affine semigroup S has pseudo-Frobenius elements if and only if the length of the graded minimal free resolution of the corresponding semigroup ring is maximal. Affine semigroups having pseudo-Frobenius elements are called maximal projective dimension (MPD) semigroups. The cardinality of the set of pseudo-Frobenius elements is called the type of S. Note that this is not the Cohen–Macaulay type of S since MPD-semigroups, when $d \geq 2$, are not Cohen–Macaulay.

One of the widely studied class of numerical semigroups is symmetric semigroups — numerical semigroups of type 1. The motivation to study these semigroups comes from the work E. Kunz, who proved that a one-dimensional analytically irreducible Noetherian local ring is Gorenstein if and only if its value semigroup is symmetric. In other words, a numerical semigroup is symmetric if and only if the associated semigroup ring is Gorenstein. Symmetric numerical semigroups have odd Frobenius number. A numerical semigroup S is called pseudo-symmetric if $\mathrm{F}(S)$ is even and $\mathrm{PF}(S) = \{\mathrm{F}(S)/2, \mathrm{F}(S)\}$. We study a generalization of these notions to the case of MPD-semigroups. Let S be an MPD-semigroup and let $\mathrm{cone}(S)$ denote the rational polyhedral cone of S. Set $\mathcal{H}(S) = (\mathrm{cone}(S) \setminus S) \cap \mathbb{N}^d$. For a fixed term order $\prec$ on $\mathbb{N}^d$, we define the Frobenius element as $\mathrm{F}(S)_\prec = \max_\prec \mathcal{H}(S)$. Note that in the case of MPD-semigroups, Frobenius elements may not always exist, with respect to any term order. But if there is a term order $\prec$ such that $\mathrm{F}(S)_\prec$ exists and $|\mathrm{PF}(S)| = 1$, then we say that S is a $\prec$-symmetric semigroup. Further, if $\mathrm{PF}(S) = \{\mathrm{F}(S)_\prec/2, \mathrm{F}(S)_\prec\}$, then we say that S is a $\prec$-pseudo-symmetric semigroup.

Any affine semigroup S has a unique minimal generating set whose cardinality is known as the embedding dimension of S, and it is denoted by $e(S)$. In 1978, Wilf proposed a conjecture related to the Diophantine Frobenius Problem that claims that the inequality

$$\mathrm{F}(S) + 1 \leq e(S) \cdot |\{s \in S \mid s < \mathrm{F}(S)\}|$$

is true for every numerical semigroup. While this conjecture still remains open, a potential extension of Wilf's conjecture to affine semigroups is studied in [7]. We prove that $\prec$-symmetric and $\prec$-pseudo-symmetric semigroups satisfy the generalized Wilf's conjecture under suitable assumptions.

In [12], A. Moscariello introduced the notion of a row-factorization (RF) matrices associated to the pseudo-Frobenius elements of a numerical semigroup. He used this object to investigate the type of almost symmetric semigroups of embedding dimension four and prove a conjecture given by T. Numata in [14], which states that the type of an almost symmetric semigroup of embedding dimension four is at most three. In recent years, RF-matrices have been studied by K. Eto and J. Herzog–K. Watanabe in ([4, 5, 11]). We extend the definition of RF-matrices of the pseudo-Frobenius elements to the setting of MPD-semigroups. We also give a description of pseudo-Frobenius elements and their RF-matrices in a glued MPD-semigroup. For an affine semigroup S, let $G(S)$ denote the group generated by S in $\mathbb{Z}^d$. Recall that an affine semigroup S is said to be a gluing if there exists a non-trivial partition of its minimal generating set, $A_1 \amalg A_2$, and $d \in \langle A_1 \rangle \cap \langle A_2 \rangle$ such that $G(\langle A_1 \rangle) \cap G(\langle A_2 \rangle) = d\mathbb{Z}$.

Let k be a field and $S = \langle a_1, \ldots, a_n \rangle$ be a finitely generated submonoid of $\mathbb{N}^d$. Then the semigroup ring $k[S] = k[\mathbf{t}^{a_1}, \ldots, \mathbf{t}^{a_n}]$ of S can be represented as a quotient of a polynomial ring using a canonical surjection $\pi \colon k[x_1, \ldots, x_n] \to k[S]$ given by $\pi(x_i) = \mathbf{t}^{a_i}$ for all $i = 1, \ldots, n$. The kernel of this k-algebra homomorphism π, denoted by I_S, is a toric ideal, called defining ideal of $k[S]$, and the ring $k[S]$ is called a toric ring. A toric ideal is called generic if it has a minimal generating set consisting of binomials of full support. The notion of genericity of lattice ideal is introduced by I. Peeva and B. Sturmfels in [15]. The authors give a minimal free resolution, namely the Scarf complex, for the toric ring when the defining toric ideal is generic. The generic toric ideals are further studied in [9], where the authors prove that if I_S is generic toric ideal, then it has a unique minimal system of generators of indispensable binomials up to the sign of binomials. In [5], K. Eto gave a necessary and sufficient condition for the defining ideal I_S of a numerical semigroup ring $k[S]$ to be generic. For MPD-semigroups, we give a necessary condition for the generic nature of the defining ideal using RF-matrices. In this article we will be giving an overview of results, detailed proofs can be seen in [1].

2 Pseudo-Frobenius elements

Let S be a finitely generated submonoid of $\mathbb{N}^d$, say minimally generated by $a_1, \ldots, a_n \subseteq \mathbb{N}^d$. Such submonoids are called affine semigroups. Consider the cone of S in $\mathbb{Q}^d_{\geq 0}$,

$$\operatorname{cone}(S) := \left\{ \sum_{i=1}^{n} \lambda_i a_i \mid \lambda_i \in \mathbb{Q}_{\geq 0}, i = 1, \ldots, n \right\}$$

and set $\mathcal{H}(S) := (\operatorname{cone}(S) \setminus S) \cap \mathbb{N}^d$. An element $f \in \mathcal{H}(S)$ is called a pseudo-Frobenius element of S if $f + s \in S$ for all $s \in S \setminus \{0\}$. The set of pseudo-Frobenius elements of S is denoted by $\operatorname{PF}(S)$. In particular,

$$\operatorname{PF}(S) = \{ f \in \mathcal{H}(S) \mid f + a_j \in S, \text{ for all } j \in [1, n] \}.$$

Observe that the set $\operatorname{PF}(S)$ may be empty. Therefore, in this article, we consider such class of rings where $\operatorname{PF}(S)$ is non-empty. Let k be a field. The semigroup ring $k[S]$ of S is a k-subalgebra of the polynomial ring $k[t_1,\ldots,t_d]$. In other words, $k[S]=k[\mathbf{t}^{a_1},\ldots,\mathbf{t}^{a_n}]$, where $\mathbf{t}^{a_i}=t_1^{a_{i1}}\cdots t_d^{a_{id}}$ for $a_i=(a_{i1},\ldots,a_{id})$ and for all $i=1,\ldots,n$. Set $R=k[x_1,\ldots,x_n]$ and define a map $\pi\colon R\to k[S]$ given by $\pi(x_i)=\mathbf{t}^{a_i}$ for all $i=1,\ldots,n$. Set $\deg x_i=a_i$ for all $i=1,\ldots,n$. Observe that R is a multi-graded ring and that π is a degree preserving surjective k-algebra homomorphism. We denote by I_S the kernel of π. Then I_S is a homogeneous ideal, generated by binomials, called the defining ideal of S. Note that a binomial $\phi=\prod_{i=1}^n x_i^{\alpha_i}-\prod_{j=1}^n x_j^{\beta_j}\in I_S$ if and only if $\sum_{i=1}^n \alpha_i a_i=\sum_{j=1}^n \beta_j a_j$. With respect to this grading, $\deg\phi=\sum_{i=1}^n \alpha_i a_i$.

We say that S satisfies the **maximal projective dimension (**MPD**)** if $\mathrm{pdim}_R k[S]=n-1$. Equivalently, $\mathrm{depth}_R k[S]=1$. In [8, Theorem 6], the authors proved that S is a MPD-semigroup if and only if $\mathrm{PF}(S)\neq\emptyset$. In particular, they prove that if S is a MPD-semigroup then $b\in S$ is the S-degree of the $(n-2)$th minimal syzygy of $k[S]$ if and only if $b\in\{a+\sum_{i=1}^n a_i \mid a\in\mathrm{PF}(S)\}$. Moreover, $\mathrm{PF}(S)$ has finite cardinality.

Example 2.1. Let $S=\langle(2,11),(3,0),(5,9),(7,4)\rangle$. The minimal free resolution of $k[S]$, as a module over $R=k[x_1,\ldots,x_4]$, is

$$0\to R(-(81,93))\oplus R(-(94,82))\to R^6\to R^5\to R\to k[S]\to 0.$$

In particular, the degrees of minimal generators of the third syzygy modules are $(81,93)$, $(94,82)$. Therefore, S has two pseudo-Frobenius elements $(64,69)$ and $(77,58)$.

Let us recall the definition of gluing [16, Theorem 1.4]. Let $S\subseteq\mathbb{N}^d$ be an affine semigroup and $G(S)$ be the group spanned by S, that is, $G(S)=\{a-b\in\mathbb{Z}^d \mid a,b\in S\}$. Let A be the minimal generating system of S and $A=A_1\amalg A_2$ be a nontrivial partition of A . Let S_i be the submonoid of $\mathbb{N}^d$ generated by $A_i, i\in 1,2$. Then $S=S_1+S_2$. We say that S is the gluing of S_1 and S_2 by d if $d\in S_1\cap S_2$ and, $G(S_1)\cap G(S_2)=d\mathbb{Z}$. If S is a gluing of S_1 and S_2 by d, we write $S=S_1+_d S_2$.

Theorem 2.2. *Let S be an affine semigroup such that $S=S_1+_d S_2$, where S_1 and S_2 are* MPD*-semigroups. Then S is a* MPD*-semigroup and*

$$\mathrm{PF}(S)=\{f+g+d \mid f\in\mathrm{PF}(S_1),\ g\in\mathrm{PF}(S_2)\}.$$

Recall that a gluing two numerical semigroups is defined as follows: Let H_1 and H_2 be two numerical semigroups minimally generated by $n_1,\ldots,n_r$ and $n_{r+1},\ldots,n_e$ respectively. Let $\lambda\in H_1\setminus\{n_1,\ldots,n_r\}$ and $\mu\in H_2\setminus\{n_{r+1},\ldots,n_e\}$ be such that $\gcd(\lambda,\mu)=1$. We say that $S=\langle\mu n_1,\ldots,\mu n_r,\lambda n_{r+1},\ldots,\lambda n_e\rangle$ is a gluing of H_1 and H_2. In other words, if S is as in the definition above, then $S=S_1+_{\mu\lambda}S_2$, where $S_1=\langle\mu n_1,\ldots,\mu n_r\rangle$ and $S_2=\langle\lambda n_{r+1},\ldots,\lambda n_e\rangle$. Since $\mathrm{PF}(S_1)=\mu\mathrm{PF}(H_1)$ and $\mathrm{PF}(S_2)=\lambda\mathrm{PF}(H_2)$, the following result now follows from the Theorem 2.2.

Theorem 2.3 ([13, Proposition 6.6])**.** *If $S=\langle\mu H_1,\lambda H_2\rangle$, then*

$$\mathrm{PF}(S)=\{\mu f+\lambda g+\mu\lambda \mid f\in\mathrm{PF}(H_1), g\in\mathrm{PF}(H_2)\}.$$

On $\mathcal{H}(S)$, we define a relation : $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in S$. It is a partial order (reflexive, anti-symmetric and transitive) on $\mathcal{H}(S)$. In [17, Proposition 2.19], the authors proved that if S is a numerical semigroup, then $\mathrm{PF}(S) = \mathrm{Maximals}_{\leq}(\mathbb{Z} \setminus S)$. In other words, $x \in \mathbb{Z} \setminus S$ if and only if $f - x \in S$ for some $f \in \mathrm{PF}(S)$. However, in the case of MPD-semigroups over $\mathbb{N}^d$, $d \geq 2$, we observe that this result is not true.

Example 2.4. Let S be the semigroup generated by the columns of the following matrix

$$\begin{pmatrix} 18 & 18 & 4 & 20 & 23 & 8 & 11 & 11 & 10 & 14 & 7 & 7 \\ 9 & 3 & 1 & 8 & 10 & 3 & 5 & 2 & 3 & 3 & 2 & 3 \end{pmatrix}.$$

Then S is a MPD-semigroup and $\mathrm{PF}(S) = \{(13,4)\}$ (see [8, Example 5]). Observe that $(15,7) \in \mathrm{cone}(S) \setminus S$ but $(13,4) - (15,7) = (-2,-3) \notin S$.

Observe that if $\mathcal{H}(S)$ is a finite set and $\mathcal{H}(S) \neq \emptyset$, then $\mathrm{PF}(S) \neq \emptyset$. In particular, if $\mathcal{H}(S)$ is finite, then the following result holds.

Theorem 2.5. *Let $\mathcal{H}(S)$ be a non-empty finite set. Then*

1. $\mathrm{PF}(S) = \mathrm{Maximals}_{\leq} \mathcal{H}(S)$.
2. *Let $\mathbf{x} \in \mathbb{N}^d$. Then $\mathbf{x} \in \mathcal{H}(S)$ if and only if $f - \mathbf{x} \in S$ for some $f \in \mathrm{PF}(S)$.*

3 $\prec$-symmetric semigroups

Let $\prec$ be a term order on $\mathbb{N}^d$. Then $\mathrm{F}(S)_{\prec} = \max_{\prec}\mathcal{H}(S)$, if it exists, is called the Frobenius element of S with respect to the term order $\prec$. In particular,

$$\mathrm{F}(S) = \{\mathrm{F}(S)_{\prec} = \max_{\prec}\mathcal{H}(S) \mid \prec \text{ is a term order }\}.$$

We write $\mathrm{F}(S)$ for the set of Frobenius elements of S. Note that Frobenius elements may not exist. However, if $|\mathcal{H}(S)| < \infty$, then Frobenius elements do exist. Also, from [8, Lemma 12], we have that every Frobenius element is a pseudo-Frobenius element, *i.e.*, $\mathrm{F}(S) \subseteq \mathrm{PF}(S)$.

In the case $\mathrm{F}(S) \neq \emptyset$, we fix a term order $\prec$ such that $\mathrm{F}(S)_{\prec} = \max_{\prec}\mathcal{H}(S) \in \mathrm{F}(S)$.

1. If $|\mathrm{PF}(S)| = 1$ and $\mathrm{PF}(S) = \{\mathrm{F}(S)_{\prec}\}$, then S is called a $\prec$**-symmetric** semigroup.
2. Put $\mathrm{PF}'(S) = \mathrm{PF}(S) \setminus \{\mathrm{F}(S)_{\prec}\}$. If $\mathrm{PF}'(S) \neq \emptyset$ and if for any $g \in \mathrm{PF}'(S)$, $\mathrm{F}(S)_{\prec} - g \in \mathrm{PF}'(S)$, we say that S is $\prec$**-almost symmetric**. Further, if $|\mathrm{PF}(S)| = 2$, then S is called $\prec$**-pseudo-symmetric**. In this case, $\mathrm{PF}(S) = \{\mathrm{F}(S)_{\prec}, \mathrm{F}(S)_{\prec}/2\}$.

Observe that when S is $\prec$-pseudo-symmetric, for some term order $\prec$, then $\mathrm{F}(S)_{\prec}$ has all even coordinates. Hereafter $\mathrm{F}(S)_{\prec}$ is even means that $\mathrm{F}(S)_{\prec}$ has all even coordinates.

Example 3.1. The semigroup $S_1 = \langle (0,1),(3,0),(5,0),(1,3),(2,3)\rangle$ is a $\prec$-symmetric semigroup as $\mathrm{PF}(S_1) = \{(7,2)\}$ and $(7,2) = \max_{\prec} \mathcal{H}(S_1)$, where $\prec$ is a graded lexicographic order.
The semigroup $S_2 = \langle (0,1),(3,0),(4,0),(1,4),(5,0),(2,7)\rangle$ is $\prec$-almost symmetric semigroup of type 2 as $\mathrm{PF}(S_2) = \{(1,3),(2,6)\}$ and $(2,6) = \max_{\prec} \mathcal{H}(S_2)$, where $\prec$ is a graded lexicographic order. In particular, S_2 is a $\prec$-pseudo-symmetric semigroup.

For numerical semigroups, the concept of symmetric and pseudo-symmetric numerical semigroups is also characterized using elements in the gap set, $\mathbb{Z} \setminus S$ (see [17, Proposition 4.4]). In order to attempt such characterization in the case of MPD-semigroups, we assume that $\mathcal{H}(S)$ is a finite set. Let S be a MPD-semigroup. If $\mathcal{H}(S)$ is a non-empty finite set, then S is said to be a $\mathcal{C}$-semigroup, where $\mathcal{C}$ denotes the cone of the semigroup. The concept of $\mathcal{C}$-semigroups is introduced in [7] and several properties of these semigroups are investigated in [3]. When S is a $\mathcal{C}$-semigroup, we give a characterization of $\prec$-symmetric and $\prec$-pseudo-symmetric semigroups.

Theorem 3.2. *Let S be a $\mathcal{C}$-semigroup and let $\mathrm{F}(S)_{\prec}$ denote the Frobenius element of S with respect to an order $\prec$. Then S is a $\prec$-symmetric semigroup if and only if for each $g \in \mathrm{cone}(S) \cap \mathbb{N}^d$ we have:*

$$g \in S \textit{ if and only if } \mathrm{F}(S)_{\prec} - g \notin S.$$

Theorem 3.3. *Let S be a $\mathcal{C}$-semigroup and let $\mathrm{F}(S)_{\prec}$ denote the Frobenius element of S with respect to an order $\prec$. Then S is a $\prec$-pseudo-symmetric semigroup if and only if $\mathrm{F}(S)_{\prec}$ is even, and for each $g \in \mathrm{cone}(S) \cap \mathbb{N}^d$ we have:*

$$g \in S \textit{ if and only if } \mathrm{F}(S)_{\prec} - g \notin S \textit{ and } g \neq \mathrm{F}(S)_{\prec}/2.$$

3.1 Extended Wilf's conjecture

Let S be a $\mathcal{C}$-semigroup. Define the Frobenius number of S as $\mathcal{N}(\mathrm{F}(S)_{\prec}) = |\mathcal{H}(S)| + |\{g \in S \mid g \prec \mathrm{F}(S)_{\prec}\}|$. Observe that if S is a numerical semigroup, then $\mathcal{N}(\mathrm{F}(S)_{\prec}) = \mathrm{F}(S)$, the Frobenius number of S. For a numerical semigroup S, Wilf proposed a conjecture related to the Diophantine Frobenius Problem that claims that the inequality

$$\mathrm{F}(S) + 1 \leq e(S) \cdot |\{s \in S \mid s < \mathrm{F}(S)\}|$$

is true. While this conjecture still remains open, a potential extension of Wilf's conjecture to affine semigroups is studied in [7].

Conjecture 3.4 (**Extension of Wilf's conjecture** [7, Conjecture 14]). *Let S be a $\mathcal{C}$-semigroup. The extended Wilf's conjecture is*

$$|\{g \in S \mid g \prec \mathrm{F}(S)_{\prec}\}| \cdot e(S) \geq \mathcal{N}(\mathrm{F}(S)_{\prec}) + 1,$$

where $e(S)$ denotes the embedding dimension of S.

The following result gives a different characterization of $\prec$-symmetric and $\prec$-pseudo-symmetric $\mathcal{C}$-semigroups when $\operatorname{cone}(S) = \mathbb{N}^d$. On $\operatorname{cone}(S)$, we define a relation $\leq_c$ as follows: $g \leq_c f$ if $g_i \leq f_i$ for all $i \in [1, d]$.

Theorem 3.5. *Let S be a $\mathcal{C}$-semigroup such that* $\operatorname{cone}(S) = \mathbb{N}^d$. *Then*

1. *S is $\prec$-symmetric if and only if $|\mathcal{H}(S)| = |\{g \in S \mid g \leq_c \mathrm{F}(S)_{\prec}\}|$.*
2. *S is $\prec$-pseudo-symmetric if and only if $|\mathcal{H}(S) \setminus \{\mathrm{F}(S)_{\prec}/2\}| = |\{g \in S \mid g \leq_c \mathrm{F}(S)_{\prec}\}|$ and $\mathrm{F}(S)_{\prec}$ is even.*

Note that if S is an affine semigroup of $\mathbb{N}^d$, where $d \geq 2$, then the semigroup ring $k[S]$ is Cohen–Macaulay when $e(S) = 2$. Since our affine semigroups are MPD, we may assume that $e(S) \geq 3$.

Theorem 3.6. *In the above characterizations of $\prec$-symmetric and $\prec$-pseudo-symmetric semigroups, the extended Wilf's conjecture holds.*

4 RF-matrices and generic toric ideals

Let $S = \langle a_1, \ldots, a_n \rangle$ be a MPD-semigroup in $\mathbb{N}^d$, minimally generated by $a_1, \ldots, a_n$. We recall the notion of row-factorization matrix (RF-matrix), introduced by A. Moscariello in [12].

Definition 4.1. Let $f \in \mathrm{PF}(S)$. An $n \times n$ matrix $M = (m_{ij})$ is an RF-matrix of f if $m_{ii} = -1$ for every i, $m_{ij} \in \mathbb{N}$ if $i \neq j$ and for every $i = 1, \ldots, n$, $\sum_{j=1}^{n} m_{ij} a_j = f$.

While RF matrices were defined for pseudo-Frobenius elements in numerical semigroups, we observe that the above definition holds in the case of pseudo-Frobenius elements in MPD-semigroups over $\mathbb{N}^d$. Note that an RF-matrix of f need not be uniquely determined. Thus, the notation $\mathrm{RF}(f)$ will denote one of the possible RF-matrices of f.

Lemma 4.2. *Let $m_1, \ldots, m_n$ be the row vectors of* $\mathrm{RF}(f)$, *and set $m_{(ij)} = m_i - m_j$ for all i, j with $1 \leq i < j \leq n$. Then $\phi_{ij} = \mathbf{x}^{m^+_{(ij)}} - \mathbf{x}^{m^-_{(ij)}} \in I_S$ for all $i < j$.*

The binomials ϕ_{ij} defined in Lemma 4.2 are called $\mathrm{RF}(f)$-relations. We call a binomial relation $\phi \in I_S$ an RF-**relation** if it is an $\mathrm{RF}(f)$-relation for some $f \in \mathrm{PF}(S)$.

Example 4.3. Let $S = \langle (2,11), (3,0), (5,9), (7,4) \rangle$. Then $\mathrm{PF}(S) = \{(64,69), (77,58)\}$ and the RF-matrices $\mathrm{RF}(64,69), \mathrm{RF}(77,58)$ are respectively

$$\begin{bmatrix} -1 & 4 & 8 & 2 \\ 0 & -1 & 5 & 6 \\ 6 & 12 & -1 & 3 \\ 5 & 17 & 2 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 4 & 5 & 6 \\ 0 & -1 & 2 & 10 \\ 5 & 17 & -1 & 3 \\ 4 & 22 & 2 & -1 \end{bmatrix}.$$

Then I_S is minimally generated by RF-relations as $I_S = \langle \phi_1, \ldots, \phi_5 \rangle$, where

$$\phi_1 := x_2^5x_3^3 - x_1x_4^4,\ \phi_2 := x_1^6x_2^{13} - x_3^6x_4^3,\ \phi_3 := x_1^5x_2^{18} - x_3^3x_4^7,$$
$$\phi_4 := x_1^4x_2^{23} - x_4^{11},\ \phi_5 := x_3^9 - x_1^7x_2^8x_4.$$

We see that $\phi_1, \phi_2, \phi_3, \phi_4$ are RF$(77,58)$-relations and ϕ_5 is a RF$(64,69)$-relation.

Lemma 4.4. *Let $S = \langle a_1, \ldots, a_n \rangle$ be a* MPD*-semigroup. Suppose $S = S_1 +_d S_2$, where $S_1 = \langle a_1, \ldots, a_e \rangle$ and $S_2 = \langle a_{e+1}, \ldots, a_n \rangle$. Let $h \in$ PF(S). Then by Theorem 2.2, there exist $f \in$ PF(S_1) and $g \in$ PF(S_2) such that $h = f + g + d$. Since $f \in$ PF(S_1), $f + d \in S_1$. Write $f + d = \sum_{j=1}^{e} m_j a_j$. Similarly, as $g + d \in S_2$, write $g + d = \sum_{j=e+1}^{n} m_j a_j$. Hence the matrix*

$$\left[\begin{array}{c|c} \mathrm{RF}(f) & B \\ \hline C & \mathrm{RF}(g) \end{array}\right],$$

where each row of the matrix B is $(m_{e+1}, \ldots, m_n)$ and each row of matrix C is $(m_1, \ldots, m_e)$, serves as an RF*-matrix for h.*

Let $S = \langle a_1, \ldots, a_n \rangle \subseteq \mathbb{N}^d$ be an affine semigroup and $I_S \subset k[x_1, \ldots, x_n]$ be the defining ideal of the semigroup ring $k[S]$. For a given vector $a = (a_1, \ldots, a_d) \in \mathbb{N}^d$, the support of a is defined as

$$\mathrm{supp}(a) = \{i \mid i \in [1,d],\ a_i \neq 0\}.$$

For a monomial $\mathbf{x}^u$, define $\mathrm{supp}(\mathbf{x}^u) = \mathrm{supp}(u)$ and for a binomial $\mathbf{x}^u - \mathbf{x}^v$, define $\mathrm{supp}(\mathbf{x}^u - \mathbf{x}^v) = \mathrm{supp}(u) \cup \mathrm{supp}(v)$. In [15], Peeva and Sturmfels defined that a toric ideal $I_S \subset k[x_1, \ldots, x_n]$ is called **generic** if it is minimally generated by the binomials of full support. A binomial $\mathbf{x}^u - \mathbf{x}^v$ is called indispensable if every system of binomial generators of I_S contains $\mathbf{x}^u - \mathbf{x}^v$ or $\mathbf{x}^v - \mathbf{x}^u$. Using [2, Theorem 3.1] it follows that if I_S is generic toric ideal, then it has a unique minimal system of generators $B(I_S)$ of indispensable binomials up to the sign of binomials.

If $x = \sum_{j=1}^{n} m_j a_j$ is the unique expression for x in S, then we say x has unique factorization in S. In other words, if $x = \sum_{j=1}^{n} m_j a_j = \sum_{j=1}^{n} m'_j a_j$ are two factorizations of x in S, then $m_j = m'_j$ for all $j \in [1,n]$. We denote the set of such elements by UF(S). We define a partial order $\leq_S$ on S as $x \leq_S y$ if and only if $y - x \in S$. We denote the set of minimal elements of $S \setminus \mathrm{UF}(S)$ with respect to $\leq_S$ by $\min(S \setminus \mathrm{UF}(S))$. For $0 \neq x \in S$, we define a subset Ap(S,x) of S relative to x as

$$\mathrm{Ap}(S,x) = \{y \in S \mid y - x \in \mathcal{H}(S)\}.$$

Theorem 4.5. *Let $S = \langle a_1, \ldots, a_n \rangle$ be a* MPD*-semigroup. If $\bigcup_{j=1}^{n} \mathrm{Ap}(S, a_j) \subseteq \mathrm{UF}(S)$, then* RF$(f)$ *is unique for all $f \in$ PF(S). Moreover, if $\mathcal{H}(S)$ is finite, then the converse is also true.*

Theorem 4.6. *Let S be a* MPD*-semigroup. If I_S is generic, then* $\mathrm{RF}(f) = (m_{ij})$ *is unique for each $f \in \mathrm{PF}(S)$ and $m_{ij} \neq m_{i'j}$ for all $i \neq i'$.*

Example 4.7. Let $S = \langle (20,0), (24,1), (1,25), (0,31) \rangle$. Then

$$\mathrm{PF}(S) = \left\{ \begin{array}{c} (223,4445), (271,3145), (319,1845), (559,1256), (799,667), \\ (1375,567), (1951,467), (2527,367), (3103,267) \end{array} \right\},$$

and I_S is generated by

$$\left\{ \begin{array}{c} x_2^{24}x_3^4 - x_1^{29}x_4^4,\ x_1^{12}x_3^{24} - x_2^{11}x_4^{19},\ x_2^{13}x_3^{28} - x_1^{17}x_4^{23},\ x_1^{41}x_3^{20} - x_2^{35}x_4^{15},\ x_2^2x_3^{52} - x_1^5x_4^{42}, \\ x_1^{70}x_3^{16} - x_2^{59}x_4^{11},\ x_1^7x_3^{76} - x_2^9x_4^{61},\ x_1^{99}x_3^{12} - x_2^{83}x_4^7,\ x_1^{128}x_3^8 - x_2^{107}x_4^3,\ x_2^{131} - x_1^{157}x_3^4x_4, \\ x_1^2x_3^{128} - x_2^7x_4^{103},\ x_3^{180} - x_1^3x_2^5x_4^{145} \end{array} \right\}.$$

Hence, I_S is generic and RF-matrices for the elements of $\mathrm{PF}(S)$ are

$$\begin{bmatrix} -1 & 8 & 51 & 102 \\ 6 & -1 & 127 & 41 \\ 4 & 6 & -1 & 144 \\ 1 & 1 & 179 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 10 & 51 & 60 \\ 11 & -1 & 75 & 41 \\ 4 & 8 & -1 & 102 \\ 6 & 1 & 127 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 12 & 51 & 18 \\ 16 & -1 & 23 & 41 \\ 4 & 10 & -1 & 60 \\ 11 & 1 & 75 & -1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 23 & 27 & 18 \\ 28 & -1 & 23 & 22 \\ 16 & 10 & -1 & 41 \\ 11 & 12 & 51 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 34 & 3 & 18 \\ 40 & -1 & 23 & 3 \\ 28 & 10 & -1 & 22 \\ 11 & 23 & 27 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 58 & 3 & 14 \\ 69 & -1 & 19 & 3 \\ 28 & 34 & -1 & 18 \\ 40 & 23 & 23 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 82 & 3 & 10 \\ 98 & -1 & 15 & 3 \\ 28 & 58 & -1 & 14 \\ 69 & 23 & 19 & -1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 106 & 3 & 6 \\ 127 & -1 & 11 & 3 \\ 28 & 82 & -1 & 10 \\ 98 & 23 & 15 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 130 & 3 & 2 \\ 156 & -1 & 7 & 3 \\ 28 & 106 & -1 & 6 \\ 127 & 23 & 11 & -1 \end{bmatrix} \text{respectively.}$$

Moreover, these matrices are unique and no two entries in a column of each matrix are same.

Using Theorem 2.2, Lemma 4.4 and Theorem 4.6, we prove the following result:

Theorem 4.8. *Let $n \geq 3$ and $S = \langle a_1, \ldots, a_n \rangle$ be a gluing of* MPD*-semigroups. Then I_S is not generic.*

Acknowledgements

Experiments with the computer algebra softwares Macaulay2 [10] and GAP [6] have provided numerous valuable insights.

References

[1] O. P. Bhardwaj, K. Goel, and I. Sengupta. "Affine semigroups of maximal projective dimension". 2021. arXiv:2111.12070.

[2] H. Charalambous, A. Katsabekis, and A. Thoma. "Minimal systems of binomial generators and the indispensable complex of a toric ideal". 2006. arXiv:0607249.

[3] J. D. Díaz-Ramírez, J. I. García-García, D. Marín-Aragón, and A. Vigneron-Tenorio. "Characterizing affine $\mathcal{C}$-semigroups". 2021. arXiv:1907.03276.

[4] K. Eto. "Almost Gorenstein monomial curves in affine four space". *J. Algebra* **488** (2017), pp. 362–387. DOI.

[5] K. Eto. "Generic Toric Ideals and Row-Factorization Matrices in Numerical Semigroups". *Numerical Semigroups : IMNS 2018*. Ed. by V. Barucci, S. Chapman, M. D'Anna, and R. Fröberg. Cham: Springer International Publishing, 2020, pp. 83–91. DOI.

[6] *GAP – Groups, Algorithms, and Programming, Version 4.11.1*. The GAP Group. 2021. Link.

[7] J. I. García-García, D. Marín-Aragón, and A. Vigneron-Tenorio. "An extension of Wilf's conjecture to affine semigroups". *Semigroup Forum* **96**.2 (2018), pp. 396–408. DOI.

[8] J. I. García-García, I. Ojeda, J. C. Rosales, and A. Vigneron-Tenorio. "On pseudo-Frobenius elements of submonoids of $\mathbb{N}^d$". *Collect. Math.* **71**.1 (2020), pp. 189–204. DOI.

[9] V. Gasharov, I. Peeva, and V. Welker. "Rationality for generic toric rings". *Math. Z.* **233**.1 (2000), pp. 93–102. DOI.

[10] D. R. Grayson and M. E. Stillman. "Macaulay2, a software system for research in algebraic geometry". Link.

[11] J. Herzog and K. Watanabe. "Almost symmetric numerical semigroups". *Semigroup Forum* **98**.3 (2019), pp. 589–630. DOI.

[12] A. Moscariello. "On the type of an almost Gorenstein monomial curve". *J. Algebra* **456** (2016), pp. 266–277. DOI.

[13] H. Nari. "Symmetries on almost symmetric numerical semigroups". *Semigroup Forum* **86**.1 (2013), pp. 140–154. DOI.

[14] T. Numata. "Almost symmetric numerical semigroups generated by four elements". *Proc. Inst. Nat. Sci., Nihon Univ.* **48** (2013), pp. 197–207. Link.

[15] I. Peeva and B. Sturmfels. "Generic lattice ideals". *J. Amer. Math. Soc.* **11**.2 (1998), pp. 363–373. DOI.

[16] J. C. Rosales. "On presentations of subsemigroups of $\mathbf{N}^n$". *Semigroup Forum* **55**.2 (1997), pp. 152–159. DOI.

[17] J. C. Rosales and P. A. García-Sánchez. *Numerical semigroups*. Vol. 20. Developments in Mathematics. Springer, New York, 2009, pp. x+181. DOI.