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A Combinatorial Model for the Transition Matrix Between the Specht and Web Bases

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Abstract. We introduce a new class of permutations, called web permutations. Using these permutations, we provide a combinatorial interpretation for entries of the transition between the Specht and web bases, which answers Rhoades's question. Furthermore, we study enumerative properties of these permutations.

Keywords: Specht basis, web basis, André permutation, Euler number

1 Introduction and the main result

In this article, we study the transition matrix between two famous bases, the Specht basis and the web basis, for the irreducible representation of the symmetric group $\mathfrak{S}_{2n}$ indexed by the partition (n, n). Motivated by Rhoades's work [9], we give a combinatorial interpretation for entries of the transition matrix as a certain class of permutations, and present their interesting properties.

For an integer $n \geq 1$, let $\mathfrak{S}_{2n}$ be the symmetric group on the set $[2n] = \{1, \ldots, 2n\}$. It is well known that each irreducible representation of $\mathfrak{S}_{2n}$ can be indexed by a partition of $2n$. For a partition λ of $2n$, we then denote by $\mathcal{S}^\lambda$ the irreducible representation indexed by λ, called the *Specht module*. In this article, we narrow our focus down to the Specht module indexed by the partition (n, n), and two well-studied bases for $\mathcal{S}^{(n,n)}$.

A *standard Young tableau* of shape (n, n) is an array of integers of shape (n, n) whose entries are $[2n]$, and each row and each column are increasing.

The set of standard Young tableaux of shape (n, n), denoted by $\mathrm{SYT}(n, n)$, is an indexing set for the *Specht basis*

$$\{v_T \in \mathcal{S}^{(n,n)} \mid T \text{ is a standard Young tableau of shape } (n,n)\}$$

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for $\mathcal{S}^{(n,n)}$. For more details on the Specht basis and related combinatorics, see [4, 11].

A *(perfect) matching* on $[2n]$ is a set partition of $[2n]$ such that each block has size 2. We also depict a matching on $[2n]$ as a diagram consisting of $2n$ vertices and n arcs where any pair of arcs has no common vertex. A *crossing* is a pair of arcs $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$. A matching is called *noncrossing* if the matching has no crossing, and *nonnesting* if there is no pair of arcs $\{a, d\}$ and $\{b, c\}$ with $a < b < c < d$.

For a matching M and $\{i, j\} \in M$ with $i < j$, i is called an *opener* and j is called a *closer*. Let Mat_{2n} (NC_{2n} and NN_{2n}, respectively) stand for the set of (noncrossing and nonnesting, respectively) matchings on $[2n]$.

Note that there is an obvious bijection between $\mathrm{SYT}(n, n)$ and NN_{2n}. For $T \in \mathrm{SYT}(n, n)$, connect two vertices lying on the same column of T via an arc, then we obtain a nonnesting matching. Using this correspondence, we index the Specht basis for $\mathcal{S}^{(n,n)}$ by nonnesting matchings of $[2n]$:

$$\{v_M \in \mathcal{S}^{(n,n)} \mid M \in \mathrm{NN}_{2n}\}.$$

We now consider the $2 \times 2n$ matrix

$$z = \begin{bmatrix} z_{1,1} & z_{1,2} & \cdots & z_{1,2n} \\ z_{2,1} & z_{2,2} & \cdots & z_{2,2n} \end{bmatrix},$$

where $z_{i,j}$'s are indeterminates. For $1 \le i < j \le 2n$, let $\Delta_{ij} := \Delta_{ij}(z)$ be the maximal minor of z with respect to the ith and jth columns, *i.e.*, $\Delta_{ij} = z_{1,i}z_{2,j} - z_{1,j}z_{2,i}$. For a matching $M \in \mathrm{Mat}_{2n}$, let

$$\Delta_M := \Delta_M(z) = \prod_{\{i,j\} \in M} \Delta_{ij} \in \mathbb{C}[z_{1,1}, \ldots, z_{2,2n}].$$

It is important to note that the polynomials Δ_{ij} satisfy the following relation: For $1 \le a < b < c < d \le 2n$,

$$\Delta_{ac}\Delta_{bd} = \Delta_{ab}\Delta_{cd} + \Delta_{ad}\Delta_{bc}. \tag{1.1}$$

We define a vector space W_n to be the $\mathbb{C}$-span of Δ_M for all $M \in \mathrm{Mat}_{2n}$. In [6], it turns out that the set

$$\{\Delta_M \in W_n \mid M \in \mathrm{NC}_{2n}\} \tag{1.2}$$

forms a basis for W_n. We call this basis the *web basis*.

We are now in a position to give the main purpose of this article. Let M_0 be the unique matching which is simultaneous noncrossing and nonnesting, *i.e.*, $M_0 = \{\{1, 2\}, \ldots, \{2n-1, 2n\}\}$. Due to [10], the isomorphism maps Δ_{M_0} to v_{M_0} up to scalar. Let $\varphi \colon W_n \to \mathcal{S}^{(n,n)}$ be the unique isomorphism with $\varphi(\Delta_{M_0}) = v_{M_0}$. We also let $w_M := \varphi(\Delta_M)$ for each $M \in \mathrm{NC}_{2n}$. Then the Specht basis can expand into (the image of) the web basis: for $M \in \mathrm{NN}_{2n}$,

$$v_M = \sum_{M' \in \mathrm{NC}_{2n}} a_{MM'} w_{M'}.$$

In [10], Russell and Tymoczko initiated the combinatorial study of the transition matrix

$$A = (a_{MM'})_{M \in \mathrm{NN}_{2n}, M' \in \mathrm{NC}_{2n}}.$$

They showed the unitriangularity of the matrix. They also gave some open problems related to their results. One of them is for positivity of entries of A, and soon after, Rhoades proved the positivity.

Theorem 1.1 ([9]). *The entries $a_{MM'}$ of the transition matrix A are nonnegative integers.*

Although Rhoades established the positivity phenomenon for entries of A, he did not find an explicit combinatorial interpretation of the nonnegative integer $a_{MM'}$, c.f. [9, Problem 1.3]. Inspired by his work, we introduce a new family of permutations which are enumerated by the integers $a_{MM'}$, and study their enumerative properties.

Our strategy is based on Rhoades's observation [9]. He figured out that the entries $a_{MM'}$ are related to resolving crossings of matchings in the following sense: For a matching $M \in \mathrm{Mat}_{2n}$, let $\{a, c\}$ and $\{b, d\}$ be a crossing pair in M (if it exists) where $a < b < c < d$. Let M' and M'' be the matchings identical to M except that $\{a, b\}$ and $\{c, d\}$ in M', and $\{a, d\}$ and $\{b, c\}$ in M''. Then, by the relation (1.1), we have

$$\Delta_M = \Delta_{M'} + \Delta_{M''}. \tag{1.3}$$

In addition, the number of crossing pairs in M' (respectively, M'') is strictly less than the number of crossing pairs in M. Therefore, iterating the resolving procedure gives the expansion of Δ_M in terms of the basis (1.2). In other words, when we write

$$\Delta_M = \sum_{M' \in \mathrm{NC}_{2n}} c_{MM'} \Delta_{M'}, \tag{1.4}$$

the coefficient $c_{MM'}$ is equal to the number of occurrences of the noncrossing matching M' obtained from iterating resolving crossings in M. Note that the order of the choice of crossing pairs does not affect the expansion of Δ_M. Rhoades showed that for $M \in \mathrm{NN}_{2n}$ and $M' \in \mathrm{NC}_{2n}$, the entry $a_{MM'}$ of the transition matrix equals $c_{MM'}$. Hence, to give a combinatorial interpretation of $a_{MM'}$, we track the resolving process from a nonnesting matching to noncrossing matchings.

To state our main result, we need some preliminaries. A *Dyck path* of length $2n$ is a lattice path from $(0,0)$ to (n,n) consisting of n north steps $(1,0)$ and n east steps $(0,1)$ that does not pass below the line $y = x$. We write N and E for the north step and the east step, respectively. We therefore regard a Dyck path as a sequence of $\{\mathsf{N}, \mathsf{E}\}$ consisting of n N's and n E's. Let Dyck_{2n} be the set of Dyck paths of length $2n$. Identifying a Dyck path with the area below the path, we give a natural partial order on Dyck_{2n} by inclusion, denoted by $\subseteq$. In Section 2, we define a map $D\colon \mathrm{Mat}_{2n} \to \mathrm{Dyck}_{2n}$, and by abuse of notation, a map $D\colon \mathfrak{S}_n \to \mathrm{Dyck}_{2n}$. No confusion might be caused by this abuse. We also define a map $M\colon \mathfrak{S}_n \to \mathrm{NC}_{2n}$. Finally, we introduce a new family of permutations, called *web permutations*. With these data, we now present our main result.

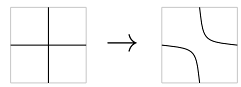


Figure 1: A crossing to an elbow.

Theorem 1.2. *For matchings $M \in \mathrm{NN}_{2n}$ and $M' \in \mathrm{NC}_{2n}$, the entry $a_{MM'}$ is equal to the number of web permutations σ such that $D(\sigma) \subseteq D(M)$ and $M(\sigma) = M'$.*

The article is organized as follows. In Section 2, we give a new model, called a grid configuration, for representing matchings. We then define web permutations from the noncrossing grid configurations, and prove the main theorem. In the next two sections, we study some properties of web permutations. In Section 3, we give a characterization of web permutations. We show that web permutations are closely related to André permutations. Section 4 provides some interesting enumerative properties of web permutations. We also give a conjecture for a relation between certain web permutations and the Seidel triangle.

2 Grid configurations and web permutations

In this section, we define grid configurations which represent matchings in a 'rigid' setting. We describe the procedure of resolving crossings within this model. We then introduce a new class of permutations, called web permutations. This provides a combinatorial interpretation for the entries $a_{MM'}$ of the transition matrix.

Consider an n by n (lattice) grid in the xy-plane with corners $(0,0), (0,n), (n,0)$ and (n,n). We denote each cell by (i,j) where i and j are the x- and y-coordinates of its upper-right corner. Let $\sigma \in \mathfrak{S}_n$ be a permutation. For each $1 \leq i \leq n$, mark the cell $(i, \sigma(i))$, and draw a horizontal line to the left and a vertical line to the top from the marked cell. We call this the *empty grid configuration* of σ. A cell (i,j) is a *crossing* if there are both a vertical line and a horizontal line through the cell, that is, $\sigma(i) < j$ and $i < \sigma^{-1}(j)$. We denote by $\mathrm{Cr}(\sigma)$ the set of all crossings of σ. For a subset $E \subseteq \mathrm{Cr}(\sigma)$, the *grid configuration* $G(\sigma, E)$ of a pair (σ, E) is defined to be the empty grid configuration of σ where each crossing in E is replaced by an elbow as shown in Figure 1. In particular, the empty grid configuration of σ is $G(\sigma, \varnothing)$.

For the n by n grid configuration, we label leftmost vertical intervals from bottom to top with 1 through n and uppermost horizontal intervals from left to right with $n+1$ through $2n$. With this label of boundary intervals, a grid configuration can be considered as a matching on $[2n]$ as follows: Each strand joining ith and jth boundary intervals represents an arc connecting i and j; see Figure 2. We denote by $M(\sigma, E)$ the matching associated to the grid configuration $G(\sigma, E)$. For short, we write $M(\sigma) = M(\sigma, \mathrm{Cr}(\sigma))$.

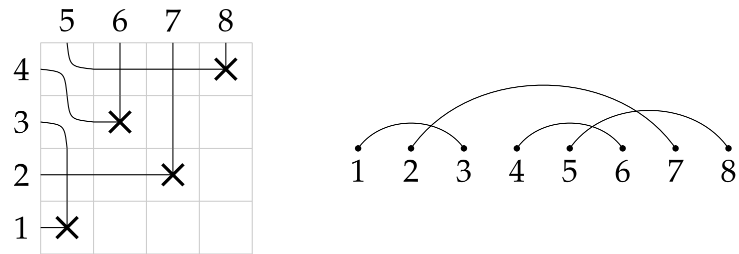


Figure 2: The grid configuration $G(1324, \{(1,3),(1,4)\})$ and the corresponding matching.

We define a partial order on cells of the n by n grid by $(x,y) \succ (x',y')$ if either $x < x'$ and $y \geq y'$, or $x = x'$ and $y > y'$.

The relation (1.3) can be interpreted as a relation between grid configurations as follows. For a permutation σ and $E \subseteq \mathrm{Cr}(\sigma)$, let $G(\sigma, E)$ be the grid configuration and $c = (i,j)$ be a maximal crossing in $G(\sigma, E)$. One way of resolving c results a grid configuration $G(\sigma, E \cup \{c\})$. This procedure of resolving a crossing is called *smoothing*. The other way of resolving c results a grid configuration $G(\sigma', E)$, where σ' is defined by

$$\begin{aligned} \sigma'(i) &= j, \\ \sigma'(\sigma^{-1}(j)) &= \sigma(i), \text{ and} \\ \sigma'(k) &= \sigma(k) \text{ for } k \neq i, \sigma^{-1}(j). \end{aligned}$$

This procedure of resolving a crossing is called *switching*. Note that the crossing sets $\mathrm{Cr}(\sigma)$ and $\mathrm{Cr}(\sigma')$ are not the same. Nevertheless, by choosing c to be maximal, crossings not smaller than c (with respect to the partial order) are left unchanged under switching. In particular, we have $E \subseteq \mathrm{Cr}(\sigma')$, so switching is well-defined. We often consider a grid configuration G as the vector $\Delta_{M(G)}$. Therefore, we can write the relation (1.3) in terms of grid configurations as

$$G(\sigma, E) = G(\sigma, E \cup \{c\}) + G(\sigma', E).$$

From the grid configuration $G(id, \emptyset)$, we obtain two grid configurations by resolving a crossing by smoothing and switching, respectively. By resolving crossings until there is no crossing left, we get grid configurations of the form $G(\sigma, \mathrm{Cr}(\sigma))$. For each remaining grid configuration $G(\sigma, \mathrm{Cr}(\sigma))$, the permutation σ is called a *web permutation* of $[n]$ and we denote the set of web permutations of $[n]$ by Web_n. In other words, we have

$$G(id, \emptyset) = \sum G(\sigma, \mathrm{Cr}(\sigma)), \tag{2.1}$$

where the right hand side is the sum of all grid configurations obtained by resolving crossings from the grid configuration $G(id, \emptyset)$ until there is no crossing left. This is

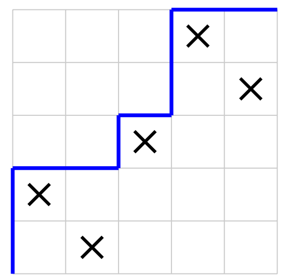
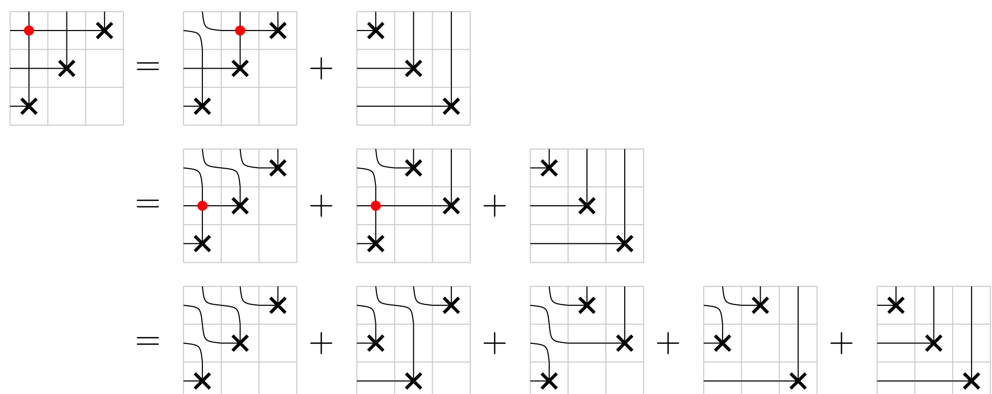


Figure 3: The Dyck path $D(\sigma)$ associated to $\sigma = 21354$ is NNEENENNEE.

reminiscent of (1.4). For example, starting from the grid configuration $G(id, \emptyset)$ for $n = 3$, we have



Therefore we conclude that $\mathrm{Web}_3 = \{123, 213, 132, 231, 321\}$. The following proposition justifies that web permutations are well-defined.

Proposition 2.1. *The expansion in* (2.1) *is unique. In other words, the grid configurations appearing in* (2.1) *does not depend on the order of resolving procedure (choice of maximal crossings). In addition, the permutations σ in* (2.1) *are all distinct.*

For a matching M, record N for openers and E for closers reading M from left to right. This gives the Dyck path $D(M)$ in the n by n grid. It is known that the two restrictions of the map $D\colon \mathrm{Mat}_{2n} \to \mathrm{Dyck}_{2n}$ to NC_{2n} and NN_{2n} are bijections. To a permutation σ, we associate the minimum Dyck path $D(\sigma)$ where every cell $(i, \sigma(i))$ lies below the path; see Figure 3.

Given a nonnesting matching $M \in \mathrm{NN}_{2n}$, let $E(M)$ be the set of cells in the n by n grid which are above the path $D(M)$. It is easy to see that the matchings M and $M(id, E(M))$ coincide.

Similarly to the definition of Web_n, we consider the equation

$$G(id, E(M)) = \sum G(\sigma, \mathrm{Cr}(\sigma)),$$

where the right hand side is the summation of grid configurations obtained by resolving crossings in $G(id, E(M))$ until there is no crossing. We then define Web_M to be the set of permutations σ appearing in the right hand side of the above equation. In particular, $\mathrm{Web}_n = \mathrm{Web}_M$ where $M = \{\{1, n+1\}, \{2, n+2\}, \ldots, \{n, 2n\}\}$.

Using the above notations, we prove one of our main results that tells us which web permutations contribute to the entry $a_{MM'}$.

Proof of Thereom 1.2. By the definition of web permutations, we have

$$a_{MM'} = |\{\sigma \in \mathrm{Web}_M : M(\sigma) = M'\}|.$$

Hence it is enough to show that

$$\mathrm{Web}_M = \{\sigma \in \mathrm{Web}_n : D(\sigma) \subseteq D(M)\}. \tag{2.2}$$

We can obtain the grid configuration $G(id, E(M))$ from $G(id, \emptyset)$ by smoothing crossings in $E(M)$. Since Proposition 2.1 says that Web_n does not depend on the order of resolving processes, we obtain $\mathrm{Web}_M \subseteq \mathrm{Web}_n$. From this, it is clear that

$$\begin{aligned}\mathrm{Web}_M &= \{\sigma \in \mathrm{Web}_n : E(M) \subseteq \mathrm{Cr}(\sigma)\} \\ &= \{\sigma \in \mathrm{Web}_n : (i, \sigma(i)) \notin E(M) \text{ for all } i\},\end{aligned}$$

which proves the claim (2.2). □

3 Characterization of web permutations

In this section, we provide a characterization of web permutations. This characterization depends only on their permutation structure. Using this characterization, we also prove the results in [5, 10].

We begin with recalling two ways to represent permutations. One way is the *one-line notation* which we have already used, that is regarding a permutation as a word. More precisely, for a permutation $\sigma\colon [n] \to [n]$, we write $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ where $\sigma_i = \sigma(i)$. Another way to write permutations is the *cycle notation*. We give an example; for the definition, see [14]. Let $\sigma = 564132 \in \mathfrak{S}_6$, then the cycle notation of σ is $(1, 5, 3, 4)(2, 6)$. We always use parentheses and commas for writing cycles.

To describe our characterization of web permutations, we review the notion of André permutations and define an analogue of them. We now think of permutations as words consisting of distinct positive integers. *André permutations* are defined recursively as follows. First, the empty word and each one-letter word are André permutations. For a permutation $w = w_1w_2 \cdots w_n$ with $n \geq 2$, let w_k be the smallest letter in w. Then w is an André permutation if both $w_1 \cdots w_{k-1}$ and $w_{k+1} \cdots w_n$ are André permutations and $\max\{w_1, \ldots, w_{k-1}\} < \max\{w_{k+1}, \ldots, w_n\}$. Using this notion, we define a cycle analogue of André permutations.

Definition 3.1. Let $C = (a_1, \ldots, a_k)$ be a cycle with $a_1 = \min\{a_1, \ldots, a_k\}$. We say that C is an *André cycle* if the permutation $a_2 \cdots a_k$ is an André permutation.

For a cycle $C = (a_1, \ldots, a_k)$, we write $\min C = \min\{a_1, \ldots, a_k\}$ and write $\max C = \max\{a_1, \ldots, a_k\}$ for short. The following lemma gives how to obtain a new André cycle from old André cycles.

Lemma 3.2. *Let $C_1 = (a_1, \ldots, a_k)$ and $C_2 = (b_1, \ldots, b_\ell)$ be André cycles with $a_1 = \min C_1$ and $b_1 = \min C_2$. If $a_1 < b_1$ and $a_k < b_\ell$, then the cycle $(a_1, \ldots, a_k, b_1, \ldots, b_\ell)$ is also an André cycle.*

We now show another main result of the article, which gives a characterization of web permutations.

Theorem 3.3. *A permutation $\sigma \in \mathfrak{S}_n$ is a web permutation if and only if each cycle of σ is an André cycle.*

As an application of the characterization, we show that the transition matrix $(a_{MM'})$ is unitriangular with respect to a certain order on NN_{2n} and NC_{2n}, and determine which entries $a_{MM'}$ vanish. These are already known due to Russell–Tymoczko [10] and Im–Zhu [5].

Before we give the vanishing condition, we first show that the set Web_n includes a well-studied class of permutations. For a permutation $\sigma = \sigma_1 \cdots \sigma_n$, we say that σ *contains a 312-pattern* if there exist three indices $1 \le i < j < k \le n$ such that $\sigma_j < \sigma_k < \sigma_i$. A permutation is *312-avoiding* if it does not contain a 312-pattern. Note that 312-avoiding permutations are a Catalan object. Furthermore, the restriction of $D\colon \mathfrak{S}_n \to \mathrm{Dyck}_{2n}$ to the set of 312-avoiding permutations of $[n]$ is a bijection.

Corollary 3.4. *A 312-avoiding permutation is a web permutation.*

Recall that the set Dyck_{2n} has a partial order $\subseteq$, and there are bijections D from NN_{2n} and from NC_{2n} to Dyck_{2n}. Then the maps D induce a partial order on NN_{2n} and NC_{2n}. Furthermore, when we choose a total order on Dyck_{2n} that completes the partial order $\subseteq$, the maps D give a total order on NN_{2n} and NC_{2n}.

We now take a total order on Dyck_{2n} which completes the partial order $\subseteq$, and thus we have the induced total order on NN_{2n} and NC_{2n}. We assume that orderings of rows and columns of the transition matrix $(a_{MM'})$ are the decreasing orders with respect to the total order on NN_{2n} and NC_{2n}. Then the entry $a_{MM'}$ is on the diagonal if and only if $D(M) = D(M')$.

We are now ready to prove the unitriangularity of the transition matrix $(a_{MM'})$ and the conjecture of Russell and Tymoczko [10, Conjecture 5.8] concerning the condition of the vanishing entries, which is later proved by Im and Zhu [5, Theorem 1.1].

Corollary 3.5 ([5, 10]). *Let $M \in \mathrm{NN}_{2n}$ and $M' \in \mathrm{NC}_{2n}$. Then $a_{MM'} > 0$ if and only if $D(M') \subseteq D(M)$. In particular, the transition matrix $(a_{MM'})$ is upper-triangular. Moreover, there are ones along the diagonal of the transition matrix, and 312-avoiding permutations contribute to the ones.*

4 Enumeration of web permutations

In this section, we focus on the number of web permutations. We also conjecture that the Seidel triangle can be recovered completely from the certain classes of web permutations.

We now present another relationship between web permutations and André cycles. Let us first review the Foata transformation $\widehat{\ }: \mathfrak{S}_n \to \mathfrak{S}_n$. For a permutation $\sigma \in \mathfrak{S}_n$, the *canonical* cycle notation of σ is a cycle notation of σ such that its cycles are sorted based on the smallest element of the cycles and the smallest element of each cycle is written in the last place of the cycle. We define $\widehat{\sigma}$ to be the permutation obtained by dropping the parentheses in the canonical cycle notation of σ. A *right-to-left minimum* is an element σ_i such that $\sigma_i < \sigma_j$ for all $j > i$. Using right-to-left minima of σ, one can easily construct the inverse of the Foata transformation. Note that the number of cycles of σ equals the number of right-to-left minima of $\widehat{\sigma}$.

We now introduce a map $\phi: \mathfrak{S}_n \to \mathfrak{S}_{n+2}$ as a slightly modification of the Foata transformation. For a permutation $\sigma \in \mathfrak{S}_n$, define the one-cycle permutation $\phi(\sigma) \in \mathfrak{S}_{n+2}$ by

$$\phi(\sigma) := (1, \widehat{\sigma}_1 + 1, \ldots, \widehat{\sigma}_n + 1, n+2).$$

It follows immediately from the bijectivity of the Foata transformation that the map ϕ is injective, and its image $\phi(\mathfrak{S}_n)$ is the set of one-cycle permutations $\sigma \in \mathfrak{S}_{n+2}$ with $\sigma(n+2) = 1$.

The following theorem gives a relation between web permutations and André cycles, and show that the numbers of web permutations equal Euler numbers.

Theorem 4.1. *For $n \geq 1$, let $\mathrm{AC}_{n+2} \subset \mathfrak{S}_{n+2}$ be the set of André cycles consisting of $[n+2]$. Then we have $\phi(\mathrm{Web}_n) = \mathrm{AC}_{n+2}$. In particular, the number of web permutations of $[n]$ is equal to the number of André cycles consisting of $[n+2]$.*

4.1 Euler and Entringer numbers

In this subsections, we give various enumerative properties of web permutations using Theorem 4.1. We start with recalling Euler numbers. The *Euler numbers* E_n are defined via the exponential generating function $E(z) := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec z + \tan z$. There are numerous combinatorial objects enumerated by Euler numbers E_n. Especially, the Euler number E_n counts André permutations of $[n]$. For details, see [13]. We provide another occurrence of Euler numbers.

Corollary 4.2. *The Euler number E_{n+1} enumerates the number of web permutations of $[n]$.*

Proof. By definition, the number of André permutations of $[n]$ is equal to the number of André cycles of $[n+1]$. Then Theorem 4.1 implies the desired result. □

For a permutation σ, let $\mathrm{c}(\sigma)$ be the number of cycles of σ, and $\mathrm{rlmin}(\sigma)$ the number of right-to-left minima of σ. By convention, we set $\mathrm{c}(\emptyset) = 0$ where $\emptyset$ is the empty permutation, and $\mathrm{Web}_0 = \{\emptyset\}$. We have the following corollary concerning the distribution of $\mathrm{c}(\sigma)$ on Web_n.

Corollary 4.3. *We have*

$$\left(\frac{1}{1-\sin z}\right)^t = \sum_{n\geq 0} \sum_{\sigma\in \mathrm{Web}_n} t^{\mathrm{c}(\sigma)} \frac{z^n}{n!}.$$

Proof. In [1, Proposition 1], the author showed that

$$\left(\frac{1}{1-\sin z}\right)^t = \sum_{n\geq 1} \sum_{\sigma} t^{\mathrm{rlmin}(\sigma)-1} \frac{z^{n-1}}{(n-1)!}$$

where the inner sum is over all André permutations of $[n]$. Therefore the proof follows immediately from Theorem 4.1. □

We also recall Entringer numbers. The *Entringer* numbers are given by the generating function

$$\frac{\cos x + \sin x}{\cos(x+y)} = \sum_{m,n\geq 0} E_{m+n,[m,n]} \frac{x^m}{m!} \frac{y^n}{n!},$$

where $[m,n]$ is m if $m+n$ is odd, and n otherwise. We have a counterpart of this refinement.

Corollary 4.4. *The Entringer number $E_{n,k}$ is equal to the number of web permutations σ of $[n]$ with $\sigma_1 = n+1-k$.*

Proof. In [3, Theorem 1.1], the authors showed that $E_{n,k}$ equals the number of André permutations σ of $[n+1]$ with $\sigma_1 = n+1-k$. Combining this fact and Theorem 4.1 gives the proof. □

4.2 Genocchi numbers and the Seidel triangle

The Genocchi numbers are well-studied numbers with various combinatorial properties; see [2, 7]. The Genocchi numbers can be defined by the *Seidel triangle* as follows [12]. Recall that the Seidel triangle is an array of integers $(s_{i,j})_{i,j\geq 1}$ such that $s_{1,1} = s_{2,1} = 1$,

$$s_{2i+1,j} = s_{2i+1,j-1} + s_{2i,j} \text{ for } j = 1,\dots,i+1, \text{ and } s_{2i,j} = s_{2i,j+1} + s_{2i-1,j} \text{ for } j = i, i-1, \dots, 1,$$

where $s_{i,j} = 0$ for $j < 0$ or $j > \lceil i/2 \rceil$. This Pascal type procedure is called the *boustrophedon algorithm*. The *Genocchi numbers* g_n are defined by

$$g_{2n-1} = s_{2n-1,n} \quad \text{and} \quad g_{2n} = s_{2n,1}.$$

Recall that we denote by M_0 for the unique matching which is simultaneous noncrossing and nonnesting, *i.e.*, $M_0 = \{\{1,2\},\dots,\{2n-1,2n\}\}$. To emphasize the size of the matchings, we denote this unique matching of $[2n]$ by $M_0^{(n)}$. Let $f(n)$ be the number of web permutations σ of $[n]$ with $M(\sigma) = M_0^{(n)}$. In [8], Nakamigawa showed the following theorem.

Theorem 4.5 ([8, Theorem 3.1])**.** *For $n \geq 1$, we have $f(n) = g_n$.*

Let $f(n,k)$ be the number of web permutations σ of $[n]$ such that $M(\sigma) = M_0^{(n)}$ and $\sigma_1 = k$. Obviously, $f(n) = \sum_{1 \leq k \leq n} f(n,k)$. Some of these numbers vanish in the following cases.

Proposition 4.6. *For $n \geq 1$ and $1 \leq k \leq \lfloor n/2 \rfloor$, we have $f(n,2k) = 0$.*

Proof. Let σ be a web permutation of $[n]$ with $\sigma_1 = 2k$. Then considering the grid configuration $G(\sigma, \mathrm{Cr}(\sigma))$, the associated matching $M(\sigma)$ has an arc connecting $2k$ and some j with $2k < j$. Since there is the arc connecting $2k-1$ and $2k$ in $M_0^{(n)}$, we deduce $M(\sigma) \neq M_0^{(n)}$. □

Proposition 4.7. *For an odd $n > 1$, we have $f(n,n) = 0$.*

Proof. Let σ be a web permutation of $[n]$ with $\sigma_1 = n$. To obtain σ from the grid configuration $G(id, \emptyset)$, the first crossing (at the cell $(1,n)$) should be resolved by switching. Then we have two markings at $(1,n)$ and $(n,1)$. Observe that the vertical line and horizontal starting from the cell $(n,1)$ cannot contain a crossing. Therefore, in the procedure of resolving crossings, the markings $(1,n)$ and $(n,1)$ remains the same. Thus we have $\sigma_n = 1$ and $\{1,2n\} \in M(\sigma)$, which implies that $M(\sigma) \neq M_0^{(n)}$. □

By Propositions 4.6 and 4.7, we have $f(n) = \sum_{1 \leq k \leq \lfloor n/2 \rfloor} f(n, 2k-1)$. We now propose a conjecture that the values appearing in the Seidel triangle are $f(n,k)$.

Conjecture 4.8 (Verified up to $n = 6$)**.** *For $n \geq 1$, we have*

$$\begin{cases} f(2n-1, 2k-1) = s_{2n-2,k}, \\ f(2n, 2k-1) = s_{2n-1,n-k+1}. \end{cases}$$

This conjecture includes Nakamigawa's result. To elaborate, let σ be a web permutation of $[n]$ such that $M(\sigma) = M_0^{(n)}$ and $\sigma_1 = 1$. Deleting the cycle (1) from σ and decreasing each letter by 1, the resulting permutation is a web permutation of $[n-1]$ with $M(\sigma) = M_0^{(n-1)}$. In addition, this correspondence is bijective, so we deduce $f(n,1) = f(n-1)$. Thus the conjecture implies Nakamigawa's result $f(n-1) = g_{n-1}$.

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