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Set Partitions, Tableaux, and Subspace Profiles of Regular Diagonal Operators

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Abstract. We introduce a family of univariate polynomials indexed by integer partitions. At prime powers, they count the number of subspaces in a finite vector space that transform under a regular diagonal matrix in a specified manner. At 1, they count set partitions with specified block sizes. At 0, they count standard tableaux of specified shape. At -1, they count standard shifted tableaux of a specified shape. These polynomials are generated by a new statistic on set partitions (called the interlacing number) as well as a polynomial statistic on standard tableaux. They allow us to express q-Stirling numbers of the second kind as sums over standard tableaux and as sums over set partitions. In a special case these polynomials coincide with those defined by Touchard in his study of crossings of chord diagrams.

Keywords: subspace dynamics, split semisimple operator, interlacing, chord diagram, q-Stirling number, crossing

1 Introduction

Let $\mathbf{F}_q$ denote a finite field with q elements. Let Δ be a linear operator on $\mathbf{F}_q^n$.

Definition 1.1. A subspace $W \subseteq \mathbf{F}_q^n$ is said to have partial Δ-profile $\mu = (\mu_1, \dots, \mu_k)$ if

$$\dim(W + \Delta W + \cdots + \Delta^{j-1}W) = \mu_1 + \cdots + \mu_j \text{ for } 1 \leq j \leq k.$$

Furthermore, if

$$\dim(W + \Delta W + \cdots + \Delta^{k-1}W) = \dim(W + \Delta W + \cdots + \Delta^{k}W),$$

then we say that W has Δ-profile μ.

For example, an m-dimensional Δ-invariant subspace is one with profile (m) while a subspace spanned by a Δ-cyclic vector has profile (1^n). It is easy to show that $\mu_j \geq \mu_{j+1}$ for all j, and so μ is a partition of some integer less than or equal to n.

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Let $\sigma_n(\mu)$ denote the number of subspaces with Δ-profile μ. Using techniques of Chen and Tseng [8] it can be shown that $\sigma_n(\mu)$ is a polynomial in q that depends only on the similarity class type of Δ. We focus on the case where Δ has n distinct eigenvalues in $\mathbf{F}_q$. Thus Δ is represented by a diagonal matrix with distinct entries on its diagonal.

In Section 3.2 we introduce a family of polynomials $b_\lambda(q)$ indexed by integer partitions such that (see Theorem 4.1)

$$\sigma_n(\mu) = \binom{n}{|\mu|}(q-1)^{\sum_{j\geq 2}\mu_j} q^{\sum_{j\geq 2}\binom{\mu_j}{2}} b_{\mu'}(q), \tag{1.1}$$

where μ' denotes the partition conjugate to μ. The polynomial $b_\lambda(q)$ is a sum over $\mathrm{Tab}_{[n]}(\lambda)$, the set of standard tableaux of shape λ:

$$b_\lambda(q) = \sum_{T\in\mathrm{Tab}_{[n]}(\lambda)} c_q(T),$$

where $c_q(T)$ is a polynomial associated to each standard tableau T of size n. The polynomial c_q arises out of a surprising new connection between two classical combinatorial classes, namely set partitions and standard tableaux. In Section 2.2 we associate a standard tableau to each partition of the set $[n] = \{1,\dots,n\}$. Set partitions that map to a given tableau are counted by a statistic c on standard tableaux. A naive substitution of certain integers that occur in the definition of c (see Equation (2.1)) by the corresponding q-integers leads to c_q.

In Section 3.3, we introduce a statistic on set partitions called the interlacing number. Although defined in the spirit of the well-known crossing number of a set partition (see, for example, [9]) these numbers coincide only in certain very special cases, an interesting case being that of chord diagrams (see Section 4.2). We show that the polynomial $c_q(T)$ is the generating polynomial of the interlacing statistic on the class of set partitions associated to the tableau T (Theorem 3.4). Moreover, noninterlacing set partitions of shape λ are in bijection with standard tableaux of shape λ (see Corollary 3.5).

In Section 3.4 we show that $b_\lambda(-1)$ counts the number of standard shifted tableaux of shape λ for a certain class of partitions.

To summarize, the polynomials $b_\lambda(q)$ have the following specializations:

1. When q is a prime power, they count subspaces of $\mathbf{F}_q^n$ with profile λ', up to a factor of the form $q^a(q-1)^b$ (Equation (1.1)).

2. When $q = 1$, they count partitions of $[n]$ with block sizes given by the parts of λ (Equation (3.3)).

3. When $q = 0$, they count standard tableaux of shape λ and also the number of noninterlacing set partitions of shape λ (Equations (3.4) and (3.6)).

4. When $q = -1$ and the parts of λ are distinct, with the possible exception of the largest part, they count standard shifted tableaux of shape λ (Theorem 3.8).

We express the q-Stirling numbers of the second kind in terms of the polynomials $b_\lambda(q)$, as sums over standard tableaux, and as sums over set partitions:

$$S_q(n,m) = \sum_{\lambda \vdash n,\, l(\lambda)=m} q^{\sum_i (i-1)(\lambda_i-1)} b_\lambda(q), \tag{1.2}$$

$$S_q(n,m) = \sum_{\lambda \vdash n,\, l(\lambda)=m} q^{\sum_i (i-1)(\lambda_i-1)} \sum_{T \in \mathrm{Tab}_{[n]}(\lambda)} c_q(T), \tag{1.3}$$

$$S_q(n,m) = \sum_{\mathcal{A} \in \Pi_{n,m}} q^{v(\mathcal{A}) + \sum_i (i-1)(\lambda_i^{\mathcal{A}}-1)}. \tag{1.4}$$

Here $\Pi_{n,m}$ denotes the collection of partitions of $[n]$ having m blocks, $(\lambda_1^{\mathcal{A}}, \lambda_2^{\mathcal{A}}, \dots)$ is the list of block sizes of $\mathcal{A}$ sorted in weakly decreasing order, and $v(\mathcal{A})$ is the interlacing number of $\mathcal{A}$. Many statistics on set partitions are known to produce the q-Stirling numbers of the second kind [5, 10, 24]. Our statistic appears to be different from all of these. Cai and Readdy [5, Theorem 3.2] express $S_q(n,m)$ as a sum of expressions of the form $q^a(q+1)^b$ over a small class of set partitions. In this vein, we also express $S_q(n,m)$ as a sum over noninterlacing set partitions but in our case the summands are powers of q times a product of q-integers:

$$S_q(n,m) = \sum_{\substack{\mathcal{A} \in \Pi_{n,m} \\ v(\mathcal{A})=0}} q^{\sum_i (i-1)(\lambda_i-1)} c_q(\mathcal{T}(\mathcal{A})).$$

Combining equations (1.1) and (1.2) allows for a bijective interpretation of an identity of Carlitz [6, Equation (8)] (see also [7, Equation (3.4)]) that expresses q-binomial coefficients in terms of q-Stirling numbers of the second kind:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \sum_{r=m}^{n} (q-1)^{r-m} \binom{n}{r} S_q(r,m).$$

The left hand side counts all m-dimensional subspaces of $\mathbf{F}_q^n$ while the right hand is obtained by counting subspaces by their Δ-profile. At $q = 1$, the identity (1.3) gives an expression for the Stirling numbers of the second kind as the sum of a statistic on standard tableaux which appears to be new. A similar identity holds for the Bell numbers.

If m divides n, say $n = md$, then a Δ-splitting subspace of $\mathbf{F}_q^n$ is a subspace W of dimension m such that

$$W \oplus \Delta W \oplus \cdots \oplus \Delta^{d-1} W = \mathbf{F}_q^n.$$

Thus an m-dimensional splitting subspace is a subspace of $\mathbf{F}_q^n$ with Δ-profile (m^d). The definition of a splitting subspace can be traced back to the work of Niederreiter [15] on pseudorandom number generation. Determining the number of m-dimensional Δ-splitting subspaces for an arbitrary operator Δ is an open problem [12, page 54]. The answer is known in the cases where Δ has an irreducible characteristic polynomial [4, 8, 11], Δ is nilpotent [2], or when the invariant factors of Δ satisfy certain degree constraints [1].

In the case where Δ has n distinct eigenvalues in $\mathbf{F}_q$, Equation (1.1) gives the number of splitting subspaces of dimension m when $\mu = (m^d)$. The case $d = 2$ is of particular interest; we find that it is associated with the study of crossings of chord diagrams that were extensively investigated by Touchard (see Section 4.2).

Detailed proofs can be found in [17]. We extensively used SageMath [20] for computations. Code for demonstrating the results in this paper is available online at https://www.imsc.res.in/~amri/set_partitions.

2 Set Partitions

2.1 Set partitions and standard notation

Let S be a finite subset of the set $\mathbf{P}$ of positive integers. A partition $\mathcal{A} = \{A_1, \dots, A_m\}$ of S is a decomposition

$$S = A_1 \cup \cdots \cup A_m,$$

where $A_1, \dots, A_m$ are pairwise disjoint non-empty subsets of S. The subsets $A_1, \dots, A_m$ are called the *blocks* of $\mathcal{A}$. The order of the blocks does not matter. Following standard conventions [14, § 2.7.1.5], the elements of each block are listed in increasing order, and the blocks are listed in increasing order of their least elements. When this is the case, we write $\mathcal{A} = A_1|\cdots|A_m$, which we call *the standard notation* for $\mathcal{A}$. The shape of a set partition is the list of cardinalities of $A_1, \dots, A_m$, sorted in weakly decreasing order. Thus the shape of a partition of S is an integer partition of $|S|$. Denote the set of all partitions of S by Π_S, and the set of all partitions of S with shape λ by $\Pi_S(\lambda)$.

2.2 The tableau associated to a set partition

Definition 2.1. Given $\mathcal{A} = A_1|\cdots|A_m \in \Pi_S$, form an array whose entry in the ith row and jth column is the jth smallest element of A_i. Then sort and top-justify the columns of this array. Denote the resulting tableau by $\mathcal{T}(\mathcal{A})$.

Example 2.2. When $\mathcal{A} = 12|389|56$, the associated array and the corresponding tableau $\mathcal{T}(\mathcal{A})$ are

$$\begin{array}{|c|c|c} \hline 1 & 2 & \multicolumn{1}{c}{} \\ \hline 3 & 8 & \multicolumn{1}{c|}{9} \\ \hline 5 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 9 \\ \hline 3 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & 8 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

Definition 2.3. A tableau T is called a *multilinear tableau* if each element of $\mathbf{P}$ occurs at most once in T, the rows of T increase from left to right, and the columns of T increase from top to bottom.

The set of integers that occur in T is called the support of T, and is denoted $\mathrm{supp}(T)$. For each $S \subseteq \mathbf{P}$, let Tab_S denote the set of multilinear tableaux with support S. For each integer partition λ, denote the set of multilinear tableaux of shape λ and support S by $\mathrm{Tab}_S(\lambda)$. Let $\mathrm{Tab}_{\subseteq S}(\lambda)$ denote the set of multilinear tableaux of shape λ whose support is a subset of S. The usual notion of a standard Young tableau coincides with that of a multilinear tableau with support $[n]$ for some $n \geq 0$.

Example 2.4. The tableau in Example 2.2 is a multilinear tableau of shape $(3, 2, 2)$ and support $\{1, 2, 3, 5, 6, 8, 9\}$.

It can be shown that Definition 2.1 gives rise to a surjective function

$$\mathcal{T} \colon \Pi_S(\lambda) \to \mathrm{Tab}_S(\lambda).$$

To see surjectivity, given $T \in \mathrm{Tab}_S(\lambda)$ simply take $\mathcal{A}$ to be the partition of S whose blocks are the rows of T. Then $\mathcal{T}(\mathcal{A}) = T$. For each $T \in \mathrm{Tab}_S(\lambda)$, let

$$\Pi(T) = \{\mathcal{A} \in \Pi_S(\lambda) \mid \mathcal{T}(\mathcal{A}) = T\},$$

the fibre of $\mathcal{T}$ over T. Our next goal is to understand the set $\Pi(T)$ for each multilinear tableau T.

Example 2.5. When $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 9 \\ \hline 3 & 6 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} \end{array}$, the set partitions in $\Pi(T)$ are

$$129|38|56,\ 129|36|58,\ 12|369|58,\ 12|38|569,\ 12|36|589,\ 12|389|56.$$

2.3 Set partitions associated to a given tableau

Suppose that $S \subseteq \mathbf{P}$ is finite, and λ is an integer partition of $|S|$.

Definition 2.6. Let T be a multilinear tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_m)$. For $1 \leq i \leq m$ and $2 \leq j \leq \lambda_i$ define

$$c_{ij}(T) = \#\{i' \mid i' \geq i \text{ and } T_{i',j-1} < T_{ij}\}.$$

Define

$$c(T) = \prod_{i=1}^{m} \prod_{j=2}^{\lambda_i} c_{ij}(T). \tag{2.1}$$

Example 2.7. For the tableau T of Example 2.5, the values of $c_{ij}(T)$ for each relevant cell are shown below:

$$\begin{array}{|c|c|c|}\hline & 1 & 3 \\ \hline & 2 \\ \cline{1-2} & 1 \\ \cline{1-2}\end{array},$$

and $c(T) = 6$.

Theorem 2.8. *For every multilinear tableau T, the cardinality of $\Pi(T)$ is $c(T)$.*

3 q-analogs

3.1 A q-analog of $c(T)$

For each multilinear tableau T of shape $\lambda = (\lambda_1, \ldots, \lambda_m)$, define $c_q(T) \in \mathbf{Z}[q]$ by

$$c_q(T) = \prod_{i=1}^{m} \prod_{j=2}^{\lambda_i} [c_{ij}(T)]_q, \tag{3.1}$$

where, for each positive integer n, $[n]_q = 1 + q + \cdots + q^{n-1}$, the q-analog of n. Clearly $c_q(T)$ is a polynomial with nonnegative integer coefficients, and substituting $q = 1$ gives

$$c_q(T)|_{q=1} = c(T),$$

the cardinality of $\Pi(T)$.

3.2 The polynomials $b_\lambda(q)$

For each integer partition λ and each positive integer n, define

$$b^n_\lambda(q) = \sum_{T \in \mathrm{Tab}_{\subseteq[n]}(\lambda)} c_q(T). \tag{3.2}$$

Write $b_\lambda(q)$ for $b^n_\lambda(q)$ when λ is a partition of n. Any order-preserving relabeling of the entries of T leaves $c_q(T)$ invariant. Therefore, for any λ and any n,

$$b^n_\lambda(q) = \binom{n}{|\lambda|} b_\lambda(q).$$

It follows from Theorem 2.8 that

$$b_\lambda(1) = |\Pi_{[n]}(\lambda)|, \tag{3.3}$$

the number of set partitions of $[n]$ of shape λ. Since, for every standard tableau T, $b_\lambda(q)$ is a product of q-integers, we have

$$b_\lambda(0) = |\mathrm{Tab}_{[n]}(\lambda)|. \tag{3.4}$$

3.3 The interlacing statistic

Let $\mathbf{P}^*$ denote the ordered set $\mathbf{P} \cup \{\infty\}$, where ∞ is deemed to be greater than every element of $\mathbf{P}$.

Definition 3.1 (Crossing arcs)**.** For $a, b, c, d \in \mathbf{P}^*$, we say that the arcs (a, b) and (c, d) cross if the intervals $[a, b]$ and $[c, d]$ are neither nested, nor disjoint. In other words:

$$\text{either } a < c < b < d \text{ or } c < a < d < b.$$

Definition 3.2 (The arcs of a set)**.** Given a set $A \subseteq \mathbf{P}$ whose elements are $a_1, \ldots, a_l$ in increasing order, its jth arc is the pair $\mathrm{arc}_j(A) = (a_j, a_{j+1})$ for $j = 1, \ldots, l-1$, and its lth arc is $\mathrm{arc}_l(A) = (a_l, \infty)$.

Definition 3.3 (Interlacing)**.** Let S be any finite subset of $\mathbf{P}$. Let $\mathcal{A} = A_1|\cdots|A_m \in \Pi_S$ with $|A_i| = l_i$. An *interlacing* of $\mathcal{A}$ is a pair $(\mathrm{arc}_j(A_i), \mathrm{arc}_j(A_{i'}))$ of crossing arcs for some $1 \leq i < i' \leq m$ and some $1 \leq j \leq \min(l_i, l_{i'})$. Let $v(\mathcal{A})$ denote the total number of interlacings of the set partition $\mathcal{A}$, called the *interlacing number* of $\mathcal{A}$.

Table 1 shows the arcs and the number of interlacings for the set partitions in Example 2.5. The first, second, and third arcs are shown in different colours. Only crossing arcs of the same colour contribute to the interlacing number.

Theorem 3.4. *For any multilinear tableau T,*

$$c_q(T) = \sum_{\mathcal{T}(\mathcal{A})=T} q^{v(\mathcal{A})}. \tag{3.5}$$

Corollary 3.5. *For each partition λ of n, the number of noninterlacing partitions of shape λ is equal to the number of standard tableaux of shape λ.*

Theorem 3.4 allows us to express $b_\lambda(q)$ in Equation (3.2) as a sum over set partitions of shape λ:

$$b_\lambda(q) = \sum_{\mathcal{A}\in\Pi_n(\lambda)} q^{v(\mathcal{A})}. \tag{3.6}$$

3.4 Value at $q = -1$

It turns out that $b_\lambda(-1)$ is always positive.

Theorem 3.6. *For every integer partition λ, $b_\lambda(-1)$ is the number of standard tableaux T of shape λ for which $c(T)$ is odd. For every non-empty*[1] *integer partition λ, $b_\lambda(-1) > 0$.*

[1] The empty partition is the unique partition of 0 with no parts.

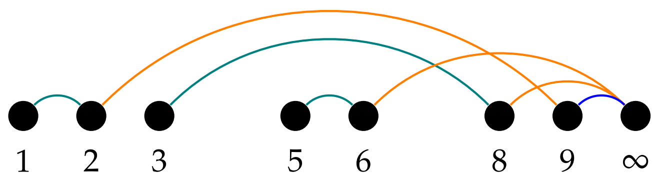
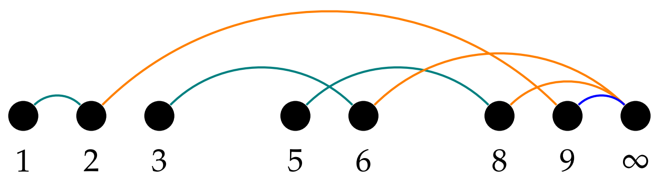
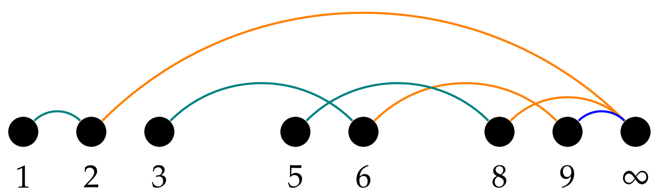
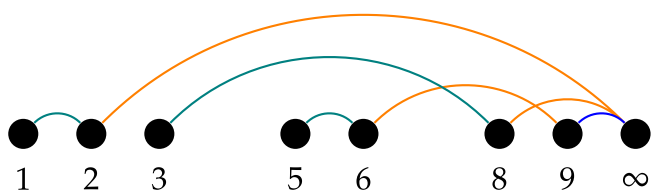
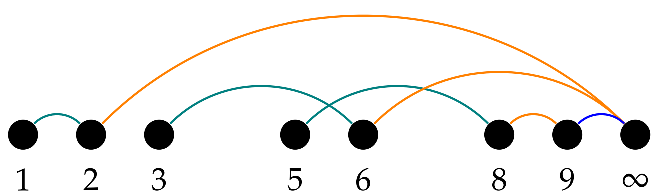
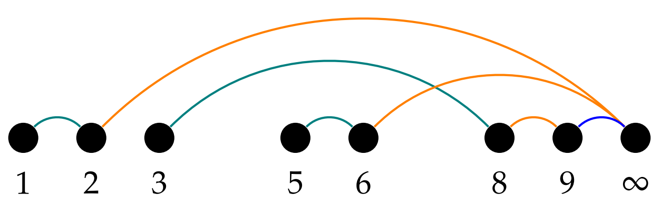
$\mathcal{A}$	Arcs of $\mathcal{A}$	$v(\mathcal{A})$
129|38|56		2
129|36|58		3
12|369|58		2
12|38|569		1
12|36|589		1
12|389|56		0

Table 1: Statistics for set partitions corresponding to $\begin{array}{|c|c|c|}\hline 1&2&9\\\hline 3&6\\\cline{1-2} 5&8\\\cline{1-2}\end{array}$.

For a large class of partitions λ, $b_\lambda(-1)$ has an interpretation in terms of shifted tableaux. Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be an integer partition of n. A standard shifted tableau of shape λ is an array

$$T = (T_{ij} \mid T_{ij} \in \mathbf{P},\ 1 \le i \le m;\ i \le j \le \lambda_i + i - 1),$$

where each integer in $[n]$ occurs exactly once. The rows of T increase from left to right, the columns of T increase from top to bottom, and the diagonals of T increase from top-left to bottom-right. We denote the set of multilinear shifted tableaux of shape λ by $\mathrm{sTab}_S(\lambda)$. When λ is a partition of $[n]$, write $\mathrm{sTab}(\lambda)$ for $\mathrm{sTab}_{[n]}(\lambda)$.

Example 3.7. The following is a standard shifted tableau of shape $(3,3,2,1)$:

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & \\ \hline & 4 & 5 & 6 \\ \hline & & 7 & 8 \\ \hline & & & 9 \\ \hline \end{array}.$$

Theorem 3.8. *For every partition λ whose parts are distinct, with the possible exception of the largest part,*

$$b_\lambda(-1) = |\mathrm{sTab}(\lambda)|.$$

Example 3.9. Take $\lambda = (3,3,1)$. Then

$$b_\lambda(q) = q^4 + 5q^3 + 15q^2 + 28q + 21,$$

and $b_\lambda(-1) = 4$. There are four standard shifted tableaux of shape $(3,3,1)$, namely

$$\begin{array}{cccc} 1 & 2 & 3 & \\ & 4 & 5 & 6 \\ & & 7 & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 4 & \\ & 3 & 5 & 6 \\ & & 7 & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & \\ & 4 & 5 & 7 \\ & & 6 & \end{array}, \quad \text{and} \quad \begin{array}{cccc} 1 & 2 & 4 & \\ & 3 & 5 & 7 \\ & & 6 & \end{array}.$$

4 Profiles of Subspaces

4.1 Counting subspaces by profile

Let Δ be a diagonal $n \times n$ matrix over $\mathbf{F}_q$ with distinct entries on its diagonal. Recall the definition of b_λ from Section 3.2.

Theorem 4.1. *For every integer partition μ, the number of subspaces of $\mathbf{F}_q^n$ with Δ-profile μ is given by*

$$\sigma_n(\mu) = \binom{n}{|\mu|}(q-1)^{\sum_{j\ge 2}\mu_j} q^{\sum_{j\ge 2}\binom{\mu_j}{2}} b_{\mu'}(q).$$

Theorem 4.1 is a consequence of a more refined counting theorem (Theorem 4.5).

For any $n \in \mathbf{P}$, let $C(n,m)$ denote the set of all subsets of $[n]$ with cardinality m. Given $C \in C(n,m)$ we always write C as a tuple $C = (c_1, \dots, c_m)$, where $c_1 < \cdots < c_m$.

Definition 4.2 (Pivots)**.** Every m-dimensional subspace $W \subseteq \mathbf{F}_q^n$ has a unique basis in reduced row echelon form, namely a basis whose elements are the rows of a matrix $A = (A_{ij})$ satisfying the following conditions:

1. There exists $C = (c_1, \dots, c_m) \in C(n,m)$ (called the pivots of W) such that the first non-zero entry in the ith row of A lies in the c_ith column and is equal to 1.
2. $A_{i'c_i} = 0$ for all $i' \neq i$ (the only non-zero entry in the c_ith column lies in the ith row).

When this happens we say that W has pivots C.

Example 4.3. The basis in reduced row echelon form for a subspace of $\mathbf{F}_q^n$ (with $n \geq 4$) with pivots $(1,3,4)$ is given by the rows of a $3 \times n$ matrix of the form

$$\begin{pmatrix} 1 & * & 0 & 0 & * & \cdots & * \\ 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 1 & * & \cdots & * \end{pmatrix},$$

where each $*$ denotes an arbitrary element of $\mathbf{F}_q$.

Recall the following well-known result [13].

Theorem 4.4. *The number of m-dimensional subspaces of $\mathbf{F}_q^n$ with pivots $C = (c_1, \dots, c_m)$ is $q^{\beta(C,[n]-C)}$, where*

$$\begin{aligned} \beta(C,[n]-C) &= |\{(c,c') \mid c < c',\, c \in C,\, c' \in [n]-C\}| \\ &= \sum_{i=1}^{m} (n - m - c_i + i). \end{aligned}$$

A simultaneous refinement of Theorems 4.1 and 4.4 is the following:

Theorem 4.5. *For every $C \in C(n,m)$ and every partition μ with $\mu_1 = m$, the number of m-dimensional subspaces of $\mathbf{F}_q^n$ with pivots C and profile μ is given by*

$$\sigma_n^C(\mu) = (q-1)^{\sum_{j\geq 2}\mu_j} q^{\sum_{j\geq 2}\binom{\mu_j}{2}} \sum_{T \in \mathrm{Tab}_{\subseteq[n]}(\mu') \text{ has first column } C} c_q(T).$$

Taking the sum over all $C \in C(n,m)$, where $m = \mu_1$ gives Theorem 4.1.

Corollary 4.6. *For any $C \in C(n,m)$,*

$$q^{\beta(C,[n]-C)} = \sum_{\{\mu \mid \mu_j = m\}} (q-1)^{\sum_{j\geq 2}\mu_j} q^{\sum_{j\geq 2}\binom{\mu_j}{2}} \sum_{T \in \mathrm{Tab}_{\subseteq[n]}(\mu') \text{ has first column } C} c_q(T).$$

4.2 Splitting subspaces and chord diagrams

Recall the definition of a splitting subspace from Ghorpade and Ram [12].

Definition 4.7. Let $n = md$. An m-dimensional subspace $W \subseteq \mathbf{F}_q^n$ is Δ-splitting if

$$\mathbf{F}_q^n = W \oplus \Delta W \oplus \cdots \oplus \Delta^{d-1} W.$$

It follows that Δ-splitting subspaces are those with Δ-profile (m^d). Theorem 4.1 gives the number of m-dimensional Δ-splitting subspaces as

$$\sigma_n(m^d) = (q-1)^{m(d-1)} q^{\binom{m}{2}(d-1)} b_{(d^m)}(q).$$

By Equation (3.6) we have

$$b_{(d^m)}(q) = \sum_{\mathcal{A} \in \Pi_n(d^m)} q^{v(\mathcal{A})}.$$

When $d = 2$, $\Pi_n(2^m)$ coincides with the set of chord diagrams on $2m$ points and the interlacing number coincides with the number of crossing chords. Thus $b_{(2^m)}(q)$ coincides with the polynomial $T_m(q)$ studied by Touchard [21, 22, 23] in the context of the stamp folding problem. For proofs of Touchard's compact expression for $T_m(q)$ see Riordan [19], Read [18] and Penaud [16]. For a beautiful exposition, see Aigner [3, page 337].

References

[1] D. Aggarwal and S. Ram. "Polynomial matrices, splitting subspaces and Krylov subspaces over finite fields". 2021. arXiv:2105.15155.

[2] D. Aggarwal and S. Ram. "Splitting subspaces of linear operators over finite fields". *Finite Fields Appl.* **78** (2022), p. 101982. DOI.

[3] M. Aigner. *A Course in Enumeration*. Vol. 238. Graduate Texts in Mathematics. Springer, Berlin, 2007, pp. x+561.

[4] A. Arora, S. Ram, and A. Venkateswarlu. "Unimodular polynomial matrices over finite fields". *J. Algebraic Combin.* **53**.4 (2021), pp. 1299–1312. DOI.

[5] Y. Cai and M. A. Readdy. "q-Stirling numbers: A new view". *Adv. in Appl. Math.* **86** (2017), pp. 50–80. DOI.

[6] L. Carlitz. "On Abelian fields". *Trans. Amer. Math. Soc.* **35**.1 (1933), pp. 122–136. DOI.

[7] L. Carlitz. "q-Bernoulli numbers and polynomials". *Duke Math. J.* **15** (1948), pp. 987–1000. Link.

[8] E. Chen and D. Tseng. "The splitting subspace conjecture". *Finite Fields Appl.* **24** (2013), pp. 15–28. DOI.

[9] W. Y. C. Chen, E. Y. P. Deng, R. R. X. Du, R. P. Stanley, and C. H. Yan. "Crossings and nestings of matchings and partitions". *Trans. Amer. Math. Soc.* **359**.4 (2007), pp. 1555–1575. DOI.

[10] A. M. Garsia and J. B. Remmel. "Q-counting rook configurations and a formula of Frobenius". *J. Combin. Theory Ser. A* **41**.2 (1986), pp. 246–275. DOI.

[11] S. R. Ghorpade and S. Ram. "Block companion Singer cycles, primitive recursive vector sequences, and coprime polynomial pairs over finite fields". *Finite Fields Appl.* **17**.5 (2011), pp. 461–472. DOI.

[12] S. R. Ghorpade and S. Ram. "Enumeration of splitting subspaces over finite fields". *Arithmetic, Geometry, Cryptography and Coding Theory*. Vol. 574. Contemp. Math. Amer. Math. Soc., Providence, RI, 2012, pp. 49–58. DOI.

[13] D. E. Knuth. "Subspaces, subsets, and partitions". *J. Combin. Theory Ser. A* **10** (1971), pp. 178–180. DOI.

[14] D. E. Knuth. *The Art of Computer Programming. Vol. 4A. Combinatorial Algorithms. Part 1*. Addison-Wesley, Upper Saddle River, NJ, 2011, pp. xv+883.

[15] H. Niederreiter. "The multiple-recursive matrix method for pseudorandom number generation". *Finite Fields Appl.* **1**.1 (1995), pp. 3–30. DOI.

[16] J.-G. Penaud. "Une preuve bijective d'une formule de Touchard-Riordan". *Discrete Math.* **139**.1-3 (1995). Formal power series and algebraic combinatorics (Montreal, PQ, 1992), pp. 347–360. DOI.

[17] A. Prasad and S. Ram. "Set partitions, tableaux, and subspace profiles under regular split semisimple matrices". 2021. arXiv:2112.00479.

[18] R. C. Read. "The chord intersection problem". *Second International Conference on Combinatorial Mathematics (New York, 1978)*. Vol. 319. Ann. New York Acad. Sci. New York Acad. Sci., New York, 1979, pp. 444–454. DOI.

[19] J. Riordan. "The distribution of crossings of chords joining pairs of $2n$ points on a circle". *Math. Comp.* **29** (1975), pp. 215–222. DOI.

[20] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.3)*. 2021. Link.

[21] J. Touchard. "Contribution à l'étude du problème des timbres poste". *Canad. J. Math.* **2** (1950), pp. 385–398. DOI.

[22] J. Touchard. "Sur un problème de configurations". *C. R. Acad. Sci. Paris* **230** (1950), pp. 1997–1998. DOI.

[23] J. Touchard. "Sur un problème de configurations et sur les fractions continues". *Canad. J. Math.* **4** (1952), pp. 2–25. DOI.

[24] M. Wachs and D. White. "p, q-Stirling numbers and set partition statistics". *J. Combin. Theory Ser. A* **56**.1 (1991), pp. 27–46. DOI.