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The Canonical Complex of the Weak Order

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Abstract. We study the canonical complex of a finite semidistributive lattice L, a simplicial complex which encodes each interval $[x, y]$ of L by recording simultaneously the canonical join representation of x and the canonical meet representation of y, and behaves properly with respect to lattice quotients of L. We then describe combinatorially the canonical complex of the weak order on permutations in terms of semi-crossing arc bidiagrams, formed by the superimposition of two non-crossing arc diagrams of N. Reading. Finally, we provide an algorithm to describe the Kreweras maps in any lattice quotient of the weak order in terms of semi-crossing arc bidiagrams.

Résumé. Nous étudions le complexe canonique d'un treillis semidistributif fini L, un complexe simplicial qui encode chaque intervalle $[x, y]$ de L par la représentation canonique par supremums de x et la représentation canonique par infimums de y, et qui se comporte bien par rapport aux treillis quotients de L. Nous décrivons ensuite combinatoirement le complexe canonique de l'ordre faible sur les permutations en termes de bidiagrammes d'arcs avec des semi-croisements, formés par la superposition de deux diagrammes d'arcs sans croisement de N. Reading. Finalement, nous présentons un algorithme pour décrire les fonctions de Kreweras dans tout treillis quotient de l'ordre faible en termes de bidiagrammes d'arcs.

Keywords: semidistributive lattices, canonical join representations, weak order, non-crossing arc diagrams, Kreweras complement

A finite lattice L is join semidistributive when any element admits a canonical join representation (see Section 1.1 or [3] for definitions). This enables us to define the canonical join complex of L [2, 7], whose vertices are the join irreducible elements of L and whose simplices are the canonical join representations in L. When L is both join and meet semidistributive, it thus admits both a canonical join complex and a canonical meet complex which are actually isomorphic flag simplicial complexes [2].

We define the canonical complex of a finite semidistributive lattice L, a larger flag simplicial complex where the canonical join complex and the canonical meet complex live and interact. More precisely, its vertex set is the disjoint union of the set of join irreducible elements of L with the set of meet irreducible elements of L, and its simplices

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are the disjoint unions $J \sqcup M$ of a canonical join representation J in L with a canonical meet representation M in L such that $\bigvee J \leq \bigwedge M$. In other words, each interval $[x, y]$ in L contributes to a simplex of the canonical complex given by the disjoint union of the canonical join representation of x with the canonical meet representation of y. This provides a model for the intervals of L compatible with lattice quotients. Namely, the canonical complex of a quotient $L/\equiv$ is the subcomplex of the canonical complex of L induced by the join and meet irreducibles of L uncontracted by the congruence $\equiv$.

We then study the combinatorics of the canonical complex of the weak order. N. Reading showed in [7] that join irreducible permutations correspond to certain arcs, and that canonical join representations of permutations correspond to non-crossing arc diagrams. We show that the elements of the canonical complex can be interpreted as semi-crossing arc bidiagrams, defined as pairs $\delta_\vee \sqcup \delta_\wedge$ of non-crossing arc diagrams where only certain types of crossings are allowed between an arc of $\delta_\vee$ and an arc of $\delta_\wedge$. It follows that the canonical complex of any quotient of the weak order is isomorphic to a subcomplex of the semi-crossing complex induced by arcs contained in an upper ideal of the subarc order. Finally, we provide an algorithm to describe the Kreweras maps in any lattice quotient of the weak order in terms of semi-crossing arc bidiagrams, generalizing the classical Kreweras complement on non-crossing partitions.

Many details and all proofs omitted in this extended abstract can be found in [1].

1 The canonical complex of a semidistributive lattice

1.1 Recollection on lattices

We start by a quick recollection on semidistributive lattices, Kreweras maps and lattice congruences. All the material covered here is classical, see for instance [2, 3, 7, 8].

Join representations and semidistributive lattices. Consider a finite lattice $(L, \leq, \vee, \wedge)$ where $\vee$ is the join and $\wedge$ is the meet. We see $\vee$ and $\wedge$ as internal binary operators on L and try to factorize the elements of L in some canonical way as products of irreducibles.

Definition 1. *An element $x \in L$ is called join (resp. meet) irreducible if it covers (resp. is covered by) a unique element denoted $x_\star$ (resp. $x^\star$). We denote by $\mathcal{JI}(L)$ (resp. $\mathcal{MI}(L)$) the subposet of L induced by the set of join (resp. meet) irreducible elements of L.*

Definition 2. *A join representation of $x \in L$ is a subset $J \subseteq L$ such that $x = \bigvee J$. Such a representation is irredundant if $x \neq \bigvee J'$ for any strict subset $J' \subsetneq J$. The irredundant join representations in L are antichains of L, and are ordered by containement of the lower sets of their elements (i.e. $J \leq J'$ if and only if for any $y \in J$ there exists $y' \in J'$ such that $y \leq y'$ in L). The canonical join representation of x, denoted $\mathbf{cjr}(x)$, is the minimal irredundant join representation of x for this order, when it exists.*

Note that when it exists, $\mathbf{cjr}(x)$ is an antichain of $\mathcal{JI}(L)$. The following statement characterizes the lattices where canonical join representations exist.

Proposition 3 ([3, Theorem 2.24, Theorem 2.56]). *A finite lattice L is join semidistributive when the following equivalent conditions hold:*

(i) $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$,

(ii) for any cover relation $x \lessdot y$ in L, the set $K_\vee(x, y) := \{z \in L \mid z \not\leq x \text{ but } z \leq y\}$ has a unique minimal element $k_\vee(x, y)$ (which is then automatically join irreducible),

(iii) any element of L admits a canonical join representation.

Moreover, the canonical join representation of $y \in L$ is $\mathbf{cjr}(y) = \{k_\vee(x, y) \mid x \lessdot y\}$.

Note that in a finite join semidistributive lattice L, we can associate to any meet irreducible element m of L a join irreducible element $\kappa_\vee(m) := k_\vee(m, m^\star)$ of L. Moreover, the existence of canonical join representations enable us to consider the following complex, initially defined for the weak order in [7] and studied in general in [2]. See Figure 1.

Definition 4. *The canonical join complex $\mathcal{CJC}(L)$ of a finite join semidistributive lattice L is the simplicial complex on $\mathcal{JI}(L)$ whose faces are the canonical join representations in L.*

The *meet semidistributivity*, the maps $K_\wedge$, $k_\wedge$ and $\kappa_\wedge$, the *canonical meet representation* $\mathbf{cmr}(x)$ and the *canonical meet complex* $\mathcal{CMC}(L)$ are all defined dually. A lattice L is *semidistributive* if it is both meet and join semidistributive.

Proposition 5 ([2, Theorem 2 & Corollary 5]). *If L is a finite semidistributive lattice, then*

(i) $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ are flag simplicial complexes (i.e. their minimal non-faces are edges, or equivalently they are the clique complexes of their graphs),

(ii) the maps $\kappa_\vee$ and $\kappa_\wedge$ induce inverse isomorphisms between $\mathcal{CMC}(L)$ and $\mathcal{CJC}(L)$.

Example 6 (Distributive lattices). *The name semidistributivity comes from the well understood class of distributive lattices. A lattice L is distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$. The fundamental theorem for distributive lattices affirms that L is distributive if and only if it is isomorphic to the lattice of lower sets of its join irreducible poset P. In other words, any antichain of join irreducible elements in P forms a canonical join representation in L. Consider, for an antichain A of P, the two lower sets $j_A := \{x \in P \mid x \leq y \text{ for some } y \in A\}$ and $m^A := \{x \in P \mid x \not\geq y \text{ for all } y \in A\}$. Said differently, A is the set of maximal elements of j_A and the set of minimal elements of $P \smallsetminus m^A$. For $y \in P$, we abbreviate $j_{\{y\}}$ into j_y and $m^{\{y\}}$ into m^y. Then*

- *the join (resp. meet) irreducibles of L are precisely the lower sets j_y (resp. m^y) for $y \in P$,*
- *the map $\kappa_\vee$ (resp. $\kappa_\wedge$) is given by $\kappa_\vee(m^y) = j_y$ (resp. $\kappa_\wedge(j_y) = m^y$),*
- *the canonical join representation of j_A is $\mathbf{cjr}(j_A) = \{j_y \mid y \in A\}$ and the canonical meet representation of m^A is $\mathbf{cmr}(m^A) = \{m^y \mid y \in A\}$.*
- *the canonical join and meet complexes $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ are both (isomorphic to) the clique complex on the incomparability graph of P.*

See [2, Exm. 10].

Kreweras maps. In a semidistributive lattice L, each element has both a canonical join representation and a canonical meet representation. It is natural to consider the maps that exchange the canonical join representations with the canonical meet representations.

Definition 7. *The Kreweras maps $\eta_\vee : \mathcal{CMC}(L) \to \mathcal{CJC}(L)$ and $\eta_\wedge : \mathcal{CJC}(L) \to \mathcal{CMC}(L)$ are defined by $\eta_\vee(M) := \mathbf{cjr}\big(\bigwedge M\big)$ and $\eta_\wedge(J) := \mathbf{cmr}\big(\bigvee J\big)$.*

The Kreweras maps are related to the Kreweras complement on non-crossing partitions for the Tamari lattice (see Example 38) and to rowmotion for distributive lattices.

Example 8 (Distributive lattices)**.** *With the notations of Example 6, for an antichain A in P, we denote by $\mathrm{row}_\vee(A)$ the set of maximal elements of m^A and by $\mathrm{row}_\wedge(A)$ the set of minimal elements of $P \smallsetminus j_A$. In other words, we have $m^A = j_{\mathrm{row}_\vee(A)}$ and $j_A = m_{\mathrm{row}_\wedge(A)}$. Hence, by Example 6, the Kreweras maps $\eta_\vee$ and $\eta_\wedge$ are given by $\eta_\vee(\{m^y \mid y \in A\}) = \big\{j_y \bigm| y \in \mathrm{row}_\vee(A)\big\}$ and $\eta_\wedge(\big\{j_y \bigm| y \in A\big\}) = \{m^y \mid y \in \mathrm{row}_\wedge(A)\}$. See [2, Rem. 32].*

Lattice congruences. We now discuss quotients of the lattice L, considered as an algebraic structure with two internal binary operators $\vee$ and $\wedge$.

Definition 9. *A congruence $\equiv$ on L is an equivalence relation on L such that $x \equiv x'$ and $y \equiv y'$ implies $x \vee y \equiv x' \vee y'$ and $x \wedge y \equiv x' \wedge y'$. Equivalently, the equivalence classes are intervals, and the maps $\pi_\downarrow^\equiv$ and $\pi^\uparrow_\equiv$ sending an element to the minimum and maximum elements in its congruence class are order preserving.*

Definition 10. *The lattice quotient $L/\!\equiv$ is the lattice structure on the congruence classes, where for any two congruence classes X and Y,*

- *the order is given by $X \le Y$ if and only if $x \le y$ for some representatives $x \in X$ and $y \in Y$,*
- *the join $X \vee Y$ (resp. meet $X \wedge Y$) is the congruence class of $x \vee y$ (resp. $x \wedge y$) for any representatives $x \in X$ and $y \in Y$.*

The lattice quotient $L/\!\equiv$ is isomorphic to the subposet of L induced by the minimal (or maximal) elements in their congruence classes. It is a join (resp. meet) subsemilattice of L but may fail to be a sublattice of L. We now consider all congruences of L.

Definition 11. *The congruence lattice $\mathrm{con}(L)$ is the set of all congruences of L ordered by refinement.*

The congruence lattice $\mathrm{con}(L)$ is a distributive lattice where the meet is the intersection of relations and the join is the transitive closure of union of relations. For any join irreducible element $j \in \mathcal{JI}(L)$, we denote by $\mathrm{con}(j)$ the unique minimal congruence of L that *contracts* j, that is with $j_\star \equiv j$. It turns out that $\mathrm{con}(j)$ is join irreducible in $\mathrm{con}(L)$ and that all join irreducible congruences in $\mathrm{con}(L)$ are of this form. Hence, any congruence of L is completely determined by the set of join irreducible elements of L that it contracts. We denote by $\mathcal{UJI}(\equiv)$ the set of join irreducible elements of L uncontracted by $\equiv$. Not all subsets of join irreducible elements of L are of the form $\mathcal{UJI}(\equiv)$ for some congruence $\equiv$ of L. The possible subsets are governed by the following relation.

Definition 12. *For $j, j' \in \mathcal{JI}(L)$, we say that j forces j', and write $j \succcurlyeq j'$, if $\mathrm{con}(j) \geq \mathrm{con}(j')$, that is if any congruence contracting j also contracts j'.*

The forcing relation is a preorder $\preccurlyeq$ (*i.e.* a transitive and reflexive, but not necessarily antisymmetric, relation) on $\mathcal{JI}(L)$, whose upper sets correspond to the congruences of L.

Proposition 13 ([8, Proposition 9-5.16])**.** *The following conditions are equivalent for $J \subseteq \mathcal{JI}(L)$:*

- *J is an upper set of the forcing preorder (i.e. $j \succcurlyeq j'$ and $j \in J$ implies $j' \in J$).*
- *$J = \mathcal{UJI}(\equiv)$ for some congruence $\equiv$ of L.*

The set $\mathcal{UJI}(\equiv)$ characterizes $\equiv$ and enables to understand the elements of L which are minimal in their congruence classes and their canonical representations as follows.

Proposition 14 ([8, Proposition 9-5.29] and [1, Proposition 16])**.** *Let $\equiv$ be a congruence of a finite join semidistributive lattice L. Then*

(i) an element $x \in L$ is minimal in its congruence class if and only if $\mathbf{cjr}(x) \subseteq \mathcal{UJI}(\equiv)$,

(ii) the quotient $L/\equiv$ is join semidistributive and the canonical joinands of a congruence class X in $L/\equiv$ are the congruence classes of the canonical joinands of the minimal element in X,

(iii) for any $x \in L$, the lower ideal of L generated by $\mathbf{cjr}(x)$ contains $\mathbf{cjr}\big(\pi_{\downarrow}^{\equiv}(x)\big)$,

(iv) the canonical join complex $\mathcal{CJC}(L/\equiv)$ of the quotient $L/\equiv$ is isomorphic to the subcomplex $\mathcal{CJC}(\equiv)$ of the canonical join complex $\mathcal{CJC}(L)$ of L induced by $\mathcal{UJI}(\equiv)$.

Dual statements hold using meets instead of joins, and we denote by $\mathcal{UMI}(\equiv)$ the meet irreducible elements of L uncontracted by $\equiv$, and by $\mathcal{CMC}(\equiv)$ the subcomplex of $\mathcal{CMC}(L)$ induced by $\mathcal{UMI}(\equiv)$ for a congruence $\equiv$ on L. Due to Proposition 14 (iv), we will always work with the subcomplexes $\mathcal{CJC}(\equiv)$ and $\mathcal{CMC}(\equiv)$ rather than with the complexes $\mathcal{CJC}(L/\equiv)$ and $\mathcal{CMC}(L/\equiv)$. When the lattice L is semidistributive, the two sets $\mathcal{UJI}(\equiv)$ and $\mathcal{UMI}(\equiv)$ and the two subcomplexes $\mathcal{CJC}(\equiv)$ and $\mathcal{CMC}(\equiv)$ are connected by the maps $\kappa_\vee$ and $\kappa_\wedge$.

Proposition 15. *Let $\equiv$ be a congruence on a finite semidistributive lattice L. Then we have $\mathcal{UJI}(\equiv) = \kappa_\vee\big(\mathcal{UMI}(\equiv)\big)$ and $\mathcal{UMI}(\equiv) = \kappa_\wedge\big(\mathcal{UJI}(\equiv)\big)$. Hence, the maps $\kappa_\vee$ and $\kappa_\wedge$ induce inverse isomorphisms between the subcomplexes $\mathcal{CMC}(\equiv)$ and $\mathcal{CJC}(\equiv)$.*

Example 16 (Distributive lattices)**.** *In a distributive lattice L, there is no forcing at all. Hence, any subset of join irreducible elements of L defines a congruence of L. In other words, with the notations of Example 6, any subset Y of P defines a congruence $\equiv_Y$ with $\mathcal{UJI}(\equiv_Y) = \{j_y \mid y \in Y\}$ and $\mathcal{UMI}(\equiv_Y) = \{m^y \mid y \in Y\}$. The lattice quotient $L/\equiv$ is again distributive and isomorphic to the lattice of lower ideals of the restriction of the poset P to Y.*

1.2 The canonical complex

We now define another complex that connects the canonical join complex $\mathcal{CJC}(L)$ to the canonical meet complex $\mathcal{CMC}(L)$ using intervals of L. See Figure 1.

Definition 17. *The canonical complex $\mathcal{CC}(L)$ of a finite semidistributive lattice L is the simplicial complex whose ground set is the **disjoint** union $\mathcal{JI}(L) \sqcup \mathcal{MI}(L)$, and whose faces are the **disjoint** unions $J \sqcup M$ where $J \in \mathcal{CJC}(L)$ while $M \in \mathcal{CMC}(L)$ and $\bigvee J \le \bigwedge M$.*

To avoid any confusion, let us insist that an element appears twice in $\mathcal{CC}(L)$ if it is both join and meet irreducible. We now gather some relevant properties of $\mathcal{CC}(L)$.

Proposition 18. *For a finite semidistributive lattice L,*

(i) the canonical join (resp. meet) complex $\mathcal{CJC}(L)$ (resp. $\mathcal{CMC}(L)$) is the subcomplex of the canonical complex $\mathcal{CC}(L)$ induced by $\mathcal{JI}(L)$ (resp. $\mathcal{MI}(L)$),

(ii) the faces of the canonical complex $\mathcal{CC}(L)$ are in bijection with the intervals of L,

(iii) the canonical complex $\mathcal{CC}(L)$ is a flag simplicial complex,

(iv) for any $j \in \mathcal{JI}(L)$, the pair $\{j, \kappa_\wedge(j)\}$ is not in $\mathcal{CC}(L)$, so that the canonical complex $\mathcal{CC}(L)$ can be embedded in the boundary of the $|\mathcal{JI}(L)|$-dimensional cross-polytope,

(v) for any congruence $\equiv$ on L, the canonical complex $\mathcal{CC}(L/\equiv)$ of the quotient $L/\equiv$ is isomorphic to the subcomplex $\mathcal{CC}(\equiv)$ of the canonical complex $\mathcal{CC}(L)$ of L induced by the disjoint union $\mathcal{UJI}(\equiv) \sqcup \mathcal{UMI}(\equiv)$ of the join and meet irreducible elements uncontracted by $\equiv$.

Example 19 (Distributive lattices)**.** *With the notations of Example 6, we have $j_y \subseteq m^z$ if and only if $y \not\ge z$. Hence, the canonical complex $\mathcal{CC}(L)$ is the clique complex of the graph whose vertex set is made of two copies $P_\vee$ and $P_\wedge$ of P and whose edge set is the union of two copies $I_\vee$ and $I_\wedge$ of the incomparability graph of P with the edges $\{y_\vee, z_\wedge\}$ for $y \not\ge z$ in P.*

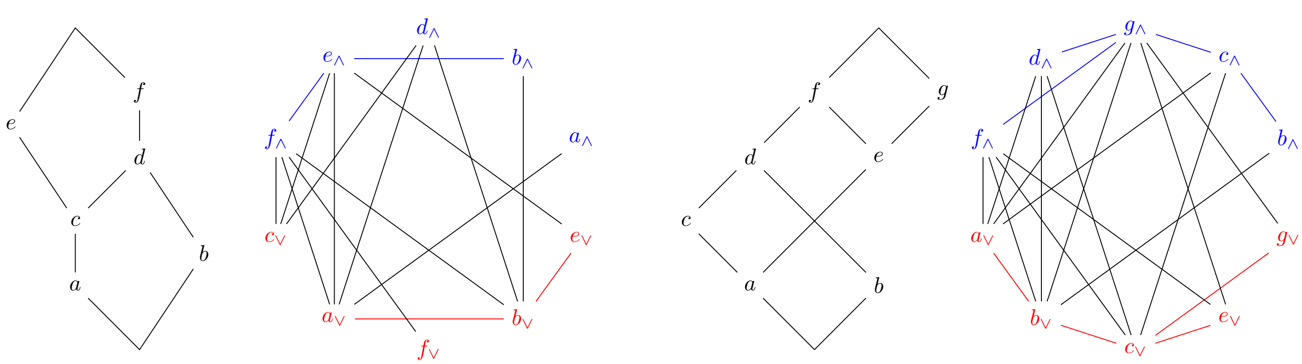

Figure 1: The canonical complexes of two semidistributive lattices. The corresponding join (resp. meet) canonical complexes are highlighted in red (resp. blue). Since the canonical complexes are flag by Proposition 18, it is sufficient to represent their graphs. The letters label all join or meet irreducible elements, and we denote by $x_\vee$ (resp. $x_\wedge$) the element x when it is considered as a join (resp. meet) irreducible.

2 Semi-crossing arc bidiagrams

2.1 Non-crossing arc diagrams

We consider the set $\mathfrak{S}_n$ of permutations of $[n] := \{1, \dots, n\}$. An *inversion* of a permutation $\sigma \in \mathfrak{S}_n$ is a pair (u, v) with $1 \leq u < v \leq n$ and $\sigma^{-1}(u) > \sigma^{-1}(v)$. The *weak order* is the lattice on $\mathfrak{S}_n$ defined by inclusion of inversion sets. Note that a cover relation corresponds to the swap of two values σ_i and σ_{i+1} at consecutive positions. The swap is increasing in the weak order if i is an *ascent* *i.e.* $\sigma_i < \sigma_{i+1}$, and decreasing if i is a *descent* *i.e.* $\sigma_i > \sigma_{i+1}$. It is classical that the weak order is semidistributive. We now describe its join (resp. meet) irreducible elements and its canonical join (resp. meet) representations in terms of the arcs and non-crossing arc diagrams introduced by N. Reading in [7].

Definition 20 ([7])**.** *An arc is a quadruple (a, b, A, B) where $1 \leq a < b \leq n$ and $A \sqcup B$ forms a partition of $]a, b[:= \{a+1, \dots, b-1\}$. Two arcs $\alpha := (a, b, A, B)$ and $\alpha' := (a', b', A', B')$ cross if there are $u \neq v$ with $u \in (A' \cup \{a', b'\}) \cap (B \cup \{a, b\})$ and $v \in (A \cup \{a, b\}) \cap (B' \cup \{a', b'\})$. A non-crossing arc diagram (NCAD) is a collection of pairwise non-crossing arcs. The non-crossing complex is the clique complex of the non-crossing relation on all arcs.*

Remark 21. *Visually, an arc (a, b, A, B) is represented by an x-monotone curve wiggling around the horizontal axis, starting at a and ending at b, and passing above points of A and below points of B. Two arcs cross if they cross in their interior or start at the same point or end at the same point (but they do not cross if one ends where the other starts). See Figure 2.*

Observe that the join (resp. meet) irreducible elements of the weak order are precisely the permutations with exactly one descent (resp. ascent). Hence, we associate to an arc $\alpha := (a, b, A, B)$ with $A := \{a_1 < \dots < a_k\}$ and $B := \{b_1 < \dots < b_\ell\}$

- a join irreducible $\boldsymbol{\sigma}_\vee(\alpha) := [1, \dots, (a-1), a_1, \dots, a_k, b, a, b_1, \dots, b_\ell, (b+1), \dots, n]$,
- a meet irreducible $\boldsymbol{\sigma}_\wedge(\alpha) := [n, \dots, (b+1), a_k, \dots, a_1, a, b, b_\ell, \dots, b_1, (a-1), \dots, 1]$.

Conversely, consider a permutation $\sigma \in \mathfrak{S}_n$ represented by its permutation table formed by dots at coordinates (σ_i, i) for $i \in [n]$. Draw segments between consecutive dots (σ_i, i) and $(\sigma_{i+1}, i+1)$, colored red for a descent $\sigma_i > \sigma_{i+1}$ and blue for an ascent $\sigma_i < \sigma_{i+1}$. Finally, flatten the picture vertically to the horizontal line, allowing segments to bend but not to pass points. The resulting picture is the superimposition of a set $\boldsymbol{\delta}_\vee(\sigma)$ of red arcs and a set $\boldsymbol{\delta}_\wedge(\sigma)$ of blue arcs. See Figure 2. More formally, $\boldsymbol{\delta}_\vee(\sigma) := \{\boldsymbol{\alpha}_\vee(\sigma, i) \mid \sigma_i < \sigma_{i+1}\}$ and $\boldsymbol{\delta}_\wedge(\sigma) := \{\boldsymbol{\alpha}_\wedge(\sigma, i) \mid \sigma_i > \sigma_{i+1}\}$ where

$$\boldsymbol{\alpha}_\vee(\sigma, i) := (\sigma_i, \sigma_{i+1}, \{\sigma_j \mid j < i \text{ and } \sigma_i < \sigma_j < \sigma_{i+1}\}, \{\sigma_j \mid j > i+1 \text{ and } \sigma_i < \sigma_j < \sigma_{i+1}\}),$$
$$\boldsymbol{\alpha}_\wedge(\sigma, i) := (\sigma_{i+1}, \sigma_i, \{\sigma_j \mid j < i \text{ and } \sigma_i > \sigma_j > \sigma_{i+1}\}, \{\sigma_j \mid j > i+1 \text{ and } \sigma_i > \sigma_j > \sigma_{i+1}\})).$$

Proposition 22 ([7])**.** *The map $\boldsymbol{\delta}_\vee$ (resp. $\boldsymbol{\delta}_\wedge$) is a bijection between the permutations and the non-crossing arc diagrams. Moreover, the canonical join (resp. meet) representation of $\sigma \in \mathfrak{S}_n$ is* $\mathbf{cjr}(\sigma) = \{\boldsymbol{\sigma}_\vee(\alpha_\vee) \mid \alpha_\vee \in \boldsymbol{\delta}_\vee(\sigma)\}$ *(resp.* $\mathbf{cmr}(\sigma) = \{\boldsymbol{\sigma}_\wedge(\alpha_\wedge) \mid \alpha_\wedge \in \boldsymbol{\delta}_\wedge(\sigma)\}$*). Hence, the canonical join (resp. meet) complex of the weak order is isomorphic to the non-crossing complex.*

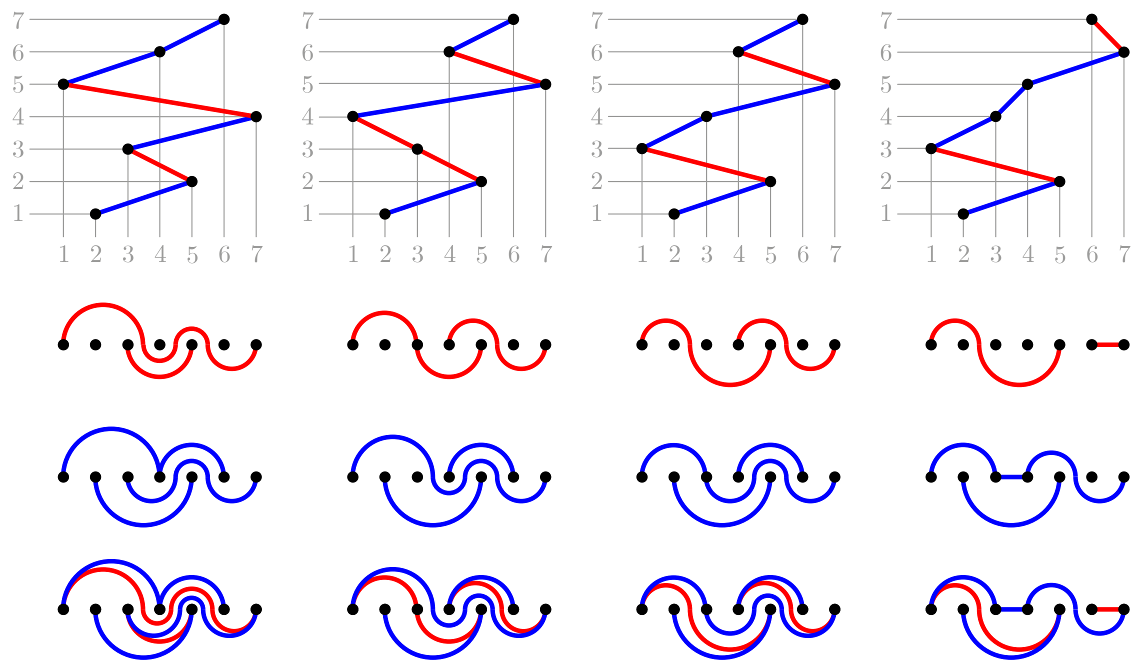

Figure 2: NCADs and SCABs of the permutations 2537146, 2531746, 2513746, and 2513476. The first line represents the table $(\sigma(i), i)$ of a permutation σ with ascents in blue and descents in red, the second line is the join diagram $\delta_\vee(\sigma)$, the third line is the meet diagram $\delta_\wedge(\sigma)$, and the fourth line is the superimposition $\delta_\vee(\sigma) \sqcup \delta_\wedge(\sigma)$.

By construction, the non-crossing arc diagrams are adapted to the maps $\kappa_\vee$ and $\kappa_\wedge$ and to quotients of the weak order.

Proposition 23. *$\kappa_\vee(\boldsymbol{\sigma}_\wedge(\alpha)) = \boldsymbol{\sigma}_\vee(\alpha)$ and $\kappa_\wedge(\boldsymbol{\sigma}_\vee(\alpha)) = \boldsymbol{\sigma}_\wedge(\alpha)$ for any arc α.*

Proposition 24 ([7])**.** *For any two arcs $\alpha := (a, b, A, B)$ and $\alpha' := (a', b', A', B')$, the join irreducible $\boldsymbol{\sigma}_\vee(\alpha)$ forces the join irreducible $\boldsymbol{\sigma}_\vee(\alpha')$ if and only if α is a subarc of α', meaning that $a' \le a < b \le b'$ and $A \subseteq A'$ while $B \subseteq B'$. Hence, to each upper ideal I of the subarc order corresponds a lattice congruence $\equiv_I$ of the weak order, and the canonical join (resp. meet) complex of the quotient of the weak order by $\equiv_I$ is isomorphic to the non-crossing complex on I.*

Remark 25. *Visually, α is a subarc of α' if the endpoints of α are weakly in between the endpoints of α', and α follows α' between its endpoints.*

Example 26. *The prototypical congruence of the weak order is the sylvester congruence $\equiv_{sylv}$ [6, 4], corresponding to the upper ideal of the subarc order given by all up arcs $(a, b,]a, b[, \varnothing)$. The quotient of the weak order by the sylvester congruence is (isomorphic to) the classical Tamari lattice [9, 5], whose elements are the binary trees on n nodes and whose cover relations are rotations in binary trees. The canonical representations in the Tamari lattice correspond to non-crossing sets of up arcs, also known as non-crossing partitions.*

2.2 Weak order on arcs

We now compare join or meet irreducible elements in the weak order in terms of arcs.

Lemma 27. *For an arc $\alpha := (a, b, A, B)$ and $u < v$, the pair (u, v) is an inversion of $\boldsymbol{\sigma}_\vee(\alpha)$ (resp. of $\boldsymbol{\sigma}_\wedge(\alpha)$) if and only if $u \in B \cup \{a\}$ and $v \in A \cup \{b\}$ (resp. $u \notin A \cup \{a\}$ or $v \notin B \cup \{b\}$).*

Corollary 28. *For any two arcs $\alpha := (a, b, A, B)$ and $\alpha' := (a', b', A', B')$, we have*

(i) $\boldsymbol{\sigma}_\vee(\alpha) \le \boldsymbol{\sigma}_\vee(\alpha')$ if and only if $a \in B' \cup \{a'\}$, $b \in A' \cup \{b'\}$, $A \subseteq A'$ and $B \subseteq B'$,

(ii) $\boldsymbol{\sigma}_\wedge(\alpha) \le \boldsymbol{\sigma}_\wedge(\alpha')$ if and only if $a' \in B \cup \{a\}$, $b' \in A \cup \{b\}$, $A' \subseteq A$ and $B' \subseteq B$,

(iii) $\boldsymbol{\sigma}_\vee(\alpha) \le \boldsymbol{\sigma}_\wedge(\alpha')$ if and only if there is no $u < v$ such that $u \in (A' \cup \{a'\}) \cap (B \cup \{a\})$ and $v \in (A \cup \{b\}) \cap (B' \cup \{b'\})$.

Remark 29. *Figure 3 shows the weak order on arcs defined by $\alpha \le \alpha'$ if $\boldsymbol{\sigma}_\vee(\alpha) \le \boldsymbol{\sigma}_\vee(\alpha')$. Visually, $\alpha \le \alpha'$ if α is a subarc of α' which starts weakly below α' and ends weakly above α'. Note that $\alpha := (a, b, A, B)$ covers at most two arcs, namely $(\min B, b, A \cap]\min b, b[, B \smallsetminus \min B)$ and $(a, \max A, A \smallsetminus \max A, B \cap]a, \max A[)$ when they are defined. Similar remarks hold for the order defined by $\boldsymbol{\sigma}_\wedge(\alpha)$ instead of $\boldsymbol{\sigma}_\vee(\alpha)$.*

Remark 30. *As illustrated in Figure 3, the weak order on join irreducible of $\mathfrak{S}_n$ has interesting enumerative properties. Let us just mention here that it has*

- *$2^n - n - 1$ elements (permutations with a single descent, or arcs) [10, A000295],*
- *$2^{n+1} - n^2 - n - 2$ cover relations (in bijection with arcs of size $n+1$ crossing the horizontal axis, or with subsets of $[n+1]$ crossing their complement) [10, A324172],*
- *$n(n+1)2^{n-2}$ intervals (including the singletons) [10, A001788].*

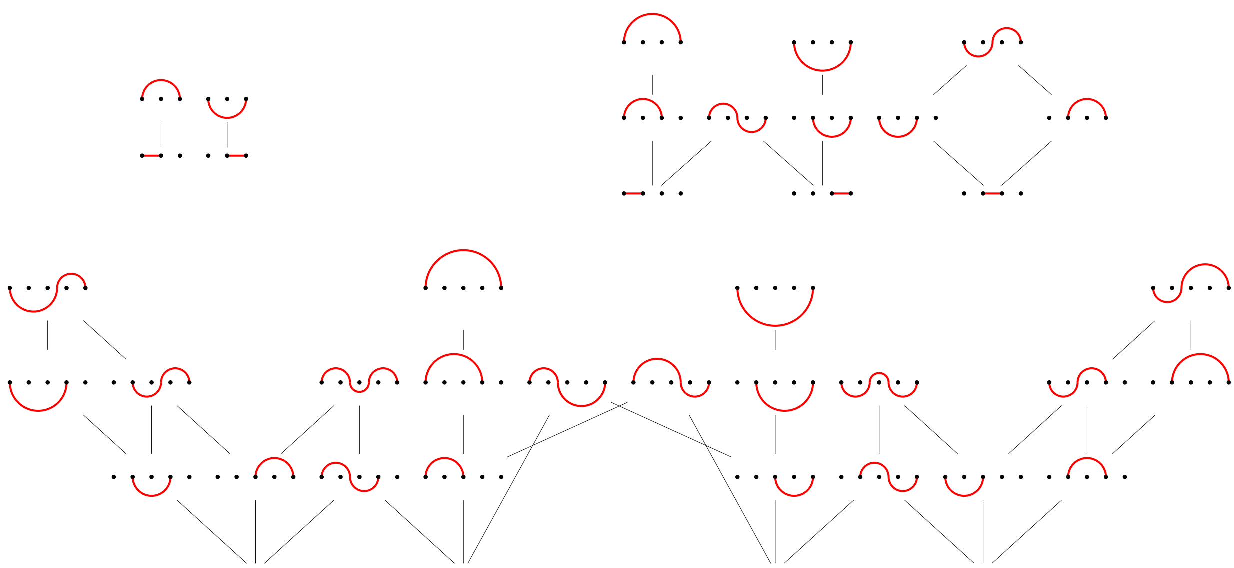

Figure 3: The weak orders of size 3 (top left), 4 (top right), and 5 (bottom) restricted to their join irreducibles represented by the corresponding arcs.

2.3 Semi-crossing arc bidiagrams

We now describe the canonical complex of the weak order as defined in Section 1.2 in terms of the following combinatorial objects, illustrated in Figure 4.

Definition 31. *A semi-crossing arc bidiagram (SCAB) is a pair $(\delta_\vee, \delta_\wedge)$ of non-crossing arc diagrams such that for any arcs $\alpha_\vee := (a_\vee, b_\vee, A_\vee, B_\vee) \in \delta_\vee$ and $\alpha_\wedge := (a_\wedge, b_\wedge, A_\wedge, B_\wedge) \in \delta_\wedge$, there is no $u < v$ with $u \in (A_\wedge \cup \{a_\wedge\}) \cap (B_\vee \cup \{a_\vee\})$ and $v \in (A_\vee \cup \{b_\vee\}) \cap (B_\wedge \cup \{b_\wedge\})$. The semi-crossing complex is the simplicial complex whose ground set contains two copies $\alpha_\vee$ and $\alpha_\wedge$ of each arc α and whose simplices are all semi-crossing arc bidiagrams.*

Remark 32. *Visually, a semi-crossing arc bidiagram $\delta_\vee \sqcup \delta_\wedge$ is a collection of arcs such that*

- *no two arcs of $\delta_\vee$ (resp. of $\delta_\wedge$) cross in their interior, or start or end at the same points,*
- *no two arcs $\alpha_\vee \in \delta_\vee$ and $\alpha_\wedge \in \delta_\wedge$ cross in their interiors with $\alpha_\vee$ going up and $\alpha_\wedge$ going down at the crossing, or start at the same point with $\alpha_\vee$ leaving above $\alpha_\wedge$, or end at the same point with $\alpha_\vee$ arriving below $\alpha_\wedge$ at this point.*

Proposition 33. *The map $[\sigma, \tau] \mapsto \boldsymbol{\delta}_\vee(\sigma) \sqcup \boldsymbol{\delta}_\wedge(\tau)$ is a bijection between the intervals of the weak order on $\mathfrak{S}_n$ and the semi-crossing arc bidiagrams. Hence, the canonical complex of the weak order is isomorphic to the semi-crossing complex.*

Some semi-crossing arc bidiagrams of intervals are illustrated in Figure 4, and the canonical complex of the weak order on $\mathfrak{S}_3$ is illustrated in Figure 5. The central symmetry corresponds to the maps $\kappa_\vee$ and $\kappa_\wedge$, which just corresponds to the exchange of color of the arcs by Proposition 23. Let us insist again here that this combinatorial model for the intervals of the weak order is adapted to the study of its quotients.

Proposition 34. *For any lower ideal I of the subarc order, the canonical complex of the quotient of the weak order by $\equiv_I$ is isomorphic to the subcomplex of the semi-crossing complex induced by $\{\alpha_\vee \mid \alpha \in I\} \sqcup \{\alpha_\wedge \mid \alpha \in I\}$.*

Remark 35. *Observe that the semi-crossing arc bidiagram corresponding to a singleton $[\sigma, \sigma]$ is just a path alternating between increasing blue steps and decreasing red steps corresponding to the ascents and descents of σ. See Figure 2. In general, we conjecture that the semi-crossing arc bidiagram corresponding to an inclusion minimal interval in a lattice quotient of the weak order does not contain any crossing in the interior of an arc.*

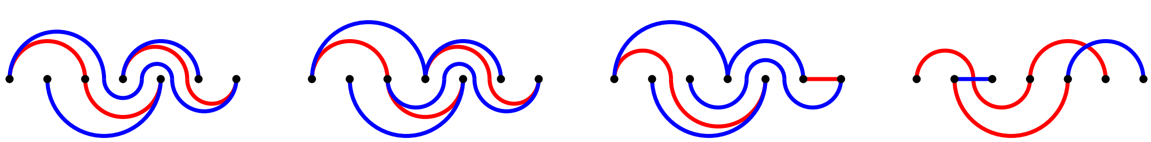

Figure 4: SCABs of the weak order intervals $[2531746, 2531746]$, $[2531746, 2537146]$, $[2513476, 2537146]$ and $[5264137, 6574231]$.

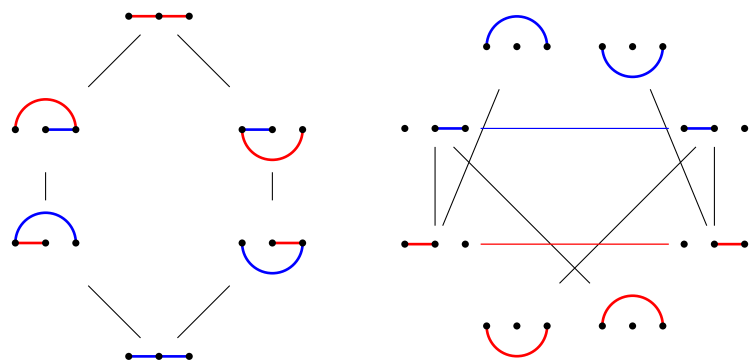

Figure 5: The weak order on $\mathfrak{S}_3$ with permutations labeled by semi-crossing arc bidiagrams, and the canonical complex of $\mathfrak{S}_3$ with join and meet irreducible permutations labeled by arcs.

2.4 Kreweras maps in quotients of the weak order

We finally describe the Kreweras maps defined in Section 1.1 in all quotients of the weak order in terms of semi-crossing arc bidiagrams. For this, we first connect the canonical join representation of a permutation to the canonical join representation of the minimal element in its class for a given congruence, improving on Proposition 14 (iii). We call weak order on arcs the order $\alpha \le \alpha'$ if $\boldsymbol{\sigma}_\vee(\alpha) \le \boldsymbol{\sigma}_\vee(\alpha')$ (see Section 2.2 and Figure 3).

Proposition 36. *Consider an upper ideal I of the subarc order and a permutation σ. Let X be the intersection of I with the lower ideal generated by the non-crossing arc diagram $\boldsymbol{\delta}_\vee(\sigma)$ in the weak order on arcs. Let Y be the set of arcs (a, b, A, B) of X such that there is $a < p < b$ such that both arcs $(a, p, A \cap]a, p[, B \cap]a, p[)$ and $(p, b, A \cap]p, b[, B \cap]p, b[)$ belong to X. Then the non-crossing arc diagram $\boldsymbol{\delta}_\vee\big(\pi_\downarrow^{\equiv_I}(\sigma)\big)$ is the set of maximal elements of $X \smallsetminus Y$ in the weak order on arcs.*

This enables us to compute the Kreweras maps in quotients of the weak order directly on non-crossing arc diagrams. For this, we extend the notations of Section 1.1 to quotients and transport them to non-crossing arc diagrams. For an upper ideal I of the subarc order, each equivalence class of $\equiv_I$ is an interval $[x, y]$ of the weak order and thus corresponds to two non-crossing arc diagrams $\delta_\vee := \boldsymbol{\delta}_\vee(x)$ and $\delta_\wedge := \boldsymbol{\delta}_\wedge(y)$. We denote by $\eta_\vee^I$ and $\eta_\wedge^I$ the two opposite maps defined by $\eta_\vee^I(\delta_\wedge) = \delta_\vee$ and $\eta_\wedge^I(\delta_\vee) = \delta_\wedge$. We just write $\eta_\vee$ and $\eta_\wedge$ when I is the set of all arcs. Note that $\eta_\vee = \boldsymbol{\delta}_\vee \circ \boldsymbol{\delta}_\wedge^{-1}$ and $\eta_\wedge = \boldsymbol{\delta}_\wedge \circ \boldsymbol{\delta}_\vee^{-1}$ are easily computed from the descriptions of the maps $\boldsymbol{\delta}_\vee$ and $\boldsymbol{\delta}_\wedge$ (see Section 2.1) and of their inverses (see [7]). Proposition 36 enables to compute $\eta_\vee^I$ and $\eta_\wedge^I$ in general.

Corollary 37. *Consider an upper ideal I of the subarc order and a non-crossing arc diagram $\delta_\wedge$ with all arcs in I. Then the non-crossing arc diagram $\eta_\vee^I(\delta_\wedge)$ is obtained from $\eta_\vee(\delta_\wedge)$ by applying the algorithm of Proposition 36.*

Example 38. *When I is the upper ideal of up arcs corresponding to the sylvester congruence, the description of Corollary 37 can be translated to the classical description of the Kreweras complement of a non-crossing partition, obtained by shifting the points and connecting the points in the same connected component. See Figure 6.*

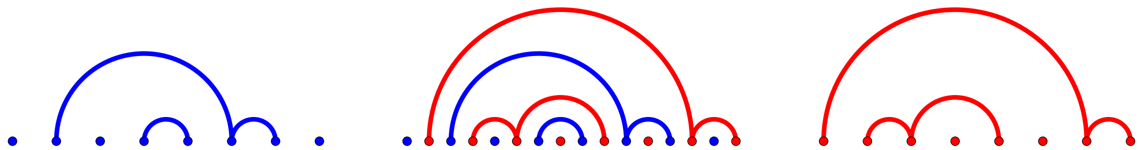

Figure 6: Classical Kreweras complement on non-crossing partitions.

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