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Rowmotion on Fences

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Abstract. A fence is a poset with elements $F = \{x_1, x_2, \ldots, x_n\}$ and covers

$$x_1 \lhd x_2 \lhd \cdots \lhd x_a \rhd x_{a+1} \rhd \cdots \rhd x_b \lhd x_{b+1} \lhd \cdots ,$$

where $a, b, \ldots$ are positive integers. We investigate rowmotion on antichains and ideals of F. In particular, we show that orbits of antichains can be visualized using tilings. This permits us to prove various homomesy results for the number of elements of an antichain or ideal in an orbit. Rowmotion on fences also exhibits a new phenomenon, which we call homometry, where the value of a statistic is constant on orbits of the same size. Along the way, we prove a homomesy result for all self-dual posets and show that any two Coxeter elements in certain toggle groups behave similarly with respect to homomesies which are linear combinations of ideal indicator functions. We end with some conjectures and avenues for future research.

Keywords: fence posets, rowmotion, homomesy, homometry, dynamical algebraic combinatorics

1 Introduction

We initiate the study of the dynamical algebraic combinatorics of fence posets. A *fence* is a poset with elements $F = \{x_1, x_2, \ldots, x_n\}$, partial order $\unlhd$, and covers

$$x_1 \lhd x_2 \lhd \cdots \lhd x_a \rhd x_{a+1} \rhd \cdots \rhd x_b \lhd x_{b+1} \lhd \cdots ,$$

where $a, b, \ldots$ are positive integers. The maximal chains of F are called *segments*. A fence with seven elements and three segments is shown in Figure 1. Throughout we will often use n for the cardinality of F which we denote by $\#F$. We also let $[n] = \{1, 2, \ldots, n\}$.

There are a number of different conventions for indicating the size of the segments of a fence F in the literature depending on the application being considered. Our results will be simplest if described in terms of unshared elements. Call $x \in F$ *shared* if it is the intersection of two segments; otherwise x is *unshared*. In Figure 1, the elements x_3 and x_6 are shared and all other elements unshared. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be a composition

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(a sequence of positive integers) with $\alpha_1, \alpha_s \geq 2$. The corresponding fence is $F = \check{F}(\alpha) = \check{F}(\alpha_1, \alpha_2, \ldots, \alpha_s)$, where $\alpha_i = 1 +$ (number of unshared elements on the ith segment) for $i \in [s]$. The fence in Figure 1 is $F = \check{F}(3,3,2)$. For any fence, $\#F = \alpha_1 + \alpha_2 + \cdots + \alpha_s - 1$.

Fences have been of recent interest because of their connections with cluster algebras, q-analogues, and unimodality. Let P be a poset with partial order $\trianglelefteq$. Recall that $I \subseteq P$ is a *(lower order) ideal* of P if $x \in I$ and $y \trianglelefteq x$ imply $y \in I$. *Upper order ideals*, U, are defined by reversing the inequality. When merely writing "ideal," we always mean a lower order ideal. Let $\mathcal{I}(P)$ and $\mathcal{U}(P)$ be the set of (lower order) ideals and upper order ideals of P, respectively. The set of ideals of a finite poset forms a distributive lattice under inclusion. If P is a fence then this lattice can be used to calculate the mutations in a cluster algebra derived from a surface with marked points on the boundary [2, 11, 13, 16, 17, 21].

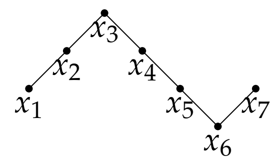


Figure 1: The fence $F = \check{F}(3,3,2)$

Let q be a variable and $r(F;q)$ be the rank generating function for $\mathcal{I}(F)$. Mourier-Genoud and Ovsienko [9] were able to define q-analogues of rational numbers which are certain rational functions of q. The numerators and denominators of these fractions are exactly the polynomials $r(F;q)$, which they conjectured were unimodal. Progress on this question can be found in [2, 4, 6, 8, 10]. Quite recently a proof was found by Ezgi Kantarcı Oğuz and Mohan Ravichandran [12].

Our focus is going to be on algebraic dynamics of fences and, in particular, on rowmotion. A subset A of a finite poset $(P, \trianglelefteq)$ is an *antichain* if no two elements of A are comparable. Let $\mathcal{A}(P)$ be the set of antichains of P. Ideals of both types and antichains are related by the maps $\Delta\colon \mathcal{I}(P) \to \mathcal{A}(P)$ where $\Delta(I) = \{x \in I \mid x \text{ is a maximal element of } I\}$, and $\nabla\colon \mathcal{U}(P) \to \mathcal{A}(P)$ where $\nabla(U) = \{x \in U \mid x \text{ is a minimal element of } U\}$. We also let $c\colon \mathcal{P} \to \mathcal{P}$ be the complement operator $c(S) = P - S$. *Rowmotion on antichains* is the group action on $\mathcal{A}(P)$ generated by the map $\rho = \nabla \circ c \circ \Delta^{-1}$ where we always compose functions from right to left. We will also consider *rowmotion on ideals*, which is generated by $\widehat{\rho} = \Delta^{-1} \circ \nabla \circ c$. There is clearly a bijection between the orbits of ρ and those of $\widehat{\rho}$. We will call the number of elements (that is, the number of antichains or, equivalently, the number of ideals) in an orbit either its *size* or its *length*. Rowmotion and its generalizations have been investigated by many authors, for example [1, 14, 19, 20]. See, in particular, the survey articles of Roby [15] and Striker [18] and the references therein. We will let $\mathcal{I}(\alpha) = \mathcal{I}(\check{F}(\alpha))$ and similarly for $\mathcal{U}$ and $\mathcal{A}$.

In addition to describing the orbits of rowmotion on fences, we will also consider properties of various statistics. If S is a finite set, then a *statistic* on S is a map $\mathrm{st}\colon S \to \mathbb{N}$ where $\mathbb{N}$ is the set of nonnegative integers. If G is a group acting on S, then statistic st is *d-mesic* if there is a constant d such that every orbit $\mathcal{O}$ of the action has average $\frac{\mathrm{st}\,\mathcal{O}}{\#\mathcal{O}} = d$, where $\mathrm{st}\,\mathcal{O} = \sum_{x \in \mathcal{O}} \mathrm{st}\, x$ and $\#\mathcal{O}$ is the size of the orbit. We say that st is *homomesic* if it is d-mesic for some d. Homomesy is a well-studied property of rowmotion; see [14] for examples. Rowmotion on fences displays a new and interesting phenomenon. We say that st is *homometric* (FKA *orbomesic*) if $\operatorname{st}\mathcal{O}$ is constant over all orbits of the same cardinality. Equivalently, for any two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ we have $\#\mathcal{O}_1 = \#\mathcal{O}_2 \implies \operatorname{st}\mathcal{O}_1 = \operatorname{st}\mathcal{O}_2$. Note that homomesy implies homometry, but not conversely. In the sequel we will see many homometries which are not homomesies.

One can break down rowmotion into smaller steps called toggles. Let S be a finite set and let $\mathcal{F}(S)$ be a family of subsets of S. If $x \in S$ then the corresponding *toggle map* is $\tau_x \colon \mathcal{F}(S) \to \mathcal{F}(S)$ defined on $T \in \mathcal{F}(S)$ by $\tau_x(T) = \begin{cases} T \oplus \{x\} & \text{if } T \oplus \{x\} \in \mathcal{F}(S), \\ T & \text{else,} \end{cases}$
where $\oplus$ is symmetric difference. Clearly τ_x is an involution. The corresponding *toggle group* is the group $\mathcal{T}$ generated by the toggles τ_x for $x \in S$. A *Coxeter element* in $\mathcal{T}$ is a product $\tau_{x_1}\tau_{x_2}\cdots\tau_{x_n}$ where $x_1, x_2, \ldots, x_n$ is some permutation of S. If P is a finite poset and $x \in P$, then we denote by τ_x and $\widehat{\tau}_x$ the corresponding toggles on $\mathcal{A}(P)$ and $\mathcal{I}(P)$, respectively. We will let $\mathcal{T}_P$ and $\widehat{\mathcal{T}}_P$ denote the toggle groups generated by the toggles τ_x and $\widehat{\tau}_x$, respectively. The relation of toggles, and in particular the corresponding Coxeter elements, to rowmotion is given in the following fundamental result of Cameron and Fon-Der-Flaass.

Theorem 1 ([1]). *Let P be a finite poset and let $x_1, x_2, \ldots, x_n$ be any linear extension of P. Then $\widehat{\rho} = \widehat{\tau}_{x_1}\widehat{\tau}_{x_2}\ldots\widehat{\tau}_{x_n}$.*

The rest of this paper is structured as follows. In the next section we introduce our principal tool in this work, which is a representation of rowmotion orbits of antichains of fences in terms of certain tilings. Section 3 is devoted to applying this model to prove various homomesy results for fences with any number of segments. We also define, for any self-dual poset, a new orbit structure which is coarser than that of rowmotion, and prove a homomesy on ideals in this setting. Section 4 is devoted to examining orbits and homometries for certain fences having at most five segments. In Section 5 we prove a general result for Coxeter elements and associated homomesies which applies to $\widehat{\mathcal{T}}_F$ for any fence F. We end with a section containing conjectures and future directions.

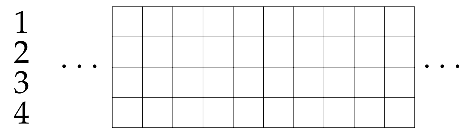
This extended abstract is a summary of result from our preprint [3], where full details (including the omitted proofs) and a few further results can be found.

2 Tilings

It turns out that the orbits of $\mathcal{A}(\alpha)$ can be nicely visualized in terms of tilings. This will be our principal tool in proving homo- and homometry results.

Consider a rectangle R_s subdivided into unit squares with rows numbered $1, 2, \ldots, s$ from top to bottom, and infinitely many columns. The rows will correspond to the segments of a corresponding fence and the columns to antichains in an orbit. For example,

R_4 is



In the following definition, an $a \times b$ *tile* is a tile which covers a rows and b columns in R_s.

Definition 2. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$, then an *$\alpha$-tiling* is a tiling of R_s using yellow 1×1 tiles, red 2×1 tiles, and black $1 \times (\alpha_i - 1)$ tiles in row i, for $1 \le i \le s$, such that the following hold for all rows.

(a) If $\alpha_i \ge 2$ and the red tiles are ignored, then the black and yellow tiles alternate in row i.

(b) There is a red tile in a column covering rows i and $i+1$ if and only if either the previous column contains two yellow tiles in those two rows when i is even, or the next column contains two yellow tiles in those two rows when i is odd.

When α is clear from the context, we will use the term *tiling* to refer to an α-tiling. We consider two tilings to be the same if one is a horizontal translate of the other. It will follow from the proof of Lemma 3 that all α-tilings are periodic and so can be viewed as lying on a cylinder. When a tiling is displayed in a figure, we will draw a bounded rectangle and assume that the two vertical edges are identified, and indicate with a jagged edge where any black tile crosses this boundary. See Figure 2 for the four possible $(4,3,4)$-tilings. Note that by considering tilings to be on a cylinder, notions such as the number of tiles of each color make sense. For example, row 1 of the first tiling in Figure 2 has four yellow tiles, four black tiles, and a red tile which also intersects row 2.

In a tiling we will call a square of R_s yellow, red, or black depending on whether the tile covering the square has the corresponding color. The *head* of a red tile is the square it covers in the higher of the two rows.

Given $\alpha = (\alpha_1, \ldots, \alpha_s)$ we now construct a bijection $\phi \colon \{\mathcal{O} \mid \mathcal{O} \text{ an orbit of } \mathcal{A}(\alpha)\} \to \{T \mid T \text{ an } \alpha\text{-tiling}\}$ as follows. Let the ith segment of $F(\alpha)$ be S_i. Given $\mathcal{O}$, we build $T = \phi(\mathcal{O})$ column-by-column. Pick any $A \in \mathcal{O}$ and any column C of R_s to correspond to A. Color the square in row i of C yellow, red or black depending upon whether $S_i \cap A$ is empty, a shared element, or an unshared element, respectively. For example, the antichain $A = \{x_4, x_9\}$ in $\check{F}(4,3,4)$ corresponds to the first column of the first tiling in Figure 2. Now color the column to the right of C in R_s in the same way using the antichain following A in $\mathcal{O}$, and similarly for the column to the left and the antichain preceding A. Continue this process until all of R_s is colored. Clearly this is a periodic tiling and so can be wrapped onto a cylinder. The tilings for the four antichain orbits in $\check{F}(4,3,4)$ are displayed in Figure 2. For example, the tiling with five columns corresponds to the orbit $\{\emptyset,\ \{x_1, x_7\},\ \{x_2, x_6, x_8\},\ \{x_3, x_5, x_9\},\ \{x_4, x_{10}\}\}$.

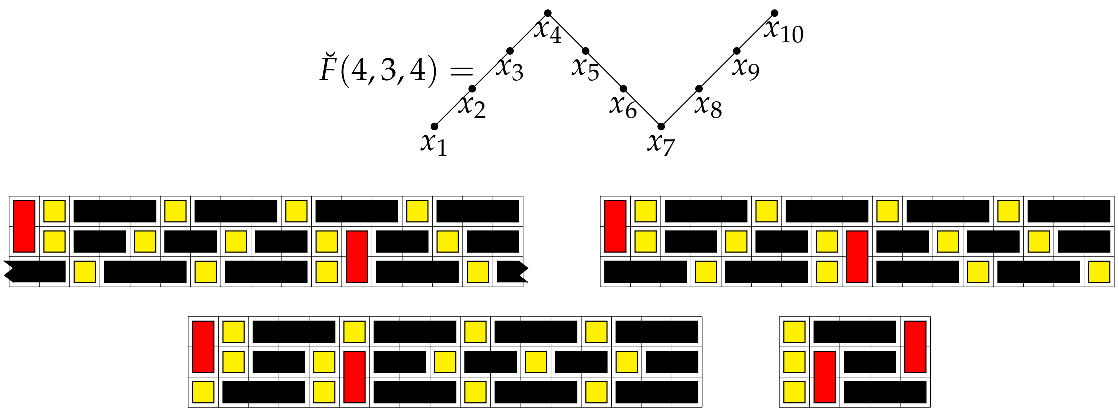


Figure 2: Antichain orbits in $\check{F}(4,3,4)$.

Lemma 3. *For every α, the map ϕ is a bijection.*

Using the tiling model it is easy to read off various statistics about orbits which will be useful in proving homomesy and homometry results. For $x \in F$, consider the *indicator function on antichains* $\chi_x \colon \mathcal{A}(F) \to \{0,1\}$ defined by $\chi_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{else.} \end{cases}$ For an orbit $\mathcal{O}$ of antichains, $\chi_x(\mathcal{O}) = \sum_{A \in \mathcal{O}} \chi_x(A)$ is the number of times x occurs in an antichain of the orbit. We also define $\chi(A) = \#A = \sum_{x \in F} \chi_x(A)$, so that $\chi(\mathcal{O})$ is the total number of antichain elements in an orbit. As usual, we add a hat for the corresponding functions on ideals I. For example, $\widehat{\chi}_x \colon \mathcal{I}(F) \to \{0,1\}$ is defined by $\widehat{\chi}_x(I) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{oterhwise.} \end{cases}$

To state our first result in this regard we will need some additional notation for fences and tilings. In F, we let $\check{S}_i$ be the set of unshared elements of segment S_i; $s_{i,j}$ be the jth smallest element of $\check{S}_i$; and s_i be the unique element of $S_i \cap S_{i+1}$. Note that we are also using s for the number of segments. But the presence or absence of a subscript will distinguish between the two uses of this notation. Note that s_i is a minimal or maximal element of F depending on whether i is even or odd, respectively. For the fence $\check{F}(4,3,4)$ in Figure 2 we have: $x_1 = s_{1,1}$, $x_2 = s_{1,2}$, $x_3 = s_{1,3}$, $x_4 = s_1$, $x_5 = s_{2,2}$, $x_6 = s_{2,1}$, $x_7 = s_2$, $x_8 = s_{3,1}$, $x_9 = s_{3,2}$, $x_{10} = s_{3,3}$.

In an α-tiling we let b_i =the number of black tiles in row i, and r_i = the number of red tiles whose head is in row i. So, in the first tiling of Figure 2 we get:

i	1	2	3
b_i	4	5	4
r_i	1	1	0

.

We let $b_i = r_i = 0$ if $i \notin [s]$, where s is the number of rows of the tiling. Finally, we need the *Kronecker function* which, given a statement R, evaluates to

$$\delta(R) = \begin{cases} 1 & \text{if } R \text{ is true,} \\ 0 & \text{if } R \text{ is false.} \end{cases}$$

The following result will show how we can use the α_i, b_i and r_i of the tilings to compute the various quantities in which we are interested.

Lemma 4. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ with corresponding fence $F = \check{F}(\alpha)$. Let $\mathcal{O}$ be an orbit of antichains A with tiling $T = \phi(\mathcal{O})$. We denote by I the ideal generated by A.*

(a) *For any $i \in [s]$ with $\alpha_i \geq 2$ we have $\#\mathcal{O} = b_i\alpha_i + r_i + r_{i-1}$.*

(b) *Considering $\mathcal{O}$ as an orbit of antichains, $\chi_x(\mathcal{O}) = \begin{cases} b_i & \text{if } x \in \check{S}_i, \\ r_i & \text{if } x = s_i, \end{cases}$*
and $\chi(\mathcal{O}) = \sum_{i=1}^{s}(b_i\alpha_i - b_i + r_i)$.

(c) *Considering $\mathcal{O}$ as an orbit of ideals, $\widehat{\chi}_x(\mathcal{O}) = \begin{cases} b_i(\alpha_i - j) + r_{i-\delta(i \text{ even})} & \text{if } x = s_{i,j}, \\ r_{2i-1} & \text{if } x = s_{2i-1}, \\ \#\mathcal{O} - r_{2i} & \text{if } x = s_{2i}, \end{cases}$*
and $\widehat{\chi}(\mathcal{O}) = \left\lfloor \frac{s-1}{2} \right\rfloor \cdot \#\mathcal{O} + \sum_{i=1}^{s}\left[b_i\binom{\alpha_i}{2} + r_{2i-1}(\alpha_{2i-1} + \alpha_{2i} - 1) - r_{2i}\right]$.

Proof. To illustrate how these results are derived, we will prove (a). Since every black tile in row i has $\alpha_i - 1$ squares, the number of black squares in that row is $b_i(\alpha_i - 1)$. From Definition 2 (a), the number of yellow tiles in row i is also b_i as long as $\alpha_i \geq 2$. Finally, there are red squares in row i from both red tiles whose head is in row $i-1$ and those whose head is in row i. Since the size of the orbit is the number of squares in row i, we have $\#\mathcal{O} = b_i(\alpha_i - 1) + b_i + r_{i-1} + r_i$, which simplifies to the given quantity. □

3 General fences

We can use Lemma 4 to demonstrate various homomesy results that hold for fences in general. For example, part (a) of the following result follows immediately from the first equation in Lemma 4 (b).

Theorem 5. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ with corresponding fence $F = \check{F}(\alpha)$ and let $\mathcal{O}$ be any rowmotion orbit of F.*

(a) *If $x, y \in \check{S}_i$ for some i, then $\chi_x - \chi_y$ is 0-mesic.*

(b) *If $x \in \check{S}_i$, $y = s_i$ and $z = s_{i-1}$, then $\alpha_i\chi_x + \chi_y + \chi_z$ is 1-mesic.*

(c) *If $x = s_{1,j}$ and $y = s_{1,k}$, then $k\widehat{\chi}_x - j\widehat{\chi}_y$ is $(k-j)$-mesic.*

(d) *Let $x = s_{2i-1}$ and $y = s_{2j}$. If $r_{2i-1} = r_{2j}$ for all orbits $\mathcal{O}$, then $\widehat{\chi}_x + \widehat{\chi}_y$ is 1-mesic.*

(e) *If s is odd and all the α_i are even, then $\#\mathcal{O}$ is even for all orbits $\mathcal{O}$.*

(f) *If $\alpha_i = 2$ for all $i \in [s]$, then χ is $s/2$-mesic.*

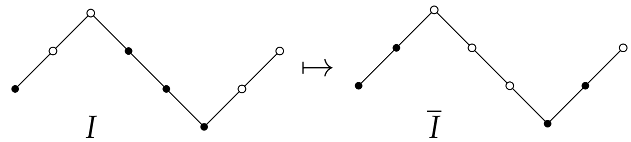


Figure 3: The ideal complement map

To state our general result on the $\widehat{\chi}$ statistic in orbits of self-dual posets, we need some definitions. Suppose our poset P is self-dual. In particular, this will be true if $P = F(\alpha)$ where α is a palindrome with an odd number of parts. So there is an order-reversing bijection $\beta\colon P \to P$. Note that $I \in \mathcal{I}(P)$ if and only if $\beta(I) \in \mathcal{U}(P)$. This permits us to define the *ideal complement* (with respect to β) of $I \in \mathcal{I}(P)$ as $\overline{I} = \beta \circ c(I)$. See Figure 3 for an example in $\check{F}(3,3,3)$ where circles are black or white depending on whether they are in the ideal or not, respectively. The relationship with rowmotion is as follows.

Lemma 6. *Let P be self-dual and fix an order-reversing bijection $\beta\colon P \to P$. Then for all $I \in \mathcal{I}(P)$ we have $\widehat{\rho}^{-1}(\overline{I}) = \overline{\widehat{\rho}(I)}$, where the ideal complements are with respect to β.*

Corollary 7. *Let P be self-dual with $n = \#P$, and fix an order-reversing bijection $\beta\colon P \to P$. Let $I \in \mathcal{I}(P)$.*

(a) If $I, \overline{I} \in \mathcal{O}$ for some orbit $\mathcal{O}$, then $\frac{\widehat{\chi}(\mathcal{O})}{\#\mathcal{O}} = \frac{n}{2}$.

(b) If $I \in \mathcal{O}$ and $\overline{I} \in \overline{\mathcal{O}}$ for some orbits $\mathcal{O}$ and $\overline{\mathcal{O}}$ with $\mathcal{O} \neq \overline{\mathcal{O}}$, then $\#\mathcal{O} = \#\overline{\mathcal{O}}$ and $\frac{\widehat{\chi}(\mathcal{O})+\widehat{\chi}(\overline{\mathcal{O}})}{\#\mathcal{O}+\#\overline{\mathcal{O}}} = \frac{n}{2}$.

Proof. We will sketch the proof of (a). It follows quickly from the definitions that for any ideal I we have $\#I + \#\overline{I} = n$. If $\mathcal{O}$ is an orbit that contains both an ideal and its ideal complement, then one can use the previous lemma to show that the orbit can be partitioned into subsets $\{I, \overline{I}\}$ where $I \neq \overline{I}$ and $\{J\}$ where $J = \overline{J}$. Since the average of $\widehat{\chi}$ is $n/2$ over each subset, this is also the average over $\mathcal{O}$. $\square$

To turn this last result into a homomesy, take a self-dual P with a given order-reversing bijection $\beta\colon P \to P$. Consider the group generated by the action of $\widehat{\rho}$ and the ideal complement map $I \mapsto \overline{I}$. The orbits of this action will be called *superorbits*. It can be shown from the proof of Corollary 7 that each superorbit is either an orbit of the action of $\widehat{\rho}$ or a union of two such orbits. From this observation and the statement of Corollary 7, the next result follows immediately.

Theorem 8. *Let P is self-dual with $n = \#P$ and fix an order-reversing bijection $\beta\colon P \to P$. Then $\widehat{\chi}$ is $(n/2)$-mesic on superorbits.* $\square$

4 Fences with few segments

In this section we will consider fences with at most 5 segments. For certain compositions α, we will completely describe the orbit sizes and the number of orbits. We will also calculate the values of χ and $\widehat{\chi}$, revealing a number of homometries.

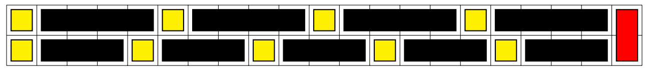


Figure 4: The orbit $\mathcal{O}$ of length 21 in $\check{F}(5,4)$, with $\chi(\mathcal{O}) = 32$ antichain elements.

We will need an expression for the number of ideals (equivalently, antichains) in $F = \check{F}(\alpha)$, which the following lemma provides.

Lemma 9. *Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ where $s \geq 3$. Then $\#\mathcal{I}(\alpha) = \alpha_s \cdot \#\mathcal{I}(\alpha_1, \ldots, \alpha_{s-1}) + \#\mathcal{I}(\alpha_1, \ldots, \alpha_{s-2})$.*

Corollary 10. *We have the following ideal counts.*

(a) If $\alpha = (a, b)$, then $\#\mathcal{I}(\alpha) = ab + 1$.

(b) If $\alpha = (a, b, c)$, then $\#\mathcal{I}(\alpha) = abc + a + c$.

We start with the case of two segments. The following result can be proved using the tiling counts in Lemma 4. Note that parts (c) and (d) are homometries which are not homomesies.

Theorem 11. *Rowmotion on $\check{F}(a, b)$ has the following properties.*

(a) All orbits $\mathcal{O}$ have size $\ell = \operatorname{lcm}(a, b)$ except for one $\mathcal{O}'$ which has size $\ell + 1$.

(b) The number of orbits is $\gcd(a, b)$.

(c) For a size ℓ orbit, $\chi(\mathcal{O}) = \frac{2ab-a-b}{\gcd(a,b)} := m$. For the size $\ell + 1$ orbit, $\chi(\mathcal{O}') = m + 1$.

(d) For a size ℓ orbit, $\widehat{\chi}(\mathcal{O}) = \frac{\ell(a+b-2)}{2}$. For the size $\ell + 1$ orbit, $\widehat{\chi}(\mathcal{O}') = \frac{(\ell+2)(a+b-2)}{2} + 1$.

In the next result we consider certain fences with three segments. We do not explicitly state the values of the χ statistic for the homometries, but these can be computed from the numbers of tiles of each color, which are given in the proof in [3].

Theorem 12. *Consider $\check{F}(a, b, a)$ and define $g = \gcd(a, b)$, $\bar{a} = a/g$, $\bar{b} = b/g$, $\ell = \operatorname{lcm}(a, b)$. Since $\bar{a}, \bar{b}$ are relatively prime, there exists a smallest positive integer m such that $m\bar{a} = q\bar{b} + 1$ for some positive integer q. Then the orbits of rowmotion on $\check{F}(a, b, a)$ can be partitioned by length into three sets $\mathcal{S}, \mathcal{M}, \mathcal{L}$, which we call* small, medium, *and* large, *having the following properties:*

(a) We have $\#\mathcal{O} = \begin{cases} \ell & \text{if } \mathcal{O} \in \mathcal{S}, \\ a(2b - 2\bar{b} + m) + g & \text{if } \mathcal{O} \in \mathcal{M}, \\ a(2b - \bar{b} + m) + g & \text{if } \mathcal{O} \in \mathcal{L}, \end{cases}$

with $\#\mathcal{S} = \bar{a}(g-1)^2$, $\#\mathcal{M} = \frac{\bar{a}m-1}{\bar{b}}$, $\#\mathcal{L} = \frac{\bar{a}(\bar{b}-m)+1}{\bar{b}}$.

(b) *For rowmotion on antichains, χ is homometric.*

(c) *For rowmotion on ideals, $\widehat{\chi}$ is $n/2$-mesic where $n = \#\check{F}(a,b,a) = 2a+b-1$.*

In [3] we give similar results for 4-segment fences $\check{F}(a,a,a,a)$ and for 5-segment fences $\check{F}(a,1,a,1,a)$.

5 Toggling

Recall that the rowmotion map can be written as a Coxeter element in the ideal toggle group $\mathcal{T}$ (Theorem 1). A natural question is whether other toggling orders might change the homomesies or homometries of the map. In this section we show that any map defined as a Coxeter element in the toggle group of certain finite posets P (in particular, any fence) has the same homomesies and homometries as any other. Our theorem generalizes [7, Theorem 20], which proves a similar result in a different context, but one that is essentially the same as order-ideal toggling in zigzag posets which are fences of the form $\check{F}(2,1,1,\ldots,1,2)$. The crucial concept in determining which toggle groups have this property is a graph associated with the group.

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set and let $\mathcal{T}$ be the toggle group associated with some family of subsets of S. The *base graph* of $\mathcal{T}$, denoted $G_{\mathcal{T}}$, has as vertices the toggles τ_x for $x \in S$, and it has an edge $\tau_x\tau_y$ if τ_x and τ_y do not commute. If P is a poset, we let $G_P = G_{\mathcal{T}_P}$ and $\widehat{G}_P = G_{\widehat{\mathcal{T}}_P}$ be the graphs for the toggle groups for $\mathcal{A}(P)$ and $\mathcal{I}(P)$, respectively. We will reserve τ_i for the toggle with respect to x_i in $\mathcal{A}(P)$ and use $\widehat{\tau}_i$ for the corresponding toggle in $\mathcal{I}(P)$, and similarly for other notation involving fences.

Figure 5 displays the fence $F = \check{F}(2,2,2)$ as well as G_F and $\widehat{G}_F$. We will need another result of Cameron and Fon-Der-Flaass.

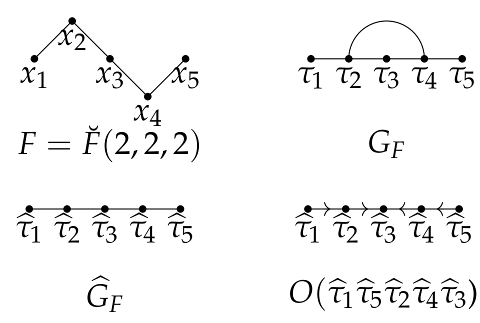


Figure 5: The fence $F = \check{F}(2,2,2)$, its graphs G_F and $\widehat{G}_F$, and an orientation.

Theorem 13 ([1])**.** *If P is a finite poset then $\widehat{G}_P$ is the Hasse diagram of P considered as an undirected graph, i.e., $\widehat{\tau}_x$ and $\widehat{\tau}_y$ commute if and only if x and y are not related by a cover. In particular, if the Hasse diagram of P (considered as a graph) is acyclic, then so is $\widehat{G}_P$.* □

The acyclicity of $G_{\mathcal{T}}$ has consequences for the corresponding Coxeter elements. In particular, in this case any two Coxeter elements can be connected by a sequence of conjugations of a certain type. Let $\mathcal{T}$ be a toggle group and w a Coxeter element of $\mathcal{T}$. We call τ_x *admissible* with respect to w if all the elements to one side (either left or right) of τ_x in w commute with τ_x. If τ_x is admissible with respect to w, then the conjugate $\tau_x w \tau_x$

is again a Coxeter element. Indeed, suppose all elements to the left of τ_x in w commute with τ_x. Then τ_x can be moved via commutation relations to the left end of w and so cancel in the conjugation. Continuing the example in Figure 5, if we let $w = \widehat{\tau}_1\widehat{\tau}_5\widehat{\tau}_2\widehat{\tau}_4\widehat{\tau}_3$, then $\widehat{\tau}_5$ is admissible, since the only element to its left is $\widehat{\tau}_1$, which commutes with $\widehat{\tau}_5$. So $\widehat{\tau}_5 w\widehat{\tau}_5 = \widehat{\tau}_5(\widehat{\tau}_1\widehat{\tau}_5\widehat{\tau}_2\widehat{\tau}_4\widehat{\tau}_3)\widehat{\tau}_5 = \widehat{\tau}_5(\widehat{\tau}_5\widehat{\tau}_1\widehat{\tau}_2\widehat{\tau}_4\widehat{\tau}_3)\widehat{\tau}_5 = \widehat{\tau}_1\widehat{\tau}_2\widehat{\tau}_4\widehat{\tau}_3\widehat{\tau}_5$, which is still a Coxeter element.

One can relate admissible toggles to the base graph as follows. Given a Coxeter element w, we will form an orientation $O(w)$ of $G_{\mathcal{T}}$. Given an edge in $G_{\mathcal{T}}$ between two toggles, we orient it from the toggle which is further left in w to the one which is further right. The orientation for $\widehat{\tau}_1\widehat{\tau}_5\widehat{\tau}_2\widehat{\tau}_4\widehat{\tau}_3$ is shown in Figure 5. It is now easy to see that τ_x is admissible with respect to w if and only if τ_x is either a source or a sink in $O(w)$. Indeed, the first case corresponds to all elements to the left of τ_x commuting with the toggle and the second to this property on the right. When the base graph is acyclic, then conjugation by admissible toggles is all that is needed to get between any two Coxeter elements, as proved by Eriksson and Eriksson.

Theorem 14 ([5]). *Let $\mathcal{T}$ be a toggle group with $G_{\mathcal{T}}$ acyclic, and let w, w' be any two Coxeter elements of $\mathcal{T}$. Then w' can be obtained from w by a sequence of conjugations where at each step the conjugating toggle is admissible with respect to the current Coxeter element.* □

From all this follows the main result of this section.

Theorem 15. *Let S be a finite set and $\mathcal{T}$ be a toggle group on S with $G_{\mathcal{T}}$ acyclic. Let w, w' be any two Coxeter elements of $\mathcal{T}$, and let W, W' be the respective groups they generate. Then any linear combination of indicator functions χ_y for $y \in S$ is d-mesic or homometric under the action of W if and only if it is d-mesic or homometric, respectively, under the action of W'.*

Combining the previous theorem with Theorem 13, we get the following result

Corollary 16. *Let F be a fence. Let w, w' be any two Coxeter elements of $\widehat{\mathcal{T}}_F$ and let W, W' be the respective groups they generate. Then any linear combination of indicator functions $\widehat{\chi}_y$ for $y \in S$ is d-mesic or homometric under the action of W if and only if it is d-mesic or homometric, respectively, under the action of W'.* □

6 Conjectures and an open problem

We now state some conjectures and an open problem; more details appear in [3]. Call a sequence of real numbers $a_1, a_2, \ldots, a_n$ *palindromic* if $a_i = a_{n+1-i}$ for all $1 \leq i \leq n$. Any α-tiling with s rows has an associated *black tile sequence* $b_1, b_2, \ldots, b_s$ and an associated *red tile sequence* $r_1, r_2, \ldots, r_{s-1}$.

Question 17. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be palindromic and $F = \check{F}(\alpha)$. Find necessary and/or sufficient conditions on α for the black sequence or the red tile sequence to be palindromic for all rowmotion orbits.*

For palindromic α, we have the following relationship between the sequences.

Proposition 18. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be palindromic where $\alpha_i \geq 2$ for all $i \in [s]$, and let $F = \check{F}(\alpha)$. Then for any orbit $\mathcal{O}$ of F, the black tile sequence $b_1, b_2, \ldots, b_s$ is palindromic if and only if the red tile sequence $r_1, r_2, \ldots, r_{s-1}$ is palindromic.*

One reason to care about the palindromic case is the following result.

Proposition 19. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ where $\alpha_i \geq 2$ for all $i \in [s]$. Also let $F = \check{F}(\alpha)$ and $n = \#F$. If α as well as the black and red tile sequences are all palindromic, then one has the following homomesies.*

(a) For all $k \in [n]$ the statistic $\chi_k - \chi_{n-k+1}$ is 0-mesic.

(b) If s is odd, then for all $k \in [n]$ the statistic $\widehat{\chi}_k + \widehat{\chi}_{n-k+1}$ is 1-mesic.

We conjecture that even more is true for constant α.

Conjecture 20. *Let $\alpha = (a^s)$ and let $F = \check{F}(\alpha)$.*

(a) The statistic χ is homometric.

(b) If s is odd then the statistic $\widehat{\chi}$ is $n/2$-mesic where $n = \#F$.

We end with a conjecture which is the antichain analogue of Corollary 16.

Conjecture 21. *Let F be a fence. Let w, w' be any two Coxeter elements of the group of antichain toggles $\mathcal{T}$, and let W, W' be the respective groups they generate. Then any linear combination of indicator functions χ_y for $y \in S$ is d-mesic or homometric under the action of W if and only if it is d-mesic or homometric, respectively, under the action of W'.*

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