Séminaire Lotharingien de Combinatoire **86B** (2022)
Article #41, 12 pp.

Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)

Ideal Lattices of Fence Posets and Rank Unimodality

Ezgi Kantarcı Oğuz*[1] and Mohan Ravichandran†[1]

[1]*Deptartment of Mathematics, Boğaziçi University, İstanbul*

Abstract. We prove a conjecture of Morier-Genoud and Ovsienko that says that rank polynomials of the distributive lattices of lower ideals of fence posets are unimodal. We do this by introducing a related class of *circular* fence posets and proving a stronger version of the conjecture due to McConville, Sagan and Smyth. We show that the rank polynomials of circular fence posets are symmetric and conjecture that unimodality holds except in some particular cases. We also apply the recent work of Elizalde, Plante, Roby and Sagan on rowmotion on fences and show many of their homomesy results hold for the circular case as well.

Keywords: fence poset, circular fence poset, unimodality, rank symmetry, rowmotion

1 Introduction

Fence posets are a natural class of posets that appear in the study of cluster algebras, quiver representations and other areas of enumerative combinatorics, see [5] for an overview. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be a composition of n. The fence poset of α, denoted $F(\alpha)$ is the poset on $x_1, x_2, \ldots, x_{n+1}$ with the order relations:

$$x_1 \lhd x_2 \lhd \cdots \lhd x_{\alpha_1+1} \rhd x_{\alpha_1+2} \rhd \cdots \rhd x_{\alpha_1+\alpha_2+1} \lhd x_{\alpha_1+\alpha_2+2} \lhd \cdots \lhd x_{\alpha_1+\alpha_2+\alpha_3+1} \rhd \cdots$$

We call maximal chains of this poset *segments*. The poset $F(\alpha)$ has $n+1$ nodes and s segments, where $n = \alpha_1 + \ldots + \alpha_s$ is the *size* of α. See Figure 1 for the fence poset corresponding to $(2,1,1,3)$. Lower order ideals of $F(\alpha)$ ordered by inclusion give a distributive lattice which we denote by $J(\alpha)$. The lattice $J(\alpha)$ is ranked by the size of the ideals, with generating polynomial $R(\alpha; q) = \sum_{I \in J(\alpha)} q^I$, called the *rank polynomial*. We will use $r(\alpha)$ to denote the corresponding *rank sequence* given by the coefficients of the powers of q in $R(\alpha; q)$.

Example 1.1. The fence poset for $\alpha = (2,1,1,3)$ is illustrated in Figure 1, left. Note that the ideals of maximal and minimal rank are unique. Ideals of rank 1 and rank 7 are given by minima and complements of maxima respectively and the five ideals of rank 2 are depicted. The full rank sequence is $(1,3,5,6,6,5,3,2,1)$.

*ezgikantarcioguz@gmail.com
†mohan.ravichandran@boun.edu.tr

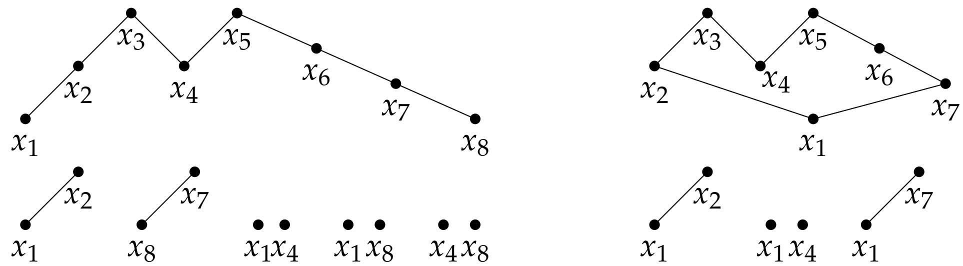


Figure 1: The fence poset $F(2,1,1,3)$ (top left) has five ideals of rank 2 (bottom left). The circular analogue $\overline{F}(2,1,1,3)$ (top right) has only 3 (bottom right).

The rank sequences of fence posets were used by Morier-Genoud and Ovsienko in [6] in their recent work defining q-analogues of the rational numbers. They use the continued fraction expansion of a rational number to first construct two fence posets. Their q-rationals are then defined as the ratio of the rank polynomials of these posets. These q-rationals enjoy several interesting properties including a type of convergence which allows them to extend the definition to obtain q-real numbers. They also proposed the following conjecture in their paper, the proof of which is the main result in this paper.

Theorem 1.2 ([6, Conjecture from Section 7])**.** *The rank polynomials of fence posets are unimodal.*

Here and elsewhere, when we say that a polynomial is unimodal or symmetric, we mean that its sequence of coefficients is respectively unimodal or symmetric. Recall that a sequence is called *unimodal* if there exists an index m such that $a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_n$.

While there was no a priori reason for the authors to expect that this conjecture holds, there was ample numerical evidence. Results predating the conjecture itself were given by Salvi and Munarini [8], who considered the case when all parts are equal to 1. Claussen [1] showed that the conjecture holds when the composition has at most 4 parts. Further partial progress was made by McConville, Sagan and Smyth [5], who proved the conjecture in the case where the first segment is larger than the sum of the others and proposed the following strengthening of this conjecture. The various interlacing properties referred to in the next theorem are defined Section 3, where we also give a sketch of its proof.

Theorem 1.3 ([5, Conjecture 1.4])**.** *Suppose* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$.

(a) If $s = 1$ *then* $r(\alpha) = (1, 1, \ldots, 1)$ *is symmetric.*

(b) If s *is even, then* $r(\alpha)$ *is bottom interlacing.*

(c) If $s \geq 3$ *is odd we have:*

(i) If $\alpha_1 > \alpha_s$ then $r(\alpha)$ is bottom interlacing.

(ii) If $\alpha_1 < \alpha_s$ then $r(\alpha)$ is top interlacing.

(iii) If $\alpha_1 = \alpha_s$ then $r(\alpha)$ is symmetric, bottom interlacing, or top interlacing depending on whether $r(\alpha_2, \alpha_3, \ldots, \alpha_{s-1})$ is symmetric, top interlacing, or bottom interlacing, respectively.

We will describe the main ideas in our proof later but it is noteworthy that our proof is purely combinatorial. Unimodality of combinatorial sequences is often deduced by first proving stronger properties of the sequence such as log concavity, ultra log concavity or even real rootedness, but for this problem, none of these stronger properties need hold. Indeed to see that even log concavity need not hold, we see that for the fence poset $F(\alpha) = F(2,1,1,3)$ described in Example 1.1, we have

$$9 = r(\alpha)[6]^2 < r(\alpha)[5]\, r(\alpha)[7] = 5 \cdot 2 = 10,$$

where $r(\alpha)[k]$ denotes the number of k element ideals of the fence poset $F(\alpha)$.

They key idea in our proof is to navigate between the properties of fence posets and those of a closely related class of objects we introduce called the *circular fence posets*. For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2s})$ of n we define the *circular fence poset* of α, denoted $\overline{F}(\alpha)$ as the fence poset of α where x_{n+1} and x_1 are taken to be equal. Note that this is a poset with n nodes.

Example 1.4. The circular fence poset $\overline{F}(2,1,1,3)$ given in Figure 1 right is obtained from the regular fence poset $F(2,1,1,3)$ by identifying the vertices x_1 and x_8, yielding a poset on 7 elements. Given that we have identified x_1 and x_8 two pairs of the five ideals of size two become identical and the ideal (x_1, x_8) does not appear. Thus, the number of rank 2 ideals in $\overline{F}(2,1,1,3)$ is 3. The full rank sequence for $\overline{F}(2,1,1,3)$ is $(1,2,3,4,4,3,2,1)$.

We will use $\overline{J}(\alpha)$ to refer to the lattice of lower ideals of $\overline{F}(\alpha)$ and denote the corresponding rank polynomial and rank sequence by $\overline{R}(\alpha;q)$ and $\overline{r}(\alpha)$ respectively. Rank polynomials for circular fence posets behave slightly differently from those for regular fence posets; there are examples where they fail to be unimodal, see Section 4 for a discussion and a characterization. However, they do satisfy a highly convenient property.

Theorem 1.5. *Rank polynomials of circular fence posets are* symmetric.

Given a fence poset, there are several naturally related circular fence posets. Our proof consists of relating the rank polynomials of these various posets and inductively proving a number of ancillary results. One of the byproducts of our proof is the following result that might be of independent interest.

Theorem 1.6. *Let $\alpha = (\alpha_1, \ldots, \alpha_{2s})$ be a composition with an even number of parts and consider any cyclic shift of α, $\beta = (\alpha_k, \alpha_{k+1}, \ldots, \alpha_{2s}, \alpha_1, \alpha_2, \ldots, \alpha_{k-1})$. Then*

$$\overline{R}(\alpha;q) = \overline{R}(\beta;q).$$

That is, the rank polynomial of a circular fence poset is well-defined over circular *compositions.*

As mentioned above, when it comes to circular fence posets, unimodality may fail.

Example 1.7. Let $\alpha = (1, a, 1, a)$ or $(a, 1, a, 1)$ be a composition. A direct calculation shows that the rank sequence is $\bar{r}(\alpha) = (1, 2, \ldots, a, a+1, a, a+1, a, a-1, \ldots, 1)$. This sequence has a dip in the middle term and is not unimodal.

We conjecture that these are the only cases when unimodality fails for circular fences and provide support for this in Section 3:

Conjecture 1.8. *For any $\alpha \neq (1, k, 1, k)$ or $(k, 1, k, 1)$ for some k, the rank polynomial $\overline{R}(\alpha; q)$ is unimodal.*

Section 4 is devoted to the behaviour of the rowmotion operation on cyclic fences, and related homomesy results. We also note that cases of orbomesy observed in the regular case break down for cyclic fences. Section 5 discusses possible future directions. The interested reader is referred to [10] for more details.

2 Circular Fences

Let P be a finite poset. A subset I of P is said to be a lower order ideal (resp. upper order ideal) if when $x \in I$, any $y \leq x$ (resp. any $y \geq x$) lies in I as well. We will use the word "ideal" to denote a lower order ideal, unless stated otherwise. Ideals (or upper order ideals) of a poset P ordered by inclusion give the structure of a distributive lattice $J(P)$, ranked by the number of elements. See [11, Chapter 3.4] for a detailed discussion. For the purposes of this work, we will use the word "rank" exclusively to refer to the rank structure of the order ideal lattice. Note that taking the setwise complement of an ideal gives an upper order ideal of complementary rank.

We will be interested in the case where P is a fence, or a circular fence. Recall that for a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2s})$ of n, the *circular fence poset* of α, denoted $\overline{F}(\alpha)$ is the fence poset of α where x_{n+1} and x_1 are taken to be equal, so we get a circular poset with n nodes. Circular fences have substantial intrinsic symmetry; shifting the parts cyclically by two steps or reversing the order of the parts both preserve the rank sequence. In the special case when all the segments are of size 1, the object we obtain is called a *crown*. Crowns were previously studied in [8] where it was shown that their rank polynomials are symmetric, and that they are unimodal when the number of segments is different than 4. Examining the one step shift allows us to directly say that the symmetry holds when one of the segments is larger as well. This will serve as the basis to prove that in fact, for any circular fence poset we get a rank symmetric lattice.

Lemma 2.1. *Shifting the parts of α cyclically by one step reverses the rank sequence $\bar{r}(\alpha)$. In particular $\overline{R}((k, 1, \ldots, 1); q)$ where the number of segments is even is symmetric for any $k \in \mathbb{N}$.*

In general, rank polynomials for circular fences are no easier to calculate than their non circular counterparts and we only have formulas for a limited number of cases. The case when $\alpha = (1, a, 1, a, \ldots, 1, a)$ was considered in [7]. They were able to formulate the rank polynomial in terms of Chebyshev polynomials of the first kind. Some other small cases that can be easily calculated by hand are listed in Table 1 below.

α	Ideal Count	Rank Polynomial
(a, b)	$ab + 2$	$1 + q[a]_q[b]_q + q^{a+b}$
$(a, 1, b, 1)$	$ab + a + b + 1$	$[a+1]_q[b+1]_q - q^{a+1} - q^{b+1}$
(a, b, c, d)	$abcd + ab + cd + ad + bc + 2$	$1 + q[a]_q[d]_q + q[b]_q[c]_q + q^{a+b+1}[c]_q[d]_q$ $+q^{c+d+1}[a]_q[b]_q + q^{a+b+c+d}$
(a, a, a, a)	$a^4 + 4a^2 + 2$	$1 + ([a]_q)^4 + (2q^{2a+1} + 2q)([a]_q)^2 + q^{4a}$

Table 1: Ideal count and rank polynomial for small cases

The cases of (a, b) and $(1, a, 1, b)$ are straightforward. The lattice formed by the ideals of $\overline{F}(a, b)$ is formed by the direct product of two chains of lengths $a-1$ and $b-1$, with an added minimum element (the empty ideal) and maximum element (the full ideal): $\widehat{0} \oplus (C_{a-1} \times C_{b-1}) \oplus \widehat{1}$. Here, the position on C_{a-1} describes unshared elements in the left segment, whereas the position on C_{b-1} describes the number of unshared elements in the right segment. The natural symmetric chain decomposition on $C_{a-1} \times C_{b-1}$ can easily be extended to accommodate the two added nodes, as seen in Figure 2 for the example of $(5, 8)$. The case of $(1, a, 1, b)$ is similar, and the lattice we obtain can be visualised as $C_a \times C_b$ with two opposite corners deleted. When $a \neq b$ this also has a natural symmetric chain decomposition. When $a = b$ however, we have no such decomposition and indeed the resulting rank polynomial is not unimodal, see Figure 2.

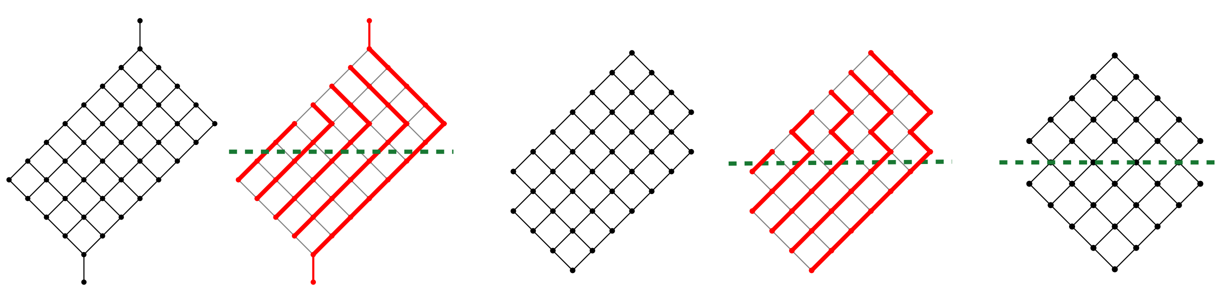


Figure 2: The lattices $\overline{J}((5, 7))$ (left) and $\overline{J}((1, 3, 1, 6))$ (middle) have (natural) symmetric chain decompositions whereas $\overline{J}((1, 4, 1, 4))$ (right) does not.

Theorem 1.5 shows that the rank polynomials of circular fences are always symmetric. Our proof starts with the observation that symmetry holds in the case $(k, 1, 1, \ldots, 1)$

for any k. We then show that if $\alpha = (\ldots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots)$ is a composition with a symmetric rank polynomial, then the composition $(\ldots, a_{i-1}, a_i - 1, a_{i+1} + 1, a_{i+2}, \ldots)$ also has a symmetric rank polynomial. Our proof of this latter fact uses the interplay between circular and non-circular fences. We also make use of the following auxiliary statement about the structure of non-circular fences that follows as a byproduct of our proof.

Corollary 2.2. *For a composition $\beta = (\beta_1, \beta_2, \ldots, \beta_{2t})$ of $n-1$, let $\mathfrak{I}_L$ denote the set of ideals of $F(\beta)$ that include the leftmost node x_1, but not the rightmost node x_n. Similarly, let $\mathfrak{I}_R$ denote the set of ideals of $F(\beta)$ that include the rightmost node but not the leftmost. The following two polynomials are symmetric with center of symmetry $n/2$:*

$$\sum_{I \in \mathfrak{I}_L} q^{|I|} - \sum_{J \in \mathfrak{I}_R} q^{|J|}, \qquad R((\beta_1 + 1, \beta_2, \ldots, \beta_{2t}); q) - R((\beta_1, \beta_2, \ldots, \beta_{2t} + 1); q).$$

Another consequence of 1.5 is the following.

Corollary 2.3. *Let* sh *denote the operation that shifts a composition one step cyclically. We then have that $\overline{R}(\alpha, q) = \overline{R}(\text{sh}(\alpha), q)$.*

3 Rank Unimodality

A sequence is called *unimodal* if there exists an index m such that $a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \ldots \geq a_n$. Theorem 1.3 involves a refined version of unimodality described in [5]. A sequence is called *top interlacing* if $a_0 \leq a_n \leq a_1 \leq a_{n-1} \leq \ldots \leq a_{\lceil n/2 \rceil}$ where $\lceil \cdot \rceil$ is the ceiling function. Similarly, the sequence is *bottom interlacing* if $a_n \leq a_0 \leq a_{n-1} \leq a_1 \leq \ldots \leq a_{\lfloor n/2 \rfloor}$ with $\lfloor \cdot \rfloor$ being the floor function. Note that top interlacing as well as bottom interlacing sequences are unimodal.

Given a rank sequence $(r_0, r_1, \ldots, r_n)$, the properties of it being top interlacing, bottom interlacing or symmetric and unimodal are determined by the relationship between elements whose indices are equidistant from $n/2$, which we will call $\text{mid}(\alpha)$. In all three cases, if $|j - \text{mid}(\alpha)| > |i - \text{mid}(\alpha)|$ we have $a_j \leq a_i$. To this end, we will partition the inequalities that correspond to interlacing into two parts: the part that separates bottom and top interlacing sequences and the part that holds for both, namely

$$\text{(ineqB)} \qquad r_0 \geq r_n, r_1 \geq r_{n-1}, \ldots, r_{\lfloor n/2 \rfloor} \geq r_{\lceil n/2 \rceil}$$

$$\text{(ineqT)} \qquad r_0 \leq r_n, r_1 \leq r_{n-1}, \ldots, r_{\lfloor n/2 \rfloor} \leq r_{\lceil n/2 \rceil}$$

$$\text{(ineqA)} \qquad r_0 \leq r_{n-1}, r_1 \leq r_{n-2}, \ldots, r_n \leq r_1, r_{n-1} \leq r_2, \ldots$$

Bottom interlacing sequences are ones that satisfy (ineqB) and (ineqA), top interlacing sequences are ones that satisfy (ineqT) and (ineqA), and sequences satisfying all three sets of inequalities are symmetric and unimodal.

To prove Theorem 1.3 we use induction on the size of α. For a composition α of n we write down two different decompositions of $R(\alpha;q)$, in each case as a sum of a symmetric rank polynomial coming from a circular fence of size $n+1$ and a unimodal polynomial from a circular fence of strictly smaller size. The first method involves adding a new node a_0 to the fence of α lying above (or below) both a_1 and a_{n+1}. The ideals of $F(\alpha)$ are in bijection with a subset of the ideals of the resulting circular fence, and the rest are in bijection with ideals of a fence poset with fewer elements. Examining how the symmetric entries of the circular fence are affected shows that the rank sequence of α satisfies (ineqA).

The second method is similar, but involves adding the relation $x_1 \geq x_{n+1}$ to α. The resulting circular fence contains all ideals of $F(\alpha)$ such that $x_n \in I$ whenever $x_1 \in I$. The ones that are left over are in bijection with ideals of a fence poset with fewer elements. Combined with the induction assumption, we recover the inequality set (ineqA) or (ineqB), depending on the properties of α.

Unlike the regular case, the rank polynomial of circular fences is not always unimodal. In the case of $\alpha = (1,k,1,k)$, we get the rank sequence $[1,2,\ldots,k,k+1,k,k+1,k,\ldots,2,1]$ which makes a slight dip in the middle (Refer to Figure 2 for the rank lattice of $(1,5,1,5)$). We will next see that this issue can only happen when we have an even number of nodes, and a dip can only happen in the middle term of the rank sequence.

Proposition 3.1. *If α has an odd number of nodes, then $\overline{R}(\alpha;q)$ is unimodal. If α is of size $2t$ for some $k \in \mathbb{N}$, then we have $r_i \geq r_{i-1}$ for all $i < t$.*

Though we were unable to prove our conjecture that $\overline{F}(\alpha)$ is unimodal except for the cases of $(1,k,1,k)$ and $(k,1,k,1)$ in all generality, the next result shows that if there are exceptions to unimodality they are indeed very rare.

Lemma 3.2. *Let T be a maximal node in the cyclic fence $\overline{F}(\alpha)$, and let F_{T^-} be the (possibly upside down) fence obtained by deleting T. If the rank polynomial $R_{T^-}(q)$ corresponding to F_{T^-} is top interlacing, then $\overline{R}(\alpha;q)$ is rank unimodal.*

Corollary 3.3. *If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2s})$ has two consecutive segments larger than one, or 3 consecutive segments $k,1,l$ with $|k-l| > 1$, then $\overline{R}(\alpha;q)$ is unimodal.*

These three results, Proposition 3.1, Lemma 3.2 and Corollary 3.3 sharply constrict the possible cases of circular fences that are not rank unimodal. We ran computer experiments and checked all circular fences coming from compositions with up to 8 parts and with the sizes of these parts being at most 20. The results support our Conjecture 1.8.

4 Rowmotion on Circular Fences

We can identify the ideals of a fence with antichains on that fence, as any ideal is uniquely described by its maximal elements. Rowmotion acts on ideals by taking an ideal I to the ideal $\partial(I)$ corresponding to the antichain given by the minimal elements of the complement of I. In their recent paper [2], Elizalde, Plante, Roby and Sagan explored rowmotion on fences, and gave homomesy and orbomesy results, many of which hold for the circular case as well. In particular they gave a bijection between rowmotion orbits on $F(\alpha)$ and an object called an α-tiling. Here, we introduce a natural analogue, the class of *circular* α-tilings:

Definition 4.1. For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2s})$ of n, a circular α-tiling is a tiling of a rectangle R_{2s} with $2s$ rows labeled $1, 2, \ldots, 2s$ from top to bottom and an infinite number of columns with yellow 1×1 tiles, red 2×1 tiles which are allowed to wrap around and black $1 \times (\alpha_i - 1)$ tiles in row i satisfying the following properties:

- **(a)** If $\alpha_i \geq 2$ then when the red tiles are ignored, the black and yellow tiles alternate in row i.
- **(b)** If i is odd, there is a red tile in a column covering rows i and $i+1$ if and only if the next column contains two yellow tiles in those two rows.
- **(c)** If i is even, there is a red tile covering rows i and $i+1$ if $i < 2s$ and wrapping around to cover $2s$ and 1 if and only if the previous column contains two yellow tiles in those rows.

We say that a red tile *starts* at row i if it covers $i, i+1$ or $i = 2s$ and it covers $2s$ and 1, and denote by r_i the number of red tiles starting on row i. Though it is by no means clear from the definition, the connection with rowmotion orbits which we will describe in Lemma 4.2 implies that all such tilings are periodic. We will represent tilings by drawing one period and identify tilings that are cyclic shifts of each other horizontally and denote the period of an orbit $\mathcal{O}$ by $|\mathcal{O}|$.

Let the map $\overline{\phi}$ take an ideal I of $\overline{F}(\alpha = (\alpha_1, \ldots, \alpha_{2s}))$ to a $2s \times 1$ rectangle where box i is colored yellow if the ith segment contains no maximal element of I, red if it shares its maximal element with segment $i-1$ or $i+1$ (considered cyclically) and black otherwise. We can then see $\overline{\phi}$ as a map taking orbits of rowmotion to infinite rectangles of $2s$ rows by seeing each iteration of the rowmotion operation as a new column (See Figure 3).

Lemma 4.2. *The map $\overline{\phi}$ is a bijection between orbits of rowmotion on $\overline{F}(\alpha)$ and circular α-tilings.*

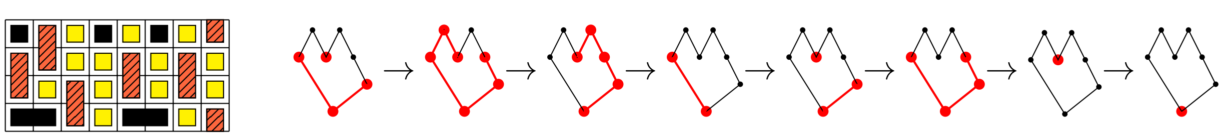
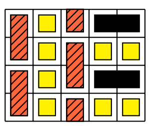
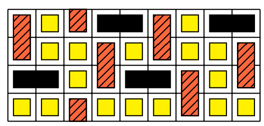
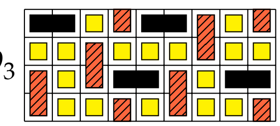


Figure 3: A circular $(2,1,1,3)$-tiling and the corresponding orbit on $\overline{F}(2,1,1,3)$

In what follows, we will identify each tiling with its corresponding orbit and use the two interchangeably. A statistic st is said to be d-mesic (with respect to a group operation) if its average is d on every orbit, and it is said to be *homomesic* if it is d-mesic for some d. We will be interested in two particular statistics in connection to fences. Let $\overline{\mathcal{M}}_x(\mathcal{O})$ stand for the number of times x occurs as a maximal element in an orbit $\mathcal{O}$, with $\overline{\mathcal{M}}(\mathcal{O}) = \sum_x \overline{\mathcal{M}}_x(\mathcal{O})$ denoting the sum over all nodes x and let $\overline{\chi}_x(\mathcal{O})$ denote the total number of times x occurs in $\mathcal{O}$ with $\overline{\chi}(\mathcal{O})$ denoting similarly the sum over all x.

As each ideal occurs exactly once as a column, the symmetry of the rank polynomial for circular fences implies that if the statistic $\overline{\chi}$ is homomesic (for example when there is a unique orbit), it is necessarily $n/2$-mesic, where $n = |\alpha|$. In particular, if $\alpha_i + \alpha_{i+1}$ is the same for all i, as in the example of $(3,1,3,1)$ below, then $\overline{\chi}$ is $n/2$-mesic if and only if $\sum_{i \le s}(r_{2i} - r_{2i-1}) = 0$ for all orbits.

Example 4.3 ($\overline{F}(3,1,3,1)$.). In the case $(3,1,3,1)$, rowmotion has the 3 orbits depicted below. We have $\overline{\mathcal{M}}(\mathcal{O}_1) = 8$, $\overline{\mathcal{M}}(\mathcal{O}_2) = \overline{\mathcal{M}}(\mathcal{O}_3) = 14$, $\overline{\chi}(\mathcal{O}_1) = 20$ and $\overline{\chi}(\mathcal{O}_2) = \overline{\chi}(\mathcal{O}_3) = 36$. As the second 9-orbit can be obtained from the first by shifting rows cyclically by 2, it makes sense that they have the same statistics. Note that $\overline{\chi}$ is 4-mesic.

$\mathcal{O}_1$  $\mathcal{O}_2$  $\mathcal{O}_3$ 

Calculating the $\overline{\mathcal{M}}$ and $\overline{\chi}$ statistics, we see that many homomesy results from the non-circular fences also apply for the circular ones:

Proposition 4.4. *For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2s})$ of n, the rowmotion operation on the circular fence $\overline{F}(\alpha)$ has the following properties:*

1. *If x and y are unshared elements on the same segment, $\overline{\mathcal{M}}_x - \overline{\mathcal{M}}_y$ is 0-mesic.*
2. *For an unshared element x of segment i that lies between a maximal element T and a minimal element B, $\overline{\mathcal{M}}_x \alpha_i + \overline{\mathcal{M}}_T + \overline{\mathcal{M}}_B$ is 1-mesic.*
3. *For a maximal element T lying between segments $2i+1$ and $2i+2$, and a minimal element B lying between segments $2j$ and $2j+1$ (cyclically), if $r_{2i+1} = r_{2j}$ for all orbits $\mathcal{O}$, then $\overline{\chi}_T + \overline{\chi}_B$ is 1-mesic.*
4. *If $\alpha_i = 2$ for all i, then $\overline{\mathcal{M}}$ is s-mesic.*

Let κ denote the setwise complement map, taking the ideals of $\overline{F}(\alpha)$ to the ideals of $\overline{\text{sh}}(\alpha)$, the fence corresponding to the cyclic shift of α by one step. We can also see κ as the map taking a circular α-tiling, doing a vertical cyclic shift of one step and a horizontal flip to get a circular sh(α)-tiling. Figure 4 shows the action of κ on the orbit seen in Figure 3. As the rowmotion is defined via the complement operation, it is quite well behaved under this map.

Composition	$\lvert\mathcal{O}\rvert$	Orbit Count	$\overline{\mathcal{M}}(\mathcal{O})$	$\overline{\chi}(\mathcal{O})$
(a,b):	$m+2$	1	$2m(a+b)(m+2)/ab$	$\lvert\mathcal{O}\rvert n/2$
$\gcd(a,b)=m$	m	$(ab/m)-1$	$2m-(a+b)m/ab$	$\lvert\mathcal{O}\rvert n/2$
$(a\neq 3t+1,1,1,1)$	$3a+4$	1	$5a+6$	$\lvert\mathcal{O}\rvert n/2$
	$a+2$	1	$5t+4$	$\lvert\mathcal{O}\rvert n/2$
$(a=3t+1,1,1,1)$	$a+1$	1	$5t+2$	$(\lvert\mathcal{O}\rvert+1)n/2$
	$a+1$	1	$5t+2$	$(\lvert\mathcal{O}\rvert-1)n/2$
$(a,1,a,1)$	$a+2$	$a-2$	$2a+2$	$\lvert\mathcal{O}\rvert n/2$
	$2a+3$	2	$4a+2$	$\lvert\mathcal{O}\rvert n/2$

Table 2: The behaviour of rowmotion on some small cases. Rows list the different types of orbits we get in each case. For example, for $(a,1,a,1)$ we get a orbits in total.

Lemma 4.5. *For any ideal I we have $\kappa(\partial(I))=\partial^{-1}(\kappa(I))$, meaning κ maps orbits to orbits. In particular, if $|\alpha|=n$, for any orbit $\mathcal{O}$ of rowmotion on $\overline{F}(\alpha)$ we have:*

$$\overline{\mathcal{M}}(\mathcal{O})=\overline{\mathcal{M}}(\kappa(\mathcal{O})) \quad \textit{and} \quad \overline{\chi}(\mathcal{O})+\overline{\chi}(\kappa(\mathcal{O}))=n|\mathcal{O}|/2.$$

While the second statement trivially follows from the fact that this is a setwise complement, the first is slightly more complicated. We do not have $\overline{\mathcal{M}}(I)=\overline{\mathcal{M}}(\kappa(I))$ for all ideals I, for example if I is the empty ideal, $\overline{\mathcal{M}}(I)=0$ whereas $\overline{\mathcal{M}}(\kappa(I))=s$, where $2s$ is the length of α.

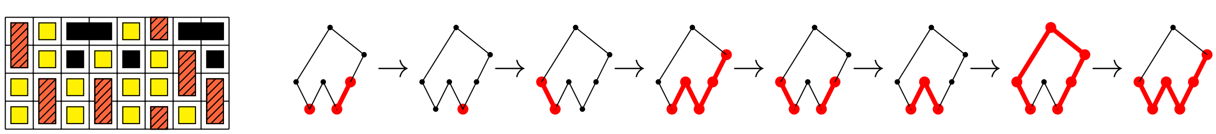


Figure 4: Setwise complement of the orbit from Figure 3 gives an orbit on $\overline{F}((3,2,1,1))$

When a statistic has the same average on all orbits with the same period, it is called *orbomesic*. Extending the idea of homomesy, the orbomesy phenomenon is introduced in [2] and illustrated through a number of cases it applies to. The orbit structure of rowmotion on fences is periodic in nature and only depends on how often we get shared elements, that is, how in sync the action on different segments are. The examples of orbomesy given in [2], though not isomorphic in a well-defined sense, are structurally equivalent and are formed by picking different pairings of moduli that are out of sync. As a result, they naturally have the same length, $\overline{\mathcal{M}}$ value and $\overline{\chi}$ value, resulting in an apparent orbomesy. In the circular case, we see that the orbomesy of $\overline{\chi}$ breaks down completely, in that we either get a full homomesy or we get two orbits of the same period with different $\overline{\chi}$ values (See Table 2. Indeed, in the case of $(3r+1,1,1,1)$ we get orbits that are *complementary*, that is, they have the same period and their $\overline{\chi}$ statistics sum up to $n(|\mathcal{O}|)$. A similar situation occurs in the case of (a,a,a,a), where the orbit situation is more complex (see [10] for a full description). This behaviour is probably a result of how the setwise complement map affects the orbits and the symmetry of the rank lattice. It might be interesting to pursue if this phenomenon continues in larger examples. In particular, are there any examples of circular lattices where $\overline{\chi}$ is orbomesic but not homomesic?

5 Comments, Questions and Future Directions

We list some questions and observations here that are of natural interest.

- **A Bijective Proof:** After a preliminary version of this paper was shared on the arXiv, Sagan and Elizalde came up with a lovely bijective proof of rank polynomial symmetry for circular fences [3].

- **Connection to $\mathrm{PSL}_q(2,\mathbb{Z})$:** In [4], Leclere and Morier-Genoud define matrices $M_q(c_1,\ldots,c_k) \in \mathrm{PSL}_q(2,\mathbb{Z})$ whose traces, up to a multiple of $\pm q^N$, are symmetric polynomials in q with non-negative integer entries. Similarly to circular rank polynomials, they are invariant under cyclic shifts of the sequence $[[c_1,\ldots,c_k]]$. Indeed, when the rational number with continued fraction representation $[c_1;c_2,\ldots,c_k]$ is ≥ 1, the traces turn out to equal circular rank polynomials [9].

- **A Polyhedral Perspective:** Given a composition α of n, consider the polytope $\overline{P}_\alpha \subset \mathbb{R}^n$ given by the indicator vectors of $\overline{J}(\alpha)$, the set of all lower ideals of the associated circular fence poset. Consider the sections of the polytope:

  $$\overline{P}^t_\alpha = \overline{P}_\alpha \cap \{x \in \mathbb{R}^n, \sum_i x_i = t\}.$$

  The symmetry of the rank polynomial for circular fences implies that for every positive integer t, the polytopes $\overline{P}^t_\alpha$ and $\overline{P}^{\frac{n}{2}-t}_\alpha$ have the same number of lattice points. We performed several computer assisted calculations on these polytopes and one of the curious facts we observed that the function $t \to \mathrm{Vol}\ \overline{P}^t_\alpha$ is symmetric about the point $n/2$, that is, $\mathrm{Vol}(\overline{P}^t_\alpha) = \mathrm{Vol}(\overline{P}^{n/2-t}_\alpha)$, despite these polytopes not necessarily being combinatorially equivalent. A natural explanation of this would be interesting.

- **Refinements of Unimodality:** McConville, Sagan and Smyth [5] investigated the existence of chain decompositions as a possible method for proving unimodality. The examples we considered lead us the believe that for circular fences (apart from the case $\alpha = (a, 1, a, 1)$, see Figure 2), the associated lattices admit *symmetric chain decompositions* and are thus *strongly Sperner*. We leave this as an open question.

Acknowledgements

The authors would like to thank Bruce Sagan for several helpful comments on a preliminary version of this paper. Both authors were supported by the grant SUP-17483 from the Boğaziçi University Scientific Research Office.

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