Séminaire Lotharingien de Combinatoire **86B** (2022)
Article #42, 12 pp.

Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)

Blowup Polynomials and delta-Matroids of Graphs

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Abstract. For every finite simple connected graph $G = (V, E)$, we introduce an invariant, its blowup-polynomial $p_G(\{n_v : v \in V\})$. This is obtained by dividing the determinant of the distance matrix of its blowup graph $G[\mathbf{n}]$ (containing n_v copies of v) by an exponential factor. We show that $p_G(\mathbf{n})$ is indeed a polynomial function in the sizes n_v, which is moreover multi-affine and real-stable. This associates a hitherto unexplored delta-matroid to each graph G; and we provide a second novel one for each tree. We also obtain a new characterization of complete multipartite graphs, via the homogenization at -1 of p_G being completely/strongly log-concave, *i.e.*, Lorentzian. (These results extend to weighted graphs.) Finally, we show p_G is indeed a graph invariant, *i.e.*, p_G and its symmetries (in the variables $\mathbf{n}$) recover G and its isometries.

Keywords: distance matrix, blowup-polynomial, real-stable polynomial, Zariski density, delta-matroid

Fifty years ago, Graham and Pollak [17] showed the following striking result in algebraic combinatorics: *Given a tree $T = (V, E)$ with distance matrix D_T, the scalar* $\det D_T$ *is independent of the tree structure, and depends only on $|V| = |E| + 1$.* Here, D_G for a finite connected, simple graph G denotes its distance matrix, with the (v, w) entry given by the length of the shortest path connecting $v \neq w \in V$, and $(D_G)_{vv} = 0$ for all $v \in V$. This result has been extended to multiple other settings, including q-distance matrices, multiplicative distances, and even combinations of these — see, *e.g.* [14] and its references for details and for an overarching generalization. The area has remained active ever since.

Graham then explored the spectral side with Lovász [16], including computing the characteristic polynomial (and roots) and inverse of D_T. This line of research too remains active, and has led to the study of "distance spectra" of graphs — see, *e.g.*, the survey [2].

Our work was motivated by both directions. On the algebraic side, we sought natural graph families $\{G_i : i \in I\}$ — *e.g.* trees on n vertices — such that the map $i \mapsto \det D_{G_i}$ is a

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"nice" function from I to $\mathbb{R}$. On the analysis side, it is well-known that the characteristic polynomial $\det(x \operatorname{Id} - D_G)$ of the distance matrix of G does not recover G, *i.e.*, there are graphs $G \not\cong H$ with the same number of nodes, which are "distance co-spectral". Thus, we were interested in finding a different byproduct of D_G that recovers G.

The purpose of this note is to describe such a byproduct of D_G (or of G), which we introduce in the work [15], and which we term the *(multivariate) blowup-polynomial* of G. We then explain how this polynomial achieves the above two goals. A third, interesting byproduct of our work is a — to our knowledge — novel family of delta-matroids, one for each graph G. This holds because the blowup-polynomial turns out to be multi-affine and real-stable. We further introduce another novel delta-matroid for every tree.

1 The blowup-polynomial of a graph, and its symmetries

We begin by introducing the key ingredient needed to define the blowup-polynomial: the family of *blowup graphs* of G:

Definition 1.1. *Given a finite simple connected (unweighted) graph $G = (V_G, E_G)$, and a set of positive integers $\mathbf{n} = \{n_v : v \in V_G\}$, the* blowup graph $G[\mathbf{n}]$ *is the finite simple connected graph with n_v copies of the vertex v, such that a copy of v is adjacent to one of w if and only if $v \neq w$ and $(v, w) \in E_G$. Define $M_G := D_G + 2 \operatorname{Id}_{V_G}$, where D_G is the distance matrix of G.*

These are studied in extremal and probabilistic graph theory; see, *e.g.*, [20, 21, 22].

We now claim that — akin to trees on n vertices for any fixed $n \geq 1$ — the family of blowups of a fixed graph G is well-behaved vis-a-vis computing $\det D_{G[\mathbf{n}]}$:

Theorem 1.2. *Given a finite simple connected (unweighted) graph G, there exists a polynomial $p_G(\mathbf{n})$ in the* sizes n_v, *with integer coefficients, such that*

$$\det D_{G[\mathbf{n}]} = (-2)^{\sum_v (n_v - 1)} p_G(\mathbf{n}), \qquad \mathbf{n} \in \mathbb{Z}_{>0}^V.$$

Also, p_G is multi-affine in $\mathbf{n}$, with constant term $(-2)^{|V|}$ and linear term $-(-2)^{|V|} \sum_{v \in V} n_v$.

Here and below, we mildly abuse notation and refer to both the integer sizes as well as indeterminates by n_v; this will be clear from context. Also recall, a polynomial $p(\{n_v\})$ is multi-affine if $\deg_{n_v}(p) \leq 1$ for all v.

Definition 1.3. *For a graph G as in Theorem 1.2, define its* (multivariate) blowup-polynomial *to be $p_G(\mathbf{n}) \in \mathbb{Z}[\mathbf{n}]$, where we think of the n_v as indeterminates. Also define the* univariate blowup-polynomial *of G to be $u_G(n) := p_G(n, n, \ldots, n)$.*

We clarify this definition with a remark. The polynomial function (by Theorem 1.2)

$$\mathbf{n} \mapsto (-2)^{-\sum_v (n_v - 1)} \det D_{G[\mathbf{n}]}, \qquad \mathbf{n} \in \mathbb{Z}_{>0}^V$$

has to first be extended to $\mathbb{R}^V$ from its Zariski dense subset $\mathbb{Z}^V_{>0}$. It can then be identified with a polynomial in $\mathbb{R}[\mathbf{n}]$ (with integer coefficients), and it is this polynomial that we denote here and below by $p_G(\mathbf{n})$ as well.

Proof of Theorem 1.2. We provide a quick sketch; the key ingredient is again algebraic here: Zariski density. (In fact, this result holds over a general commutative ring, and we refer the reader to the full paper [15] for details.) Let $k := |V|$, **fix** (throughout this note) an enumeration $(n_1, \dots, n_k)$ of $\{n_v : v \in V\}$, let $D_G = (d_{ij})_{i,j=1}^k$, and define

$$K := \sum_{i=1}^k n_i, \qquad \mathcal{W}_{K\times k} := \begin{pmatrix} \mathbf{1}_{n_1\times 1} & 0_{n_1\times 1} & \cdots & 0_{n_1\times 1} \\ 0_{n_2\times 1} & \mathbf{1}_{n_2\times 1} & \cdots & 0_{n_2\times 1} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n_k\times 1} & 0_{n_k\times 1} & \cdots & \mathbf{1}_{n_k\times 1} \end{pmatrix}.$$

Given an integer tuple $\mathbf{n} \in \mathbb{Z}^k_{>0}$, recall that $D_{G[\mathbf{n}]} = M_{G[\mathbf{n}]} - 2\,\mathrm{Id}_K$. Notice that $M_{G[\mathbf{n}]}$ is a block $k \times k$ matrix with (i,j) block $d_{ij} \cdot \mathbf{1}_{n_i \times n_j}$ for $i \neq j$ and $2 \cdot \mathbf{1}_{n_i \times n_i}$ for $i = j$; in particular, $M_{G[\mathbf{n}]} = \mathcal{W} M_G \mathcal{W}^T$. We now employ Zariski density, by first considering the entries of M_G as well as the sizes n_i to be variables, and working over the field $\mathbb{F}$ of rational functions in these, with coefficients in $\mathbb{Q}$. In particular, $\det M_G \in \mathbb{F}^\times$. We compute, using Schur complements repeatedly:

$$\begin{aligned} \det D_{G[\mathbf{n}]} = {} & \det(\mathcal{W} M_G \mathcal{W}^T - 2\,\mathrm{Id}_K) = \det \begin{pmatrix} -2\,\mathrm{Id}_K & -\mathcal{W} \\ \mathcal{W}^T & M_G^{-1} \end{pmatrix} \det(M_G), \qquad (1.1) \\ & = (-2)^K \det(M_G^{-1} - 2^{-1}\mathcal{W}^T\mathcal{W}) \det(M_G) = (-2)^{K-k} \det((-2)\,\mathrm{Id}_k + \Delta_{\mathbf{n}} M_G), \end{aligned}$$

where $\Delta_{\mathbf{n}} = \mathcal{W}^T\mathcal{W}$ is the diagonal matrix with (i,i) entry n_i. Now (1.1) proves the result over the field $\mathbb{F}$ of rational functions, hence — by Zariski density — in the subring of polynomials in the same variables, since both sides of (1.1) are polynomial *functions*. As $\mathbb{Q}$ is infinite, we obtain an equality of polynomials, both of which have integer coefficients. Finally, specialize the sizes n_i and the entries of M_G to the given graph-data. □

Remark 1.4. It also follows from the above proof that $p_G(\mathbf{n}) = \det(\Delta_{\mathbf{n}} M_G - 2\,\mathrm{Id}_k)$.

Theorem 1.2 and its proof enable us to do more: we can compute the coefficient of every monomial in p_G, and relate p_G to p_H for certain induced subgraphs H of G:

Proposition 1.5. *Using the same notation as above:*

1. *Given a subset $I \subset V$, the coefficient in $p_G(\mathbf{n})$ of $\prod_{i\in I} n_i$ is $(-2)^{|V\setminus I|} \det(M_G)_{I\times I}$, where $(M_G)_{I\times I}$ is the principal submatrix of M_G formed by the rows and columns indexed by I.*

2. *Let H be an induced subgraph of G with vertex set $I \subset V$ and no isolated nodes. Then,*

$$p_H(\{n_i : i \in I\}) = p_G(\mathbf{n})|_{n_j = 0 \textit{ for all } j \notin I} \cdot (-2)^{-|V\setminus I|}.$$

Thus if some monomial $\prod_{i\in I_0} n_i$ (for $I_0 \subset I$) does not occur in p_H, it does not occur in p_G.

3. *Suppose H, K are induced subgraphs of G, say on node sets $I, J \subset V$ respectively, and each without isolated nodes. If H, K are isomorphic, then the coefficients in $p_G(\mathbf{n})$ of $\prod_{i\in I} n_i$ and $\prod_{j \in J} n_j$ are equal.*

4. *The iterated blowup of a graph $G = (V, E)$ is also a blowup of G. In particular, the blowup-polynomial of $p_{G[\mathbf{n}]}$ has total degree at most $|V|$, for all $\mathbf{n} \in \mathbb{Z}^V_{>0}$.*

As a simple illustration of the final assertion here, notice that the path graph P_3, the cycle C_4, and all star graphs $K_{1,n}$ are instances of complete bipartite graphs $K_{r,s}$. As $K_{r,s} = K_2[(r,s)]$ is a blowup of the edge K_2, the blowup-polynomials of all of these graphs are multi-affine of degree 2, and can be easily computed.

Proposition 1.5 has multiple applications; we provide two here. First, it makes tractable the computation of $p_G(\cdot)$ for certain more involved graphs. Here is an example.

Example 1.6. Given integers k, l with $0 \leq l \leq k-2$, let $K_k^{(l)}$ denote the graph on vertices $\{1, \dots, k\}$, with all edges connected except for $(1,2), \dots, (1, l+1)$. These form a family of chordal graphs, with isomorphism/isometry group $S_l \times S_{k-l-1}$ corresponding to the partition of the vertex set $V = \{1\} \sqcup \{2, \dots, l+1\} \sqcup \{l+2, \dots, k\}$. Now we have:

$$\begin{aligned} p_{K_k^{(l)}}(\mathbf{n}) = & \sum_{r=0}^{l} \sum_{s=0}^{k-l-1} \Big[(-2)^{k-r-s}(1+r+s)\Big] e_r(n_2, \dots, n_{l+1}) e_s(n_{l+2}, \dots, n_k) \\ & + n_1 \sum_{r=0}^{l} \sum_{s=0}^{k-l-1} \Big[(-2)^{k-r-s-1}(1-r)(s+2)\Big] e_r(n_2, \dots, n_{l+1}) e_s(n_{l+2}, \dots, n_k), \end{aligned} \tag{1.2}$$

with $e_r(\cdot)$ the elementary symmetric polynomial. (The graphs $K_k^{(1)}$ were crucially used in [18].)

The above decomposition of the nodes of $K_k^{(l)}$ is into subsets, each containing nodes that are all isomorphic to one another. These auto-isometries (*i.e.*, adjacency-preserving bijections) of the underlying graph translate into *symmetries* of the blowup-polynomial, as seen in (1.2). (We may thus call p_G a *partially symmetric polynomial*.) Conversely, it is natural to ask if p_G can recover the auto-isometries of G — and more strongly, if p_G recovers the graph G itself. Our next result provides a positive answer.

Proposition 1.7. *Given G as above, the symmetries of p_G coincide with the auto-isometries of G. More strongly, the polynomial p_G recovers G. However, this is not true for the univariate specialization u_G.*

Proof-sketch. The first claim follows from the second, which holds because the Hessian equals

$$\mathcal{H}(p_G) := ((\partial_{n_i} \partial_{n_j} p_G)(\mathbf{0}))_{i,j=1}^k = (-2)^k \mathbf{1}_{k \times k} - (-2)^{k-2} M_G^{\circ 2},$$

where given a matrix $M = (m_{ij})$, $M^{\circ 2} := (m_{ij}^2)$ is its entrywise square. Finally, to study u_G, define the graphs H, K in Figure 1, both with vertices $\{1, \dots, 6\}$. Next, we define:

Figure 1: Two non-isomorphic graphs on six vertices with co-spectral blowups.

$$H' := H[(2,1,1,2,1,1)], \qquad K' := K[(2,1,1,1,1,2)].$$

It is easily checked by direct computations that H', K' are not isomorphic, but

$$u_{H'}(n) = u_{K'}(n) = -320n^6 + 3712n^5 - 10816n^4 + 10880n^3 - 1664n^2 - 2048n + 256. \quad \square$$

Thus, $H' \ncong K'$ (both with $|V| = 8$) are graphs whose distance matrices have the same characteristic polynomial and equal univariate polynomials $u_{H'} = u_{K'}$; but $p_{H'} \neq p_{K'}$.

Our second application of Proposition 1.5 involves a special case of the graphs $K_k^{(l)}$ — namely, for $l = 0$, in which case $K_k^{(l)} = K_k$, a complete graph. In this case, one checks:

$$p_{K_k}(n_1, \dots, n_k) = \prod_{i=1}^{k}(n_i - 2) + \sum_{i=1}^{k} n_i \prod_{i' \neq i}(n_{i'} - 2). \tag{1.3}$$

This is "fully" symmetric in the n_i. In fact, there are no other graphs with this property:

Proposition 1.8. *Given a graph G as above, the blowup-polynomial $p_G(\mathbf{n})$ is symmetric in the variables $\{n_i : 1 \leq i \leq k\}$ if and only if G is complete.*

2 Real-stability and related properties

Our next goal is to explain how the blowup-polynomial gives rise to a hitherto unexplored delta-matroid for every graph. (More generally, one obtains such a delta-matroid from every finite metric space — see Remark 2.5.) This will follow from the polynomial p_G possessing additional desirable features, which we describe in this section.

As a motivating example, note that specializing the polynomial $p_{K_k}(\mathbf{n})$ in (1.3) yields the univariate blowup-polynomial $u_{K_k}(n) = (n-2)^{k-1}(kn + n - 2)$, and this is real-rooted. More generally, this turns out to hold for all graphs G — in fact, far more is true. Real-rootedness is the one-variable manifestation of a more general, and far more powerful notion: a polynomial $p(z_1, \dots, z_k)$ with real coefficients and complex arguments is said to be *real-stable* if it is non-vanishing whenever $\Im(z_j) > 0$ for all j. Real-stable polynomials and their generalizations are the focus of tremendous recent research, see, *e.g.*, the well-known papers by Borcea–Brändén [3, 4, 5] and Marcus–Spielman–Srivastava [23, 24], in which longstanding conjectures of Bilu–Linial, Johnson, Kadison–Singer, Lubotzky, and others are resolved, and the Laguerre–Pólya–Schur program from the early 20th century is significantly extended (among other remarkable achievements).

In combinatorics, the importance of real-rootedness and (strong) log-concavity is very well known, see, *e.g.*, [12, 26]. Recently, there has been much work in going beyond these notions and studying the connections of stability to combinatorics and statistical physics; see, *e.g.*, [10, 25]. Our next result shows that graph blowup-polynomials $p_G(\cdot)$ are indeed real-stable (which is what will yield novel delta-matroids, below):

Theorem 2.1. *Given a finite simple connected graph G, its blowup-polynomial $p_G(\mathbf{n})$ is real-stable in the variables $\{n_v : v \in V\} = \{n_1, \dots, n_k\}$. (In particular, $u_G(n)$ is always real-rooted.)*

Recall from [9, 27] that a multi-affine polynomial $f(z_1, \dots, z_n)$ is real-stable if and only if $\partial_i f \cdot \partial_j f \geq f \cdot \partial_i \partial_j f$ on $\mathbb{R}^n$, for all i, j. The class of real-stable multi-affine polynomials is also connected to matroids; see [9, 13]. Theorem 2.1 says that graph blowup-polynomials $p_G(\mathbf{n})$ provide novel (to our knowledge) examples of such maps.

Proof. As the goal is to prove real-stability, in this proof we write $p_G(z_1, \dots, z_k)$ to indicate that the variables are complex (rather than algebraic). From Remark 1.4,

$$\begin{aligned} p_G(\mathbf{z}) = & \det(\Delta_{\mathbf{z}} M_G - 2\,\mathrm{Id}_k) = \prod_{j=1}^k z_j \cdot \det(2^{-1} M_G - \Delta_{\mathbf{z}}^{-1}) \cdot 2^k \\ & = 2^k \prod_{j=1}^k z_j \cdot \det\left(2^{-1} M_G + \sum_{j=1}^k (-z_j^{-1} E_{jj})\right), \end{aligned} \tag{2.1}$$

where $E_{jj} \in \mathbb{Z}^{k \times k}$ is the elementary matrix with (j, j)-entry 1. Now we recall a fundamental determinantal example of real-stable polynomials by Borcea–Brändén — see [3] (or [9, Lemma 4.1]). The authors show that if $A_1, \dots, A_k, B$ are real symmetric matrices, with all A_j positive semidefinite, then the polynomial

$$f(z_1, \dots, z_k) := \det\left(B + \sum_{j=1}^k z_j A_j\right) \tag{2.2}$$

is real-stable or identically zero. Moreover, "inversion preserves stability": if $g(z_1, \dots, z_k)$ is a polynomial with z_j-degree $d_j \geq 1$ that is real-stable, then so is $z_1^{d_1} g(-z_1^{-1}, z_2, \dots, z_k)$. (This is because the map $z \mapsto -1/z$ preserves the upper half-plane.) Now apply (2.2) to $A_j = E_{jj}$ and $B = 2^{-1} M_G$, and then apply inversion in each variable, to conclude via (2.1) that p_G is real-stable. □

Returning to u_G, which we now know is real-rooted, we also note that it is indeed related to the distance spectrum of G (*i.e.*, to the characteristic polynomial of D_G):

Proposition 2.2. *For any finite simple connected (unweighted) graph G, a real number n is a root of u_G if and only if $n \neq 0$ and $2n^{-1} - 2$ is an eigenvalue of D_G (with the same multiplicity).*

2.1 A novel characterization of complete multipartite graphs

We next mention two other notions related to stability, which have been greatly studied in recent years, and which are not satisfied by p_G. By the final assertion in Theorem 1.2, the coefficients of the multi-affine polynomial p_G cannot be normalized to form a probability distribution, since they are not all of the same sign. Similarly, the polynomial p_G is clearly not homogeneous. In two fundamental and important papers, stable polynomials with these two properties have been studied (in broader settings) by Borcea–Brändén–Liggett [6] and Brändén–Huh [11], under the name of *strongly Rayleigh measures/polynomials* and *Lorentzian polynomials*, respectively. Our next result explains that while p_G is neither strongly Rayleigh nor Lorentzian, a suitable normalization/homogenization can be. In fact, we completely characterize all such graphs:

Theorem 2.3. *Given a graph G as above, define its* homogenized blowup-polynomial

$$\widetilde{p}_G(z_0, z_1, \dots, z_k) := (-z_0)^k p_G \left(\frac{z_1}{-z_0}, \dots, \frac{z_k}{-z_0} \right) \in \mathbb{R}[z_0, z_1, \dots, z_k]. \tag{2.3}$$

The following are equivalent.

1. *The homogenized polynomial $\widetilde{p}_G(z_0, z_1, \dots, z_k)$ is real-stable.*

2. *The polynomial $\widetilde{p}_G(z_0, z_1, \dots, z_k)$ is* Lorentzian. *That is, $\widetilde{p}_G(\cdot)$ is homogeneous of degree k with non-negative coefficients, and given indices $0 \leq j_1, \dots, j_{k-2} \leq k$, if*

   $$g(z_0, z_1, \dots, z_k) := \left(\partial_{z_{j_1}} \cdots \partial_{z_{j_{k-2}}} \widetilde{p}_G \right) (z_0, z_1, \dots, z_k),$$

   then its Hessian matrix $\mathcal{H}_g := (\partial_{z_i} \partial_{z_j} g)_{i,j=0}^k \in \mathbb{R}^{(k+1)\times(k+1)}$ is Lorentzian (i.e., $\mathcal{H}_g$ is nonsingular and has exactly one positive eigenvalue).

3. *$\widetilde{p}_G(\cdot)$ has all coefficients non-negative (i.e., of the monomials $z_0^{k-|J|} \prod_{j \in J} z_j$).*

4. *$(-1)^k p_G(-1, \dots, -1) > 0$, and the normalized "reflected" polynomial*

   $$(z_1, \dots, z_k) \quad \mapsto \quad \frac{p_G(-z_1, \dots, -z_k)}{p_G(-1, \dots, -1)}$$

   is strongly Rayleigh. *That is, this multi-affine polynomial is real-stable, has non-negative coefficients (of all monomials $\prod_{j \in J} z_j$), and these sum up to 1.*

5. *The matrix $M_G = D_G + 2\,\mathrm{Id}_k$ is positive semidefinite.*

6. *The graph G is a blowup of a complete graph — that is, G is a complete multipartite graph.*

Theorem 2.3 characterizes the complete multipartite graphs in terms of stability. We refer the reader to the full paper [15] for the proof.

It turns out that two additional, related notions in the literature also characterize the complete multipartite graphs, and we mention them here for completeness. Suppose a polynomial $p \in \mathbb{R}[z_1, \dots, z_k]$ has non-negative coefficients. In [19], Gurvits defines p to be *strongly log-concave* if for every $\alpha \in \mathbb{Z}^k_{\geq 0}$, either the derivative $\partial^\alpha(p) := \prod_{i=1}^{k} \partial^{\alpha_i}_{x_i} \cdot p$ is identically zero, or $\partial^\alpha p > 0$ and $\log(\partial^\alpha(p))$ is concave on $(0,\infty)^k$. Next in [1], Anari, Oveis Gharan, and Vinzant define p to be *completely log-concave* if for all $m \in \mathbb{Z}_{>0}$ and matrices $A = (a_{ij}) \in [0,\infty)^{m\times k}$, either the derivative $\partial_A(p) := \prod_{i=1}^{m} \left(\sum_{j=1}^{k} a_{ij} \partial_{x_j} \right) \cdot p$ is identically zero, or $\partial_A(p) > 0$ and $\log(\partial_A(p))$ is concave on $(0,\infty)^k$. We now have:

Corollary 2.4. *We use the notation as in Theorem 2.3. Then G is complete multipartite if and only if either of the following holds:*

7. *The polynomial $\widetilde{p}_G(z_0, \dots, z_k)$ is strongly log-concave.*

8. *The polynomial $\widetilde{p}_G(z_0, \dots, z_k)$ is completely log-concave.*

Proof. For arbitrary real homogeneous polynomials, [11, Theorem 2.30] shows that both of these assertions are equivalent to: $\widetilde{p}_G$ is Lorentzian. Now use Theorem 2.3. □

Remark 2.5. As a concluding remark concerning the results mentioned until this point, we discuss how these results hold in greater generality. First, the definitions of a blowup and the blowup-polynomial extend to all finite metric spaces (X, d). Now Theorems 1.2, 2.1, and 2.3, Corollary 2.4, as well as Propositions 1.5 and 1.8 extend to arbitrary finite metric spaces, possibly with some modifications. We refer the reader to [15] for the details.

3 A blowup delta-matroid for graphs, and one for trees

In addition to being a graph invariant and a multi-affine polynomial, p_G also yields a novel delta-matroid for every graph G. Delta-matroids were introduced by Bouchet [7], and consist of a finite "ground set" E and a nonempty subset of its power set $\mathcal{F} \subset 2^E$. The elements F of $\mathcal{F}$ are called *feasible subsets*, and satisfy: (1) $\bigcup_{F\in\mathcal{F}} F = E$; (2) the *symmetric exchange axiom:* Given $A, B \in \mathcal{F}$ and $x \in A\Delta B$ (their symmetric difference), there exists $y \in A\Delta B$ such that $A\Delta\{x,y\} \in \mathcal{F}$.

Brändén has shown [9] that the set of monomials occurring in a real-stable multi-affine polynomial forms a delta-matroid. In particular:

Definition 3.1. *The* blowup delta-matroid of G *is denoted by* $\mathcal{M}_{M_G}$*; it has ground set* V *and feasible subsets corresponding to the nonzero monomials in* p_G.

In fact, more is true: this delta-matroid is *linear* [8], in that its feasible subsets are precisely the sets of indices $I \subset \{1, \dots, k\}$ for which the principal matrix $(M_G)_{I \times I}$ is nonsingular (by Proposition 1.5(1)). This delta-matroid appears to be unexplored in the literature, and was not known to experts.

The goal of this section is to construct another delta-matroid $\mathcal{M}'(T)$, this time for all trees T. We begin by taking a closer look at $\mathcal{M}_{M_G}$ for G a "small" path graph $P_k = \{(1,2), \dots, (k-1,k)\}$. Indeed, one can verify that, for $k \leq 4$,

$$\mathcal{M}_{M_{P_k}} = 2^{\{1,\dots,k\}} \setminus \{ \ \{i, i+1, i+2\}, \ \ \{i, i+2\} \ : \ 1 \leq i \leq k-2 \}. \tag{3.1}$$

Let us explain why the sets $\{i, i+1, i+2\}$ and $\{i, i+2\}$ are infeasible — *i.e.*, why the coefficients of the monomials $n_i n_{i+1} n_{i+2}, n_i n_{i+2}$ in p_{P_k} vanish — for all $k \geq 3$. This happens because the points $\{i, i+2\}$ are part of a graph $\{i, i+1, i+2\} \cong P_3$, which is a blowup of $K_2 = P_2$ — and in this blowup, $i, i+2$ are copies of a vertex. More generally:

Proposition 3.2. *Suppose* G, H *are finite simple connected graphs, and the tuple* $\mathbf{n} \in \mathbb{Z}_{>0}^{V(G)}$ *is such that* $G[\mathbf{n}]$ *is an induced subgraph of* H*. If some* $n_v \geq 2$ *and* $v_1, v_2 \in G[\mathbf{n}]$ *are copies of* v*, then the coefficient of* $\prod_{i \in I} n_i$ *in* $p_H(\cdot)$ *is zero whenever* $\{v_1, v_2\} \subset I \subset V(G[\mathbf{n}])$.

Proof. By Proposition 1.5(1), it suffices to show that $(M_H)_{I \times I}$ is singular. In turn, this holds because one verifies that the rows of M_H indexed by v_1, v_2 are identical. □

As a consequence of Proposition 3.2, the assertion preceding it, which involved $n_i n_{i+1} n_{i+2}$, now extends to arbitrary graphs containing two independent nodes a, c with a common neighbor b. It is thus natural to return to (3.1), and ask two things: (a) Does this equality hold for all k? (b) Independent of (a), is the right-hand side also a delta-matroid? It is also natural to ask if (c) the converse to Proposition 3.2 holds: namely, if a monomial does not occur in p_G, does the induced subgraph on those vertices contain two copies of a vertex inside some blowup? The next result answers these questions.

Proposition 3.3. *Using the same notation as above:*

1. *The right-hand side of* (3.1) *is a delta-matroid for every* k.
2. *The equality in* (3.1) *holds if and only if* $k \leq 8$.
3. *The converse to Proposition* 3.2 *does not hold, even for path graphs.*

Proof. The first part is presently explained in greater generality, for all trees. Second, the equality in (3.1) holds for $k \leq 8$ by explicit computations (*e.g.*, using a computer). One also computes: $\det(M_{P_9}) = 0$. Hence by Proposition 1.5(3), the coefficient of

$n_i n_{i+1} \cdots n_{i+8}$ in $p_{P_k}(\mathbf{n})$ is zero for all $1 \leq i \leq k-8$. It follows that the left-hand side of (3.1) is a strict subset of the right-hand side, for $k \geq 9$. The third/final assertion now follows from this computation, since P_9 is not the blowup of a smaller graph. □

We now explore if the right-hand delta-matroid in (3.1) can be generalized to other graphs. This indeed turns out to hold for all trees; to describe it, recall that the *Steiner tree* $T(I)$ of a subset of vertices I of a tree is the unique smallest sub-tree containing I.

Theorem 3.4. *Suppose T is any tree, and we define a subset of vertices I to be infeasible if its Steiner tree $T(I)$ has two leaves which are in I and have the same parent. (All other subsets are feasible.) Then the set $\mathcal{M}'(T)$ of feasible subsets is a delta-matroid.*

(See [15] for the proof.) We term this delta-matroid the *tree-blowup delta-matroid* $\mathcal{M}'(T)$. Notice by Proposition 3.3(2) that $\mathcal{M}'(P_k) \neq \mathcal{M}_{M_{P_k}}$ for $k \geq 9$, so this is not the blowup delta-matroid of P_k. Moreover, $\mathcal{M}'(T)$ also appears to not be known to experts.

Our final result answers a natural question: *Akin to the delta-matroid $\mathcal{M}_{M_{P_k}}$, can the definition of $\mathcal{M}'(T)$ also be extended to yield a delta-matroid for every graph?* In this regard, a key observation is that in Theorem 3.4, a set of nodes I is infeasible if and only if its Steiner tree $T(I)$ is a blowup of a graph with a strictly smaller vertex set. We therefore introduce the following two possible extensions of this version of infeasibility to general graphs, which are both natural choices:

Definition 3.5. *Let $G = (V, E)$ be a finite simple connected graph. Say that a subset $I \subset V$ is*

1. infeasible of the first kind *if there are vertices $v_1 \neq v_2 \in I$ and a subset $I \subset \widetilde{I} \subset V$, satisfying: (a) the induced subgraph $G(\widetilde{I})$ on $\widetilde{I}$ of G is connected, and (b) v_1, v_2 have the same set of neighbors in $G(\widetilde{I})$.*
2. infeasible of the second kind *if there exist $v_1 \neq v_2 \in I$ and $I \subset \widetilde{I} \subset V$, with: (a) the induced graph $G(\widetilde{I})$ has: $M_{G(\widetilde{I})} = (M_G)_{\widetilde{I} \times \widetilde{I}}$ and (b) v_1, v_2 have the same neighbors in $G(\widetilde{I})$.*

Also define $\mathcal{M}'_1(G)$ (respectively, $\mathcal{M}'_2(G)$) to comprise all subsets of V that are not infeasible of the first (respectively, second) kind.

As an example, if $G = T$ is a tree, then one checks that $\mathcal{M}'_1(T) = \mathcal{M}'_2(T) = \mathcal{M}'(T)$. It is now natural to ask if either $\mathcal{M}'_1(G)$ or $\mathcal{M}'_2(G)$ is a delta-matroid for all graphs G. It turns out that this is not the case:

Proposition 3.6 ([15])**.** *For the graph $G = \mathbf{G}_\circ$ (see Figure 2), neither $\mathcal{M}'_1(G)$ nor $\mathcal{M}'_2(G)$ is a delta-matroid.*

In closing, we note the above results describe several novel invariants associated to finite simple connected graphs (in fact, finite metric spaces). These include the polynomials $p_G(\mathbf{n})$, $u_G(n)$; the delta-matroid $\mathcal{M}_{M_G}$ (and $\mathcal{M}'(G)$ for G a tree); but also "simpler"

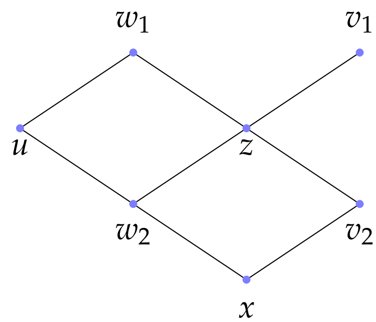

Figure 2: The graph $\mathbf{G}_\circ$.

invariants like $\deg p_G$, $\deg u_G$. (These degrees are not necessarily $|V|$ even if G is not a blowup of a smaller graph; *e.g.*, $G = P_k$ for $k \geq 9$, by Proposition 3.3.) It would be desirable and interesting to explore if these are relatable to more "familiar" combinatorial graph invariants.

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