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New Companions to Gordon Identities from Commutative Algebra

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Abstract. We survey the ideas of a proof of a recent conjecture on partitions due to the first author, which was discovered in the study of arc spaces and differential algebra. This result gives rise to new companions to the famous Andrews–Gordon identities. Our tools involve graded quotient rings, new Durfee-type dissections for integer partitions, and q-series identities.

Keywords: integer partitions, Andrews–Gordon identities, q-series, Bailey lemma, Durfee dissections, monomial ideals, graded rings

1 Introduction

Two of the most famous q-series formulas are the Rogers–Ramanujan identities:

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q)\cdots(1-q^k)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \tag{1.1}$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(1-q)\cdots(1-q^k)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}, \tag{1.2}$$

where empty products obtained with $k=0$ on the left-hand sides are taken to be 1. These identities appear in many fields such as combinatorics, statistical mechanics, number theory, representation theory, or algebraic geometry (see, *e.g.*, [7, 9, 12, 15]).

Recall that a partition of a positive integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_s = n$. The integers λ_i are called

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the parts of λ and s is its length. After their discovery, (1.1) and (1.2) were interpreted combinatorially by MacMahon [16] and Schur [17], giving rise to the following partition identities.

Theorem 1.1 (Rogers–Ramanujan identities, combinatorial version)**.** *Let n be a nonnegative integer and set $i \in \{1, 2\}$. Denote by $T_{2,i}(n)$ the number of partitions of n such that the difference between consecutive parts is at least 2 and the part 1 appears at most $i-1$ times. Let $E_{2,i}(n)$ be the number of partitions of n into parts congruent to $\pm(2+i) \mod 5$. Then we have*

$$T_{2,i}(n) = E_{2,i}(n).$$

A famous family of partition identities, which generalizes the previous ones and plays a central role in this article, is due to Gordon [14].

Theorem 1.2 (Gordon's identities)**.** *Let r and i be integers such that $r \geq 2$ and $1 \leq i \leq r$. Let $\mathcal{T}_{r,i}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ where $\lambda_j - \lambda_{j+r-1} \geq 2$ for all j, and at most $i-1$ of the parts λ_j are equal to 1. Let $\mathcal{E}_{r,i}$ be the set of partitions whose parts are not congruent to $0, \pm i \mod (2r+1)$. Let n be a nonnegative integer, and let $T_{r,i}(n)$ (respectively $E_{r,i}(n)$) denote the number of partitions of n which belong to $\mathcal{T}_{r,i}$ (respectively $\mathcal{E}_{r,i}$). Then we have*

$$T_{r,i}(n) = E_{r,i}(n).$$

Identity (1.1) (resp. (1.2)) is Theorem 1.2 with $r = i = 2$ (resp. $r = i + 1 = 2$).

Our main goal in this article is to prove new companions to the Gordon identities; in particular, we will consider a new set of partitions that we will call $\mathcal{C}_{r,i}$ and prove that for all nonnegative integers n, the number $C_{r,i}(n)$ of partitions of n belonging to $\mathcal{C}_{r,i}$ is equal to $T_{r,i}(n)$ and $E_{r,i}(n)$. This settles positively a conjecture made by the first author in [2]. We first discuss the algebro-geometric origin of this conjecture.

Let $\mathcal{R} := \mathbf{K}[x_i, i \geq 1]$ be the ring of polynomials in countably-many variables over a field $\mathbf{K}$ of characteristic 0. We consider the graded structure on $\mathcal{R}$ obtained from assigning to x_i the weight i, *i.e.*, $\mathcal{R} = \oplus_{n\geq 0} R_n$ where $R_0 = \mathbf{K}$ and R_n is the $\mathbf{K}$-vector space with a basis given by the monomials $x_{i_1} \cdots x_{i_s}$ (we can assume that $i_1 \geq i_2 \cdots \geq i_s > 0$) such that $i_1 + \cdots + i_s = n$. Monomials of weight n are therefore trivially in bijection with partitions of n, so that the exponent of a variable x_i in some monomial is identified with the multiplicity of the part i in the corresponding partition. It follows that the Hilbert–Poincaré series $HP_{\mathcal{R}}$ of $\mathcal{R}$ is given by

$$HP_{\mathcal{R}}(q) := \sum_{n\geq 0} \dim_{\mathbf{K}} R_n q^n = \sum_{n \geq 0} p(n) q^n,$$

where $p(n)$ is the number of partitions of n.

Let $[x_1^r]$ be the differential ideal generated by x_1^r and its iterated derivative with respect to the derivation D defined by $D(x_j) := x_{j+1}$. Thus we have

$$[x_1^r] = (x_1^r, r x_1^{r-1} x_2, r(r-1) x_1^{r-2} x_2^2 + r x_1^{r-1} x_3, \ldots).$$

It follows from [11, 12] that for integers $1 \leq i \leq r-1$, the leading ideal of $J := (x_1^i, [x_1^r])$ with respect to the "weighted reverse lexicographical order", that is the ideal generated by the leading monomials of all the elements in J, is

$$J_{r,i} = (x_1^i, x_k^{r-s} x_{k+1}^s; k \geq 1; s = 0, \ldots, r-1).$$

One should recall that the leading ideal is in general not generated by the leading monomials of a system of generators; a system of generators of an ideal I such that the leading monomials of its members generate the leading ideal of I is called a Gröbner basis. Note that the Hilbert–Poincaré series of a graded ring quotiented by an ideal I is equal to the Hilbert–Poincaré series of the ring quotiented by the leading ideal of I with respect to any monomial ordering which is compatible with the grading. As the monomials in $\mathcal{R}/J_{r,i}$ (i.e the monomials in $\mathcal{R}$ which do not belong to $J_{r,i}$) correspond exactly to the partitions in $\mathcal{T}_{r,i}$, we conclude

$$HP_{\mathcal{R}/J}(q) = HP_{\mathcal{R}/J_{r,i}}(q) = 1 + \sum_{n \geq 1} T_{r,i}(n) q^n.$$

In [2] (see [3] for the case $r = 2$), the first author predicted (from experimentations) that the leading ideal of J with respect to the weighted lexicographical order is equal to the ideal $I_{r,i} \subset \mathbf{K}[x_1, x_2, \ldots]$ generated by x_1^i and the monomials of the following form:

$$\underbrace{x_{n_{1,1}}}_{\text{first block}} \underbrace{x_{n_{2,1}} \cdots x_{n_{2,f_{r,i}(2)}}}_{\text{second block}} \underbrace{x_{n_{3,1}} \cdots x_{n_{3,f_{r,i}(3)}}}_{\text{third block}} \cdots \underbrace{x_{n_{r,1}} \cdots x_{n_{r,f_{r,i}(r)}}}_{r\text{-th block}},$$

where

$$f_{r,i}(j) := \begin{cases} 1 & \text{if } j = 1, \\ n_{j-1, f_{r,i}(j-1)} & \text{if } 2 \leq j \leq i, \\ n_{j-1, f_{r,i}(j-1)} - 1 & \text{if } i+1 \leq j \leq r. \end{cases}$$

The set of partitions $\mathcal{C}_{r,i}$ introduced in [2] corresponds to the set of monomials in the quotient ring $\mathcal{R}/I_{r,i}$. More precisely, given an integer $r \geq 2$, for $1 \leq i \leq r$ we define the (i, ℓ)-new part of $\lambda = (\lambda_1, \ldots, \lambda_s)$ as follows:

$$p_{i,\ell}(\lambda) := \begin{cases} \lambda_s & \text{if } \ell = 1, \\ \lambda_{s - \sum_{j=1}^{\ell-1} p_{i,j}(\lambda)} & \text{if } 2 \leq \ell \leq i, \\ \lambda_{s+\ell-i-\sum_{j=1}^{\ell-1} p_{i,j}(\lambda)} & \text{if } i < \ell \leq r-1, \end{cases}$$

where $\lambda_j = 0$ for $j \leq 0$, and if $p_{i,\ell}(\lambda) = 0$ then $p_{i,j}(\lambda) = 0$ for $j > \ell$. We denote the number of all non-zero (i, ℓ)-new parts of λ by $N_{r,i}(\lambda)$.

Conjecture 1.3 ([2]). *Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. Let $\mathcal{C}_{r,i}$ be the set of partitions of the form $\lambda = (\lambda_1, \dots, \lambda_s)$, such that at most $i-1$ of the parts are equal to 1 and either $N_{r,i}(\lambda) < r-1$, or $N_{r,i}(\lambda) = r-1$ and $s \leq \sum_{j=1}^{r-1} p_{i,j}(\lambda) - (r-i)$. Let n be a nonnegative integer, and denote by $C_{r,i}(n)$ the number of partitions of n which belong to $\mathcal{C}_{r,i}$. Then we have*

$$C_{r,i}(n) = T_{r,i}(n) = E_{r,i}(n).$$

One of the main results of [1] is a proof of this conjecture, which we survey here.

Theorem 1.4. *Conjecture 1.3 is true.*

The proof of Theorem 1.4 uses a characterization of the set $\mathcal{C}_{r,i}$ in terms of new types of Durfee dissections that were inspired by a companion to the Gordon identities due to Andrews, called the Andrews–Gordon identities [5]. They can be stated for all integers $r \geq 2$ and $1 \leq i \leq r$ as:

$$\sum_{s_1 \geq \cdots \geq s_{r-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{r-1}^2+s_i+\cdots+s_{r-1}}}{(q)_{s_1-s_2} \cdots (q)_{s_{r-2}-s_{r-1}} (q)_{s_{r-1}}} = \frac{(q^{2r+1}, q^i, q^{2r-i+1}; q^{2r+1})_\infty}{(q)_\infty}. \tag{1.3}$$

Here, q is a fixed complex parameter such that $0 < |q| < 1$ (actually, convergence issues are not important here, q can be treated as an indeterminate and series like above can be considered as formal power series), and for any $a \in \mathbb{C}$,

$$(a)_\infty \equiv (a;q)_\infty := \prod_{j \geq 0} (1 - aq^j), \qquad (a)_k \equiv (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty},$$

where k is any integer, and $(a_1, \dots, a_m; q)_k := (a_1)_k \cdots (a_m)_k$ for k an integer or infinity.

Note that the right-hand side of (1.3) is the generating series of $\mathcal{E}_{r,i}$, and that (1.1) (resp. (1.2)) is obtained from (1.3) by taking $r = i = 2$ (resp. $r = 2$ and $i = 1$).

In Section 2 we recall Andrews' Durfee dissection that he used to prove (1.3) and outline the two new Durfee-type dissections that we defined in [1] to rewrite Conjecture 1.3 in a more natural way. Thanks to this, we are able to compute the generating function for the partitions in Conjecture 1.3, which reduces its proof to the proof of the following result similar to (1.3).

Theorem 1.5. *For all integers $r > 0$ and $0 \leq i \leq r-1$, we have:*

$$\sum_{s_1 \geq \cdots \geq s_{r-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{r-1}^2-s_1-\cdots-s_i}(1-q^{s_i})}{(q)_{s_1-s_2} \cdots (q)_{s_{r-2}-s_{r-1}} (q)_{s_{r-1}}} = \frac{(q^{2r+1}, q^{r-i}, q^{r+i+1}; q^{2r+1})_\infty}{(q)_\infty}, \tag{1.4}$$

where for $i = 0$, the term $1 - q^{s_0}$ on the left-hand side is simply taken to be 1.

Note that $i \to r-i$ above gives the same right-hand side as in (1.3), and $(r,i) \to (2,0)$ (resp. $(2,1)$) yields (1.1) (resp. (1.2)).

In Section 3 we prove Theorem 1.5 by using the Bailey lattice established in [4], which is a generalization of the famous Bailey lemma (see for instance [7, 18]), and was used in [4] to give a new proof of the Andrews-Gordon identities (1.3).

2 New Durfee-type dissections and proof strategy

In this section, we recall Andrews' Durfee dissection and introduce the two new aforementioned Durfee-type dissections. They are all illustrated graphically on the same partition in Figure 1, and we will refer to this figure in our three definitions.

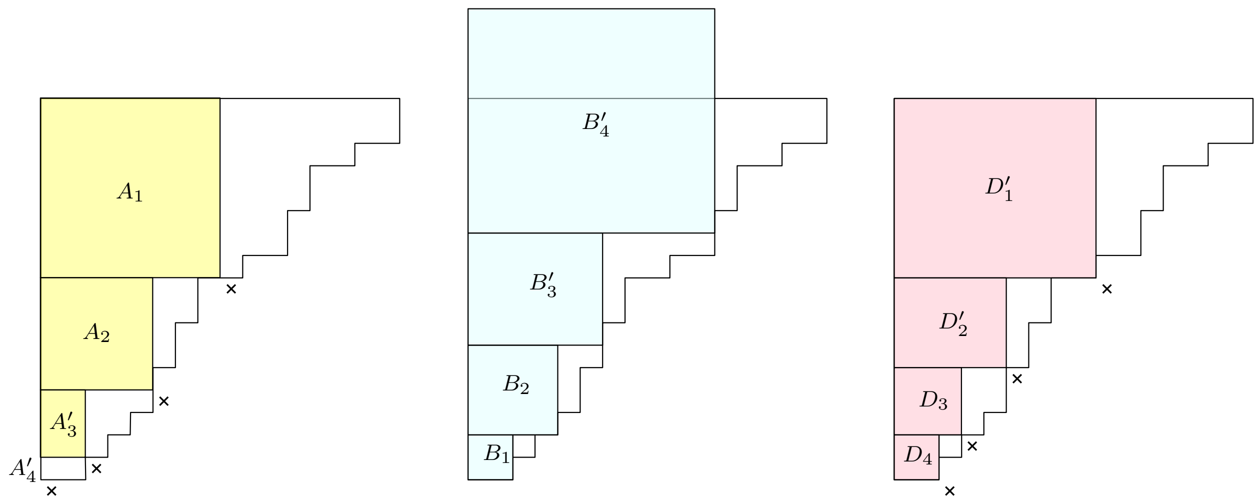

Figure 1: The three types of Durfee dissections

We start by recalling the Durfee dissection which was defined by Andrews in his combinatorial interpretation of the Andrews–Gordon identities (1.3) in [6], and is illustrated on the left of Figure 1. We use a slightly different terminology than his, which will help avoid any confusion with our new types of dissections. Define the Durfee square of a partition λ to be the largest square of size $k \times k$ fitting in the top-left corner of the Young diagram of λ. In Figure 1, A_1 is the Durfee square of the partition.

Similarly we can define its vertical Durfee rectangle to be the largest vertical rectangle of size $(k-1) \times k$ fitting in the top-left corner of its Young diagram.

It is possible to define successive Durfee squares/rectangles by drawing the first Durfee square/rectangle, and then drawing the Durfee square/rectangle of the partition restricted to the parts below it, and repeating the process until the row below a square/rectangle is empty. For convenience in our proofs in [1], we took the convention that we can still draw Durfee squares/rectangles after exiting the partition, but that they are empty.

When we choose that the first $i-1$ Durfee squares/rectangles are squares, and that all the following ones are rectangles, the sequence of non-empty Durfee squares/rectangles in λ is uniquely defined and is called the (vertical) $(i-1)$-Durfee dissection of λ. We denote the successive Durfee squares (resp. rectangles) by $A_1, \ldots, A_{i-1}$ (resp. $A'_i, A'_{i+1}, \ldots$).

The left of Figure 1 shows the vertical 2-Durfee dissection of a partition (the last rectangle A'_4 is of size 0×1, and all rectangles below are empty). The crosses represent

boxes which, by definition of Durfee squares/rectangles, can not belong to the partition when the Durfee dissection is fixed.
Andrews' combinatorial version of the Andrews–Gordon identities is the following.

Theorem 2.1 (Andrews [6]). *Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. Let $\mathcal{A}_{r,i}$ be the set of partitions such that in their vertical $(i-1)$-Durfee dissection, all vertical Durfee rectangles below A'_{r-1} are empty, and* such that the last row of each non-empty Durfee rectangle is actually a part of the partition. *For all nonnegative integers n, denote by $A_{r,i}(n)$ the number of partitions of n which belong to $\mathcal{A}_{r,i}$. Then we have*

$$A_{r,i}(n) = E_{r,i}(n).$$

The partition in Figure 1 is not in $\mathcal{A}_{5,3}$. Indeed, in its vertical 2-Durfee dissection, even though all the vertical Durfee rectangles below A'_4 are empty, the last row of the vertical rectangles A'_3 and A'_4 are not parts of the partition.

Partitions in $\mathcal{A}_{r,i}$ are generated by the left-hand side of (1.3), as shown in [1] by using q-binomial coefficients, which generate partitions in a box.

Inspired by the previous description, we aim to reformulate Conjecture 1.3 by using another type of dissection, represented in the middle of Figure 1. We define the bottom square (resp. bottom rectangle) of a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ to be the square of size $\lambda_s \times \lambda_s$ (resp. the horizontal rectangle of size $\lambda_s \times (\lambda_s - 1)$) whose bottom coincides with the bottom of the Young diagram of λ. On Figure 1, B_1 is the bottom square of the partition.

Just like for Durfee squares, we define successive bottom squares/rectangles by drawing the first bottom square/rectangle, and then drawing the bottom square/rectangle of the partition restricted to the parts above it, and repeating the process until the row above a square/rectangle is empty. For convenience, we take the convention that we can still draw bottom squares/rectangles after exiting the partition, but that they are empty. We also allow bottom rectangles of size 1×0 (this can appear if the smallest part of the partition is a 1).

When we choose that the first $i-1$ bottom squares/rectangles are squares, and all the following ones are rectangles, the sequence of non-empty bottom squares/rectangles in λ is uniquely defined and we call it the $(i-1)$-*bottom dissection* of λ. We denote the successive bottom squares (resp. rectangles) by $B_1, \ldots, B_{i-1}$ (resp. $B'_i, B'_{i+1}, \ldots$). The middle of Figure 1 shows the successive bottom squares/rectangles of a partition, with two successive bottom squares (the bottom rectangles above B'_4 are empty).

Let $\mathcal{B}_{r,i}$ be the set of partitions such that in their $(i-1)$-bottom dissection, all bottom rectangles above B'_{r-1} are empty. In other words, if one draws $i-1$ successive bottom squares $B_1, \ldots, B_{i-1}$ followed by $r-i$ bottom rectangles $B'_i, \ldots, B'_{r-1}$, then the row above B'_{r-1} is empty. Denote by $B_{r,i}(n)$ the number of partitions of n which belong to $\mathcal{B}_{r,i}$. For example, the partition in Figure 1 belongs to $\mathcal{B}_{5,3}$ but not to $\mathcal{B}_{4,3}$.

By definition of bottom squares/rectangles, for all $1 \leq i \leq r$, we have $\mathcal{B}_{r,i} = \mathcal{C}_{r,i}$, so Conjecture 1.3 can be reformulated as follows.

Conjecture 2.2 (Reformulation of Conjecture 1.3)**.** *Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. Then for all nonnegative integers n, we have $B_{r,i}(n) = A_{r,i}(n) = E_{r,i}(n)$.*

Using the same strategy as above, we computed in [1] the generating function for partitions in $\mathcal{B}_{r,i}$, which takes the following form (here empty sums are taken to be 0 and empty products are taken to be 1):

$$1 + \sum_{k=1}^{i-1} \sum_{m_k \geq \cdots \geq m_1 \geq 1} \left(\sum_{m=1}^{m_k} \frac{q^{mm_k}}{(q)_{m-1}} \right) q^{\sum_{\ell=1}^{k-1} m_\ell^2} \frac{(q)_{m_k-1}}{(q)_{m_1-1}} \prod_{\ell=1}^{k-1} \frac{1}{(q)_{m_{\ell+1}-m_\ell}} \tag{2.1}$$
$$+ \sum_{k=i}^{r-1} \sum_{m_k \geq \cdots \geq m_1 \geq 1} \left(\sum_{m=1}^{m_k-1} \frac{q^{mm_k}}{(q)_{m-1}} \right) q^{\sum_{\ell=1}^{k-1} m_\ell^2 - \sum_{\ell=i}^{k-1} m_\ell} (1 - q^{m_i-1}) \frac{(q)_{m_k-2}}{(q)_{m_1-1}} \prod_{\ell=1}^{k-1} \frac{1}{(q)_{m_{\ell+1}-m_\ell}}.$$

As it did not seem clear to us that it is the same generating series as the one in (1.3), we showed that the conjecture is equivalent to a conjecture involving successive Durfee squares and rectangles. In contrast to Andrews' dissection [6], our Durfee rectangles will be horizontal, we will start with rectangles and finish with squares, and we will not have the restriction that the last row of Durfee rectangles have to actually be parts of the partition. This third type of dissection is illustrated on the right of Figure 1.

Define the horizontal Durfee rectangle of a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ to be the largest horizontal rectangle of size $k \times (k-1)$ fitting in the top-left corner of the Young diagram of λ. *From now on, when we mention a Durfee rectangle, we mean horizontal Durfee rectangle.* In Figure 1, D'_1 is the Durfee rectangle of the partition. As we did before, we can define successive Durfee squares/rectangles by drawing the first Durfee square/rectangle, and then drawing the Durfee square/rectangle of the partition restricted to the parts below, and repeating the process until the row below a square/rectangle is empty. Again, we take the convention that we can still draw Durfee squares/rectangles after exiting the partition, but that they are empty. We also allow Durfee rectangles of size 1×0, which are not considered to be empty (this can happen when there is a part equal to 1).

When we choose that the first k Durfee squares/rectangles are rectangles, and that the following are all squares, the sequence of non-empty Durfee squares/rectangles in λ is uniquely defined and is called the *k-Durfee dissection* of λ. We denote the successive Durfee rectangles (resp. squares) by $D'_1, \ldots, D'_k$ (resp. $D_{k+1}, D_{k+2}, \ldots$). The right of Figure 1 shows the 2-Durfee dissection of our partition (the Durfee squares below D_4 are all empty). The crosses show boxes which, by definition of our Durfee squares/rectangles, must be empty.

Now define $\mathcal{D}_{r,i}$ to be the set of partitions such that in their $(r-i)$-Durfee dissection, all Durfee squares below D_{r-1} are empty. In other words, if one draws $r-i$ (horizontal)

Durfee rectangles $D'_1, \ldots, D'_{r-i}$ followed by $i-1$ Durfee squares $D_{r-i+1}, \ldots, D_{r-1}$, then the row below D_{r-1} is empty. For example, the partition in Figure 1 belongs to $\mathcal{D}_{5,3}$ but not to $\mathcal{D}_{4,3}$.

Note that Figure 1 shows a particular partition which belongs both to $\mathcal{B}_{5,3}$ and to $\mathcal{D}_{5,3}$ (but not to $\mathcal{A}_{5,3}$). We showed in [1] that this is a general phenomenon and that the following holds.

Theorem 2.3. *For $r \geq 2$ and $1 \leq i \leq r$ two integers, we have $\mathcal{B}_{r,i} = \mathcal{D}_{r,i}$.*

We omit the proof of this result here, but we refer the reader to [1] for two proofs, one purely combinatorial, and the other algebraic.

Now doing the same as we did above for $\mathcal{A}_{r,i}$ and $\mathcal{B}_{r,i}$, we can compute the generating function for partitions in $\mathcal{D}_{r,i} = \mathcal{B}_{r,i}$ to derive a simpler form than in (2.1); we get (replacing $1 - q^{d_0}$ by 1)

$$\sum_{d_1 \geq \cdots \geq d_{r-1} \geq 0} \frac{q^{d_1^2 + \cdots + d_{r-1}^2 - d_1 - \cdots - d_{r-i}}}{(q)_{d_1 - d_2} \cdots (q)_{d_{r-2} - d_{r-1}} (q)_{d_{r-1}}} (1 - q^{d_{r-i}}). \tag{2.2}$$

It should be possible to prove a weaker version of Theorem 2.3 analytically by showing that (2.1) and (2.2) are equal; however it did not seem obvious to us how it could be done, so we looked for a combinatorial proof instead, which also has the advantage of giving more insight into the different types of dissections.

Now what is left to do in order to prove the conjecture is showing that (2.2) equals the generating function for partitions in $\mathcal{A}_{r,i}$ or $\mathcal{E}_{r,i}$.

In the case of only squares ($i = r$), the partitions in $\mathcal{A}_{r,r}$ and $\mathcal{D}_{r,r}$ are the same by definition. In the case of only rectangles ($i = 1$), there is a simple bijection between $\mathcal{A}_{r,1}$ and $\mathcal{D}_{r,1}$ by rotating the horizontal Durfee rectangles in $\mathcal{A}_{r,1}$ by 90 degrees and thus obtaining partitions in $\mathcal{D}_{r,1}$, and vice versa. However this simple bijection does not work for other values of i, as some problems can appear at the transition between squares and rectangles, and because Andrews' Durfee dissection for $\mathcal{A}_{r,i}$ starts with squares and ends with rectangles while ours for $\mathcal{D}_{r,i}$ does the contrary.

We found in [1] a more complicated bijection in the particular case $i = r-1$, but finding a bijection between $\mathcal{A}_{r,i}$ and $\mathcal{D}_{r,i}$ for all i still eludes us.

Therefore our proof of Conjecture 2.2 will actually consist in showing that the generating function (2.2) of $\mathcal{D}_{r,i}$ equals the infinite product which is the generating function for $\mathcal{E}_{r,i}$. This is done by proving Theorem 1.5. Indeed, the right-hand side (resp. left-hand side) of (1.4) is the generating series for $\mathcal{E}_{r,r-i}$ (resp. $\mathcal{D}_{r,r-i}$) obtained by taking $r-i$ instead of i in the right-hand side of (1.3) (resp. in (2.2)).

This shows that Conjecture 2.2 (and therefore Conjecture 1.3) is an immediate consequence of Theorem 1.5 and Theorem 2.3.

3 Proof of Theorem 1.5 via the Bailey lattice

Fix a formal indeterminate a. Recall [7] that a Bailey pair $(\alpha_n, \beta_n)_{n\geq 0}$ related to a is a pair of sequences satisfying:

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}} \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

The Bailey lemma describes how, from a Bailey pair, one can produce infinitely many of them. We do not give this result in full generality, but only the following limit case.

Theorem 3.1 (Bailey lemma, special case)**.** *If (α_n, β_n) is a Bailey pair related to a, then so is (α'_n, β'_n), where*

$$\alpha'_n = a^n q^{n^2} \alpha_n \quad \textit{and} \quad \beta'_n = \sum_{j=0}^{n} \frac{a^j q^{j^2}}{(q)_{n-j}} \beta_j.$$

In [7], the following unit Bailey pair (related to a) is considered:

$$\alpha_n^{(0)} = \frac{(-1)^n q^{n(n-1)/2}(1-aq^{2n})(a)_n}{(1-a)(q)_n}, \qquad \beta_n^{(0)} = \delta_{n,0}, \tag{3.2}$$

and two iterations of Theorem 3.1 applied to (3.2) yield (1.1) and (1.2) by taking $a = 1$ and $a = q$. More generally, iterating $r \geq 2$ times Theorem 3.1 for the unit Bailey pair (3.2) yields a new Bailey pair $(\alpha_n^{(r)}, \beta_n^{(r)})$ with

$$\alpha_n^{(r)} = a^{rn} q^{rn^2} \alpha_n^{(0)},$$

and

$$\beta_n^{(r)} = \sum_{n\geq s_1\geq\cdots\geq s_r\geq 0} \frac{a^{s_1+\cdots+s_r} q^{s_1^2+\cdots+s_r^2}}{(q)_{n-s_1}(q)_{s_1-s_2}\ldots(q)_{s_{r-1}-s_r}} \beta_{s_r}^{(0)}.$$

Applying the definition (3.1) to this Bailey pair and letting $n \to \infty$ gives

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0} \frac{a^{s_1+\cdots+s_{r-1}} q^{s_1^2+\cdots+s_{r-1}^2}}{(q)_{s_1-s_2}\ldots(q)_{s_{r-1}}} = \frac{1}{(aq)_\infty} \sum_{j\geq 0} a^{rj} q^{rj^2} (-1)^j q^{j(j-1)/2} \frac{(1-aq^{2j})(a)_j}{(1-a)(q)_j}.$$

Now taking $a = 1$, the right-hand side of this formula is equal to

$$\begin{aligned} \frac{1}{(q)_\infty}\left(1 + \sum_{j\geq 1} q^{rj^2}(-1)^j q^{j(j-1)/2}(1+q^j)\right) &= \frac{1}{(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{(2r+1)j^2/2} q^{j/2} \\ &= \frac{(q^{2r+1}, q^r, q^{r+1}; q^{2r+1})_\infty}{(q)_\infty}, \end{aligned}$$

by taking $q \to q^{2r+1}$, $z \to q^r$ in the Jacobi triple product identity [13, Appendix, (II.28)]

$$\sum_{j\in\mathbb{Z}} (-1)^j z^j q^{j(j-1)/2} = (q, z, q/z; q)_\infty. \tag{3.3}$$

Therefore we get the $i = 0$ case of (1.4) (equivalently the $i = r$ instance of (1.3)). In the same way, one gets the $i = r - 1$ case of (1.4) (equivalently the $i = 1$ instance of (1.3)) by choosing $a = q$ above.

This method is an efficient way to show these two instances of the Andrews–Gordon identities, but it fails when one aims to prove them in such a direct way for general i. The concept of Bailey lattice was therefore developed in [4] to prove (1.3) for general i in a similar fashion (see also [8, 10] for alternative methods avoiding the use of the Bailey lattice). In [4], the authors change the parameter a at some point before iterating the Bailey lemma, therefore providing a concept of Bailey lattice instead of the above classical Bailey chain. Here is the tool proved in [4] (again we only highlight the limit case that we need).

Theorem 3.2 (Bailey lattice, special case)**.** *If (α_n, β_n) is a Bailey pair related to a, then (α'_n, β'_n) is a Bailey pair related to a/q, where $\alpha'_0 = \alpha_0$,*

$$\alpha'_n = (1-a)a^n q^{n^2-n}\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right) \quad \text{and} \quad \beta'_n = \sum_{j=0}^{n} \frac{a^j q^{j^2-j}}{(q)_{n-j}}\beta_j.$$

For our purpose, we will use the following consequence of Theorem 3.2 obtained in [4, Corollary 4.2] by iterating $r - i$ times Theorem 3.1, then using Theorem 3.2, and finally $i - 1$ times Theorem 3.1 with a replaced by a/q, and at the end letting $n \to \infty$.

Corollary 3.3. *If (α_n, β_n) is a Bailey pair related to a, then for all integers $0 \le i \le r$, we have:*

$$\sum_{s_1\ge\cdots\ge s_r\ge 0} \frac{a^{s_1+\cdots+s_r} q^{s_1^2+\cdots+s_r^2-s_1-\cdots-s_i}}{(q)_{s_1-s_2}\ldots(q)_{s_{r-1}-s_r}}\beta_{s_r} = \frac{1}{(a)_\infty} \times \left(\alpha_0 + \sum_{j\ge1}(1-a)a^{ij}q^{i(j^2-j)}\right.$$
$$\left.\left(\frac{a^{(r-i)j}q^{(r-i)j^2}\alpha_j}{1-aq^{2j}} - \frac{a^{(r-i)(j-1)+1}q^{(r-i)(j-1)^2+2j-2}\alpha_{j-1}}{1-aq^{2j-2}}\right)\right). \tag{3.4}$$

By applying Corollary 3.3 to the unit Bailey pair (3.2) with $a = q$, (1.3) is proved in [4], after factorizing the right-hand side by (3.3) and replacing i by $i - 1$.

Now we are ready to prove Theorem 1.5 in the same spirit.

Proof of Theorem 1.5. First notice that the left-hand side of (1.4) can be written as $A_i(q) - A_{i-1}(q)$, where the dependence on r is omitted, $A_{-1}(q) := 0$, and for $0 \le i \le r - 1$,

$$A_i(q) := \sum_{s_1\ge\cdots\ge s_{r-1}\ge 0} \frac{q^{s_1^2+\cdots+s_{r-1}^2-s_1-\cdots-s_i}}{(q)_{s_1-s_2}\ldots(q)_{s_{r-1}}}.$$

We want to use Corollary 3.3 with the unit Bailey pair (3.2) with $a = 1$ to compute $A_i(q)$. To do this, we first rewrite the right-hand side of (3.4) by shifting the index j to $j+1$ in the summation involving α_{j-1}:

$$\frac{1-a^{i+1}}{(a)_\infty}\alpha_0 + \frac{1-a}{(a)_\infty}\sum_{j\geq 1} a^{ij}q^{i(j^2-j)}\frac{a^{(r-i)j}q^{(r-i)j^2}\alpha_j}{1-aq^{2j}}(1-a^{i+1}q^{(2i+2)j}). \tag{3.5}$$

Now using $(1-a)/(a)_\infty = 1/(aq)_\infty$ and taking the unit Bailey pair (3.2) with $a = 1$ yields

$$A_i(q) = \frac{1}{(q)_\infty}\left(i+1+\sum_{j\geq 1}(-1)^j q^{rj^2-ij+j(j-1)/2}\frac{1-q^{(2i+2)j}}{1-q^j}\right).$$

We then obtain that, for $0 \leq i \leq r-1$, $A_i(q) - A_{i-1}(q)$ is equal to

$$\begin{aligned}
&\frac{1}{(q)_\infty}\left(1+\sum_{j\geq 1}(-1)^j q^{rj^2+j(j-1)/2}\frac{q^{-ij}-q^{ij+2j}-q^{-ij+j}+q^{ij+j}}{1-q^j}\right)\\
&= \frac{1}{(q)_\infty}\left(1+\sum_{j\geq 1}(-1)^j q^{rj^2+j(j-1)/2}(q^{-ij}+q^{ij+j})\right)\\
&= \frac{1}{(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{(2r+1)j^2/2}q^{ij+j/2}\\
&= \frac{1}{(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{(2r+1)j(j-1)/2}q^{(i+r+1)j}.
\end{aligned}$$

Finally, by using (3.3) with q replaced by q^{2r+1} and $z = q^{i+r+1}$, we get our conclusion:

$$A_i(q) - A_{i-1}(q) = \frac{(q^{2r+1}, q^{i+r+1}, q^{r-i}; q^{2r+1})_\infty}{(q)_\infty}.$$

□

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