Séminaire Lotharingien de Combinatoire **86B** (2022)
Article #5, 12 pp.

Proceedings of the 34^{th} Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)

Triangulations, Order Polytopes, and Generalized Snake Posets

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Abstract. This work regards the order polytopes arising from the class of generalized snake posets and their posets of meet-irreducible elements. Among generalized snake posets of the same rank, we characterize those whose order polytopes have minimal and maximal volume. We give a combinatorial characterization of the circuits in these order polytopes and then conclude that every triangulation is unimodular. For a generalized snake word, we count the number of flips for the canonical triangulation of these order polytopes. We determine that the flip graph of the order polytope of the poset whose lattice of filters comes from a ladder is the Cayley graph of a symmetric group. Lastly, we introduce an operation on triangulations called twists and prove that twists preserve regular triangulations.

Keywords: triangulation, order polytope, meet-irreducible, generalized snake poset

1 Introduction

Stanley [8] introduced two geometric objects associated to a finite partially ordered set, or *poset*, known as the order polytope and the chain polytope. It is well-known that the set of all regular triangulations of a polytope correspond to the vertices of its secondary polytope, and that these triangulations are connected via flips. Various triangulations of order polytopes have been constructed or considered, often for special classes of posets. See, for example, Santos, Stump, and Welker for products of chains [7], Féray and Reiner for non-unimodular triangulations related to graph-associahedra [4], Bränden and Solus for **s**-lecture hall order polytopes [2], and others. However, the general space of regular

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triangulations of an order polytope, *i.e.*, the 1-skeleton of the secondary polytope of an order polytope, does not appear to have been studied in detail and motivates our work.

Our contributions in this paper add to the literature on order polytopes and further the study of the general space of regular triangulations of order polytopes. Specifically, we investigate circuits, flips, and regular triangulations of order polytopes arising from a certain class of posets, called generalized snake posets. This article is an extended abstract of [1] and is organized as follows. In Section 2, we review some background and establish notation. In Section 3, we introduce the family of generalized snake posets P and study volumes of their corresponding order polytopes. The characterization of circuits of the order polytope $\mathcal{O}(Q)$ of the poset of filters of P is given in Section 4. Lastly, Section 5 is devoted to introducing twists, and then presenting four theorems regarding twists, flips, and triangulations of $\mathcal{O}(Q)$.

2 Background and Notation

2.1 Triangulations

Definition 2.1. Given a point configuration $\mathbf{A} \subseteq \mathbb{R}^d$, let $\text{conv}(\mathbf{A})$ denote the convex hull of $\mathbf{A}$. A *triangulation* of $\mathbf{A}$ is a collection $\mathcal{T}$ of d-simplices all of whose vertices are points in $\mathbf{A}$ that satisfies that the union of all of these simplices equals $\text{conv}(\mathbf{A})$ (*Union Property*) and that any pair of these simplices intersects in a (possibly empty) common face (*Intersection Property*).

A triangulation is *unimodular* if every simplex has normalized volume one. A triangulation of a point configuration $\mathbf{A} \subseteq \mathbb{R}^d$ is *regular* if it can be obtained by projecting the lower envelope of a lifting of $\mathbf{A}$ from $\mathbb{R}^{d+1}$.

Definition 2.2. A point configuration $\mathbf{A}$ with index set J has *corank one* if and only if it has an affine dependence relation $\sum_{j \in J} \lambda_j \mathbf{v}_j = 0$ with $\sum_{j \in J} \lambda_j = 0$ that is unique up to multiplication by a constant. This affine dependence partitions J into three subsets:

$$J_+ := \{j \in J : \lambda_j > 0\}, J_0 := \{j \in J : \lambda_j = 0\}, \text{ and } J_- := \{j \in J : \lambda_j < 0\}.$$

In the case when $\mathbf{A}$ has corank one, J_+ and J_- are the only disjoint subsets of J with the property that their relative interiors intersect at the point $\sum\limits_{j \in J_+} \lambda_j \mathbf{v}_j = \sum\limits_{j \in J_-} |\lambda_j| \mathbf{v}_j$, where the λ_j are assumed to be normalized so that $\sum_{j \in J_+} \lambda_j = \sum_{j \in J_-} |\lambda_j| = 1$. The set $J_+ \bigcup J_-$ is called a *circuit* in J and the pair (J_+, J_-) is called the *oriented circuit*, or *Radon partition* , of $\mathbf{A}$.

Definition 2.3. Let $\mathbf{A}$ be a point configuration with index set J. In general, a subset Z of J is a *circuit* if it is a minimal dependent set (that is, it is dependent but every proper subset is independent). Let (Z_+, Z_-) be a partition of Z, such that $\text{conv}(Z_+) \cap \text{conv}(Z_-)$

is nonempty. The partition (Z_+, Z_-) is called an *oriented circuit*. We say the circuit is of *type* $(|Z_+|, |Z_-|)$.

From the circuits we can generate triangulations by using flips to locally transform one triangulation into another.

Lemma 2.4 ([3, Lemma 2.4.2]). *Let* $\mathbf{A}$ *be a point configuration of corank one and* $J = J_+ \cup J_0 \cup J_-$ *be its label set, partitioned by the unique oriented circuit of* $\mathbf{A}$. *Then the following are the only two triangulations of* $\mathbf{A}$: $\mathcal{T}_+ = \{J \setminus \{j\} : j \in J_+\}$, *and* $\mathcal{T}_- = \{J \setminus \{j\} : j \in J_-\}$.

Triangulation of $\mathbf{A}$ is a simplicial complex on $\mathbf{A}$. Recall that an *(abstract) simplicial complex* Δ on a set X is a collection of subsets of X such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of a simplicial complex are called *faces* and a subcomplex Δ' of Δ is a subcollection of Δ which is also a simplicial complex. The *link* of a face $\sigma \in \Delta$ is the simplicial complex $\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cup \tau \in \Delta \text{ and } \sigma \cap \tau = \varnothing\}$. If Δ and Δ' are simplicial complexes, then their *join* is $\Delta * \Delta' = \{\sigma \cup \sigma' : \sigma \in \Delta \text{ and } \sigma' \in \Delta'\}$.

Theorem 2.5 ([3, Theorem 4.4.1]). *Let* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *be two triangulations of a point configuration* $\mathbf{A}$. *Then* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *differ by a flip if and only if there is a circuit* Z *of* $\mathbf{A}$ *such that*

(i) They contain, respectively, the two triangulations $\mathcal{T}_Z^+$ *and* $\mathcal{T}_Z^-$ *of* Z.

(ii) All the maximal simplices of $\mathcal{T}_Z^+$ *and* $\mathcal{T}_Z^-$ *have the same link* L *in* $\mathcal{T}_1$.

(iii) Removing the subcomplex $\mathcal{T}_Z^+ * \mathrm{L}$ *from* $\mathcal{T}_1$ *and replacing it by* $\mathcal{T}_Z^- * \mathrm{L}$ *gives* $\mathcal{T}_2$.

Two triangulations of $\mathbf{A}$ are *adjacent* if they differ by a flip. The set of all triangulations of $\mathbf{A}$, under adjacency by flips, forms the *graph of triangulations*, or *flip graph*, of $\mathbf{A}$.

In Sections 4 and 5, we will take a look at the *secondary polytope* whose vertices are in bijection with regular triangulations of a point configuration. Recall that we can define for each triangulation of a point configuration $\mathbf{A}$ a GKZ-vector. As stated in the following definition, the convex hull of the GKZ-vectors for $\mathbf{A}$ is the secondary polytope. See [3, Section 5.1] for a further discussion of secondary polytopes and GKZ-vectors.

Definition 2.6 (Secondary Polytope). For a point configuration $\mathbf{A}$ the secondary polytope of $\mathbf{A}$ is $\mathrm{conv}\{\varphi_{\mathbf{A}}(\mathcal{T}) : \mathcal{T} \text{ regular triangulation of } \mathbf{A}\}$, where $\varphi_{\mathbf{A}}(\mathcal{T})$ represents the GKZ-vector of $\mathcal{T}$ in $\mathbf{A}$.

The flip graph, which is the graph of all triangulations connected by flips, is in general not connected, but the flip graph of regular triangulations is connected and contains the 1-skeleton of the secondary polytope as a spanning subgraph.

2.2 Order polytopes

Let P be a poset on the set of elements $[d] := \{1, \ldots, d\}$. We abuse notation and write P to denote the elements of P. The *order polytope* of P, introduced by Stanley [8], is defined as

$$\mathcal{O}(P) = \left\{ \mathbf{x} = (x_1, \ldots, x_d) \in [0,1]^d : x_i \leq x_j \text{ for } i <_P j \right\}.$$

An *upper order ideal of* P, also called a *filter*, is a set $A \subseteq P$ such that if $i \in A$ and $i <_P j$, then $j \in A$. Let $J(P)$ denote the poset of upper order ideals of P, ordered by reverse inclusion. We use $\langle p_1, \ldots, p_k \rangle$ to denote the ideal generated by elements $p_1, \ldots, p_k \in P$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the standard basis vectors of $\mathbb{R}^d$. For an upper order ideal $A \in J(P)$, define the *characteristic vector* $\mathbf{v}_A := \sum_{i \in A} \mathbf{e}_i$. The vertices of $\mathcal{O}(P)$ are given by

$$V(\mathcal{O}(P)) = \{\mathbf{v}_A : A \in J(P)\}.$$

Define a hyperplane $\mathcal{H}_{i,j} = \{\mathbf{x} \in \mathbb{R}^d : x_i = x_j\}$ for $1 \leq i < j \leq d$. The set of all such hyperplanes, called the d-dimensional braid arrangement of type A, induces a triangulation $\mathcal{T}$ of $\mathcal{O}(P)$ known as the *canonical triangulation*, which has the following three fundamental properties: $\mathcal{T}$ is unimodular, the simplices are in bijection with the linear extensions of P, so the normalized volume of the order polytope $\text{vol}(\mathcal{O}(P))$ is equal to the number of linear extensions of P, and the simplex corresponding to a linear extension $(a_1, \ldots, a_d)$ of P is

$$\sigma_{a_1,\ldots,a_d} = \left\{ \mathbf{x} \in [0,1]^d : x_{a_1} \leq x_{a_2} \leq \cdots \leq x_{a_d} \right\},$$

with vertex set $\{\mathbf{0}, \mathbf{e}_{a_d}, \mathbf{e}_{a_{d-1}} + \mathbf{e}_{a_d}, \ldots, \mathbf{e}_{a_1} + \cdots + \mathbf{e}_{a_d} = \mathbf{1}\}$.

3 Generalized snake posets

We introduce the family of generalized snake posets $P(\mathbf{w})$, which are distributive lattices with width two, and give a recursive formula for the normalized volume of the order polytope of $P(\mathbf{w})$.

Definition 3.1. For $n \in \mathbb{Z}_{\geq 0}$, a *generalized snake word* is a word of the form $\mathbf{w} = w_0 w_1 \cdots w_n$ where $w_0 = \varepsilon$ is the empty letter and w_i is in the alphabet $\{L, R\}$ for $i = 1, \ldots, n$. The *length* of the word is n, which is the number of letters in $\mathbf{w}$ which belongs to $\{L, R\}$.

Definition 3.2. Given a generalized snake word $\mathbf{w} = w_0 w_1 \cdots w_n$, we define the *generalized snake poset* $P(\mathbf{w})$ recursively in the following way:

- $P(w_0) = P(\varepsilon)$ is the poset on elements $\{0, 1, 2, 3\}$ with cover relations $1 \prec 0$, $2 \prec 0$, $3 \prec 1$ and $3 \prec 2$.

- $P(w_0w_1\cdots w_n)$ is the poset $P(w_0w_1\cdots w_{n-1}) \cup \{2n+2, 2n+3\}$ with the added cover relations $2n+3 \prec 2n+1$, $2n+3 \prec 2n+2$, and

$$\begin{cases} 2n+2 \prec 2n-1, & \text{if } n=1 \text{ and } w_n = L, \text{ or } n \geq 2 \text{ and } w_{n-1}w_n \in \{RL, LR\}, \\ 2n+2 \prec 2n, & \text{if } n=1 \text{ and } w_n = R, \text{ or } n \geq 2 \text{ and } w_{n-1}w_n \in \{LL, RR\}. \end{cases}$$

In this definition, the minimal element of the poset $P(\mathbf{w})$ is $\widehat{0} = 2n+3$, and the maximal element of the poset is $\widehat{1} = 0$.

If $\mathbf{w} = w_0w_1\cdots w_n$ is a generalized snake word of length n, then $P(\mathbf{w})$ is a distributive lattice of width two and rank $n+2$. We point out two special cases of generalized snake posets. For the length n word $\varepsilon LRLR\cdots$, $S_n := P(\varepsilon LRLR\cdots)$ is the *snake poset*, and for the length n word $\varepsilon LLLL\cdots$, $\mathcal{L}_n := P(\varepsilon LLLL\cdots)$ is the *ladder poset*. For an example, refer to Figure 1.

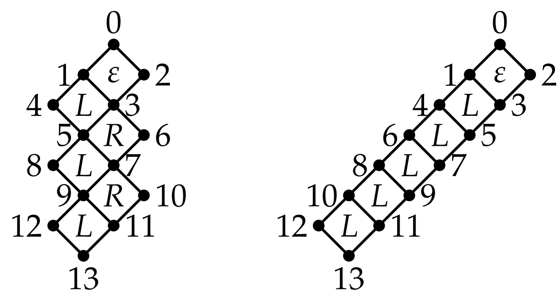

Figure 1: The snake poset $S_5 = P(\varepsilon LRLRL)$ and the ladder poset $\mathcal{L}_5 = P(\varepsilon LLLLL)$.

3.1 Volume of the order polytope of generalized snake posets

Recall that the volume of an order polytope $\mathcal{O}(P)$ is determined by the number of linear extensions of the poset P. Thus, to study the volume of $\mathcal{O}(P(\mathbf{w}))$ we consider the recursive structure of the poset of upper order ideals of $P(\mathbf{w})$. Because of the definition of the generalized snake poset $P(\mathbf{w})$, the minimal element of $J(P(\mathbf{w}))$ is $\widehat{0} = \langle 2n+3 \rangle = P(\mathbf{w})$ and the maximal element is $\widehat{1} = \varnothing$.

Lemma 3.3. *Let $\mathbf{w} = w_0w_1\cdots w_n$ be a generalized snake word. If $k \geq 0$ is the largest index such that $w_k \neq w_n$, then $J(P(\mathbf{w})) =$*

$$J(P(w_0w_1\cdots w_{n-1})) \cup \{\langle 2n+3\rangle, \langle 2n+2\rangle, \langle 2n+2, 2k+2\rangle\} \cup \{\langle 2n+2, 2k+2i+1\rangle\}_{i=1}^{n-k}.$$

Remark 3.4. Thus, we see that $J(P(\mathbf{w}))$ can be constructed by adding a chain of $n-k+3$ elements to the bottom of $J(P(w_0w_1\cdots w_{n-1}))$. In the Hasse diagram for $J(P(\mathbf{w}))$, this corresponds to drawing a strip of $n-k+1$ squares. See Figure 2 for an illustration.

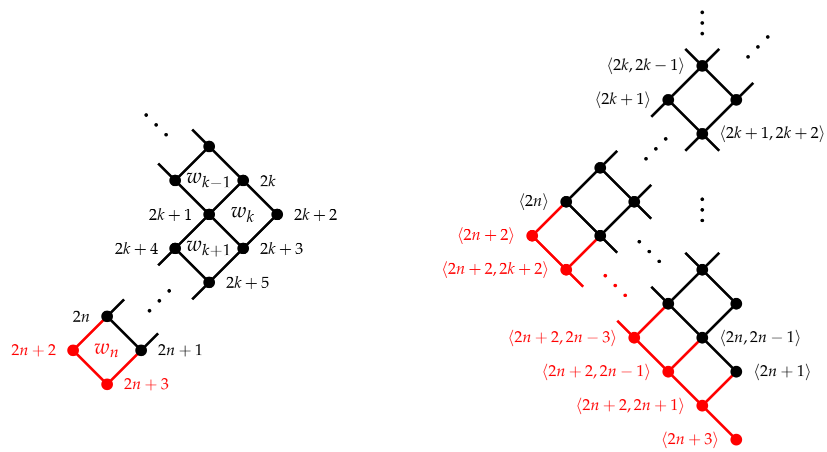

Figure 2: An illustration of Lemma 3.3. On the left is a portion of a generalized snake poset $P(\mathbf{w})$ and on the right is the corresponding poset of upper order ideals $J(P(\mathbf{w}))$. To construct $J(P(\mathbf{w}))$ from $J(P(w_0 \cdots w_{n-1})$ is to add $n-k+3$ elements, with cover relations shown in red in the Hasse diagram on the right.

The normalized volume of the order polytope $\mathcal{O}(P(\mathbf{w}))$ can be computed by a recursive formula involving Catalan numbers.

Definition 3.5. For $m \geq 0$, the *m-th Catalan number* is $\operatorname{Cat}(m) = \frac{1}{m+1}\binom{2m}{m}$.

Theorem 3.6. *For $n \geq 0$, let $\mathbf{w} = w_0 w_1 \cdots w_n$ be a generalized snake word. If $k \geq 0$ is the largest index such that $w_k \neq w_n$, then the normalized volume v_n of $\mathcal{O}(P(\mathbf{w}))$ is given recursively by $v_n = \operatorname{Cat}(n-k+1)v_k + (\operatorname{Cat}(n-k+2) - 2 \cdot \operatorname{Cat}(n-k+1))\, v_{k-1}$, with $v_{-1} = 1$ and $v_0 = 2$.*

For the snake poset $S_n = P(\varepsilon LRLR \cdots)$, the letters alternate so we have $n-k=1$ at every step. In the case of the ladder poset $\mathcal{L}_n = P(\varepsilon LLLL \cdots)$, we have $k=0$ at every step, and hence we have the following corollary.

Corollary 3.7. *The normalized volume of $\mathcal{O}(S_n)$ with $n \geq 0$ is given recursively by $v_n = 2v_{n-1} + v_{n-2}$, with $v_{-1} = 1$ and $v_0 = 2$. These are the Pell numbers. Also, the normalized volume of $\mathcal{O}(\mathcal{L}_n)$ with $n \geq 0$ is given by $v_n = \operatorname{Cat}(n+2)$.*

We end this section by showing that the normalized volume of an order polytope $\mathcal{O}(P(\mathbf{w}))$ of a generalized snake poset is bounded above and below by the volume of the order polytope of the ladder poset and the snake poset, respectively.

Theorem 3.8. *For any generalized snake word $\mathbf{w} = w_0 w_1 \cdots w_n$ of length n,*

$$\operatorname{vol} \mathcal{O}(S_n) \leq \operatorname{vol} \mathcal{O}(P(\mathbf{w})) \leq \operatorname{vol} \mathcal{O}(\mathcal{L}_n).$$

4 A combinatorial interpretation of circuits

In the remainder of this article, we study the properties of the order polytope of a poset $Q_{\mathbf{w}}$ whose lattice of filters is a generalized snake poset.

Define $\widehat{P}(\mathbf{w})$ to be the generalized snake poset $P(\mathbf{w})$ with $\widehat{0}$ and $\widehat{1}$ adjoined, and when $\mathbf{w}$ is clear from context we write $\widehat{P}$. Given $\mathbf{w} = w_0 w_1 \cdots w_n$, $\widehat{P} = \widehat{P}(\mathbf{w})$ is a distributive lattice with order $2n+6$ because $\widehat{P}$ does not contain a copy of the smallest non-modular lattice with five elements and does not contain a sublattice isomorphic to a three-element antichain with a $\widehat{0}$ and $\widehat{1}$ added. Let $Q_{\mathbf{w}} = \mathrm{Irr}_{\wedge}(\widehat{P})$ denote the poset of meet-irreducibles of $\widehat{P}$. Heuristically, $\mathrm{Irr}_{\wedge}(\widehat{P})$ is obtained from $\widehat{P}$ by removing $\widehat{1}$, and every vertex which is at the bottom of a bounded face in the Hasse diagram. See Figure 3. By the fundamental theorem of finite distributive lattices, $\widehat{P} \cong J(Q_{\mathbf{w}})$, where $J(Q_{\mathbf{w}})$ is the lattice of filters of $Q_{\mathbf{w}}$, ordered by reverse inclusion.

We construct a graph $G = G(\mathbf{w})$ associated to $\widehat{P}$ as follows. If $\mathbf{w} = w_0 w_1 \cdots w_n$, the vertex set of G is $V(G) = \{w_0, w_1, \ldots, w_n\}$. The edge set of G is given by

$$E(G) = \{(w_i, w_{i+1}) : i = 0, \ldots, n-1\} \cup \{(w_i, w_{i+2}) : \text{ if } w_i w_{i+1} w_{i+2} = LLR \text{ or } RRL\}.$$

In other words, G consists of the path of length n on the vertices $w_0, \ldots, w_n$, with a 3-cycle for each turn LLR or RRL in $\mathbf{w}$. We denote the set of nonempty connected induced subgraphs of $G(\mathbf{w})$ by $\mathcal{G}(\mathbf{w})$.

The Hasse diagram of $\widehat{P}$ can be embedded on the plane so that its edges are non-crossing where each bounded face of the embedded Hasse diagram has degree 4 given by the length of the cycle bounding the face. We call the bounded faces the *squares* of $\widehat{P}$. There is a one-to-one correspondence between the squares of $\widehat{P}(\mathbf{w})$ and the letters of $\mathbf{w}$ by realizing $G = G(\mathbf{w})$ as follows. Consider each square in the Hasse diagram $\mathrm{Hasse}(\widehat{P})$ as a vertex, then form an edge between squares when they intersect in the plane, as shown in Figure 3. For each vertex w_i of G, let $\mathrm{Sq}(w_i)$ denote the four elements of $\widehat{P}$ contained in the 4-cycle which bounds the face of $\mathrm{Hasse}(\widehat{P})$ corresponding to w_i.

Next, we study the circuits of the order polytope $\mathcal{O}(Q_{\mathbf{w}})$. Understanding this for arbitrary words $\mathbf{w}$ is a challenge, therefore we instead restrict our attention in this section to the following set of words.

Definition 4.1. Let $\mathcal{V}$ denote the subset of words which do not contain the substring LRL or RLR.

Theorem 4.2 shows that for $\mathbf{w} \in \mathcal{V}$, circuits in the vertices of $\mathcal{O}(Q_{\mathbf{w}})$ have a combinatorial interpretation as the nonempty connected induced subgraphs of the graph $G(\mathbf{w})$.

Theorem 4.2. *Let $\mathbf{w} \in \mathcal{V}$ be a generalized snake word of length n. There exists a bijection $\Gamma \colon \mathcal{G}(\mathbf{w}) \to \mathcal{C}(Q_{\mathbf{w}})$ between the set $\mathcal{G}(\mathbf{w})$ of nonempty connected induced subgraphs of $G(\mathbf{w})$ and the set $\mathcal{C}(Q_{\mathbf{w}})$ of circuits of the vertex set of the order polytope $\mathcal{O}(Q_{\mathbf{w}})$.*

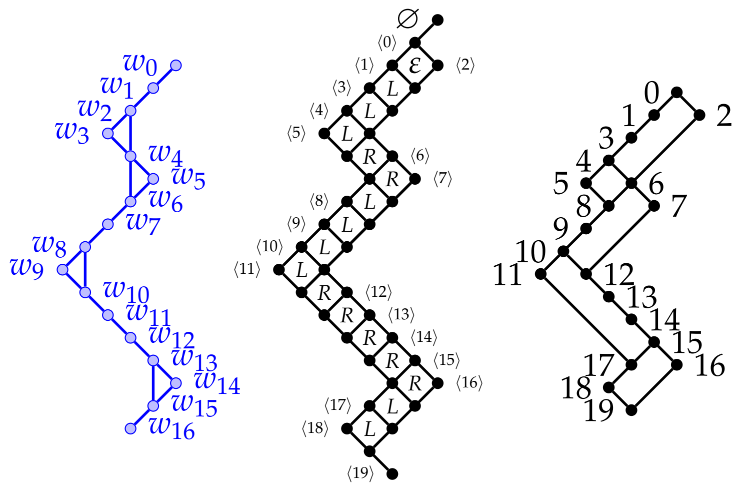

Figure 3: In the center is the lattice $\widehat{P}(\mathbf{w})$ for $\mathbf{w} = \varepsilon L^3 R^2 L^4 R^5 L^2$. Its poset of meet-irreducibles $Q_{\mathbf{w}} = \mathrm{Irr}_{\wedge}(\widehat{P})$ is shown to the right, and the associated graph $G(\mathbf{w})$ is shown to the left.

Remark 4.3. Theorem 4.2 does not hold for a generalized snake word $\mathbf{w}$ outside of $\mathcal{V}$. Computational evidence suggests that the size of $\mathcal{G}(\mathbf{w})$ is an upper bound for the number of circuits of $\mathcal{O}(Q_{\mathbf{w}})$.

Next, we obtain a number of corollaries about the structure of the circuits in the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$.

Corollary 4.4. *Let $\mathbf{w} \in \mathcal{V}$. A circuit Z with partition (Z_+, Z_-) in the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$ has an affine dependence relation of the form $\sum_{j \in Z_+} \mathbf{v}_j = \sum_{j \in Z_-} \mathbf{v}_j$. In particular, $|Z_-| = |Z_+|$.*

Corollary 4.5. *Let $H = \{w_{i_1}, \dots, w_{i_k}\}$ be a connected induced subgraph of G induced by the subword $w_{i_1} \cdots w_{i_k}$ of $\mathbf{w} = w_0 \cdots w_n \in \mathcal{V}$ such that $i_1 < \cdots < i_k$. Suppose $H' = H \cup \{w_{i_j}\}$ is a connected induced subgraph of G such that $i_k < i_j$. Then*

$$\text{if } w_{i_j} = w_{i_k}, \text{ then } |\Gamma(H')| = |\Gamma(H)| \text{ or if } w_{i_j} \neq w_{i_k}, \text{ then } |\Gamma(H')| = |\Gamma(H)| + 2.$$

In the case where $H = \{\varepsilon\}$, $|\Gamma(H')| = |\Gamma(H)|$. Thus, the smallest circuits in the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$ have four vertices. The largest circuits have $4 + 2t$ vertices where t is the number of turns (an occurrence of LLR or RRL) in $\mathbf{w}$.

Using the bijection of Theorem 4.2, we can recursively compute the number of circuits in the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$.

Corollary 4.6. *Let $\mathbf{u} = w_0 \cdots w_{n-1} \in \mathcal{V}$ and $\mathbf{w} = \mathbf{u}w_n \in \mathcal{V}$. Let N_k be the number of connected induced subgraphs of $G(\mathbf{u})$ that contain w_k but not w_{k+1}. Then $|\mathcal{G}(\varepsilon)| = 1$, $|\mathcal{G}(\varepsilon w_1)| = 3$, and (a) if $w_n = w_{n-1}$, then $|\mathcal{G}(\mathbf{w})| = |\mathcal{G}(\mathbf{u})| + N_{n-1} + 1$, or (b) if $w_n \neq w_{n-1}$, then $|\mathcal{G}(\mathbf{w})| = |\mathcal{G}(\mathbf{u})| + N_{n-1} + N_{n-2} + 1$.*

Remark 4.7. When $\mathbf{w} = \varepsilon RRLLRRLL\ldots$, the poset $Q_{\mathbf{w}} = P(\varepsilon RLRLRL\ldots) = S_k$ is the snake poset. The number of circuits of the order polytope of the snake poset is equal to the number of nonempty connected induced subgraphs of the graph TS_{2k+1}. The graph TS_n is known as a *triangular snake graph* [5].

Circuit properties imply the following result regarding triangulations of $\mathcal{O}(Q_{\mathbf{w}})$.

Theorem 4.8. *For $\mathbf{w} \in \mathcal{V}$, every vertex of the secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$ is a unimodular triangulation. Thus, every triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ is unimodular.*

Moreover, all of our computations support the following conjecture.

Conjecture 4.9. *If $\mathbf{w} \in \mathcal{V}$, all triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ are regular.*

5 Flips and a twist action on triangulations

In this section we will take a deeper look at the 1-skeleton of the secondary polytope. Starting from the canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ we will see that for a length k word there are exactly $k+1$ flips, where a single flip corresponds to a local move along an edge in the flip graph. As a consequence, we fully determine the flip graph of regular triangulations in the special case of the ladder. We will also introduce the notion of twists which act globally by inducing automorphisms on the flip graph.

Using the notation from Section 4, let $\mathbf{w}$ be a generalized snake word in $\mathcal{V}$ and consider the associated poset $Q_{\mathbf{w}}$. In this section, we present four theorems about flips of regular triangulations for $\mathcal{O}(Q_{\mathbf{w}})$. First, we classify the flips that can be made from the canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$.

Theorem 5.1. *Let $\mathbf{w} \in \mathcal{V}$ have length k. The canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ admits exactly $k+1$ flips.*

As an application, we determine the flip graph of regular triangulations for the special case of a ladder. When $\mathbf{w} = \varepsilon L^{n-1}$, $\widehat{P}\backslash\{\widehat{0},\widehat{1}\}$ is the product of a $(n+1)$-chain and a 2-chain. Thus, the next result is a rephrasing of the result that the secondary polytope of the Cartesian product of an n-simplex and 1-simplex is an n-dimensional permutahedron [6, Section 16.7.1].

Theorem 5.2. *Let $\mathbf{w} = \varepsilon L^{n-1}$, and $Q_{\mathbf{w}} = \mathrm{Irr}_{\wedge}(\widehat{P}(\mathbf{w}))$. The flip graph of triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ is the Cayley graph of the symmetric group $\mathfrak{S}_{n+1}$ with the simple transpositions as the generating set.*

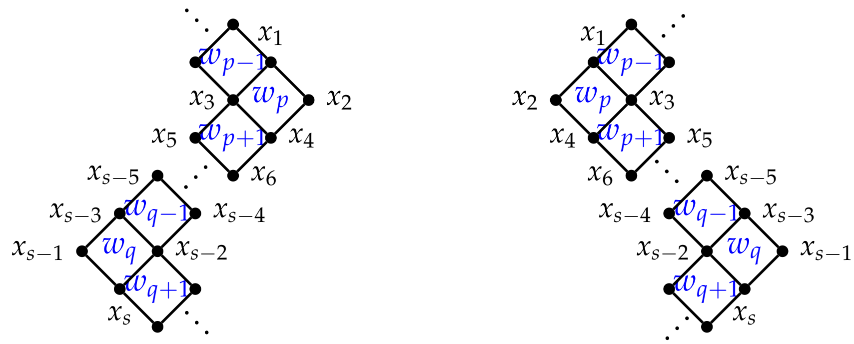

Figure 4: Ladder $\mathcal{L}^i$ in $\widehat{P}$ containing boxes with labels $w_p, \dots, w_q$, where $w_p < w_{p+1} < \cdots < w_q$. The left (right) represents the case where $w_q = L$ ($w_q = R$).

Third, we introduce the following group. Let $\widehat{P} = \widehat{P}(\mathbf{w})$ be defined as in the previous section. We can then think of $\widehat{P}$ as being made up of $\widehat{0}$, $\widehat{1}$, and ladders $\mathcal{L}^1, \dots, \mathcal{L}^t$ for $t \geq 1$ defined as follows. Given the vertices $w_0, \dots, w_k$ of $G(\mathbf{w})$, let w_{i_1} be the first index such that there is an edge from w_{i_1} to w_{i_1+2}. Then $\mathcal{L}^1$ is the ladder in $\widehat{P}$ induced by the elements of $\cup_{j=0}^{i_1+1}\mathrm{Sq}(w_j)$. Let w_{i_2} be the next vertex where there is an edge from w_{i_2} to w_{i_2+2}. Then $\mathcal{L}^2$ is the ladder in $\widehat{P}$ induced by the elements of $\cup_{j=i_1+1}^{i_2+1}\mathrm{Sq}(w_j)$. Inductively define $\mathcal{L}^i$ in a similar fashion. Note that by definition these ladders are disjoint except that $\mathcal{L}^i \cap \mathcal{L}^{i+1}$ is a single square corresponding to a corner box in $\widehat{P}$. That is, $\mathcal{L}^i \cap \mathcal{L}^{i+1}$ comes from the underlined letter $\cdots R\underline{R}L \cdots$ or $\cdots L\underline{L}R \cdots$ in the expression for $\mathbf{w}$. Moreover, we index the ladders so that y, the top element of $\mathcal{L}^1$, is covered by $\widehat{1}$ in $\widehat{P}$. That is, $y \prec \widehat{1}$. Since $\mathbf{w}$ avoids subwords LRL and RLR, each $\mathcal{L}^i$, for $1 < i < t$, consists of at least three squares and $\mathcal{L}^1, \mathcal{L}^t$ consist of at least two squares, except for the case where $\mathbf{w} = \varepsilon$, in which case we have one square and one ladder.

Let V_0 denote the set of vertices of $\widehat{P}$. Next, we define a collection of certain permutations on elements of V_0. Consider the ladder $\mathcal{L}^i$ for $i \in [t]$ in the poset $\widehat{P}$. Then $\mathcal{L}^i$ has the following structure up to a reflection of $\widehat{P}$ in a vertical axis. Label the vertices of $\mathcal{L}^i$ as $x_1, \dots, x_s$ for some even integer s as in Figure 4. In the case where $\mathbf{w} = \varepsilon$, we resolve the ambiguity of the labeling by choosing the convention that the left and right elements in the antichain of the square have labels x_2 and x_3 respectively.

Definition 5.3. Given a ladder $\mathcal{L}^i$, define $\tau_i \in \mathfrak{S}_{|V_0|}$ to be the permutation of V_0 such that for $v \in V_0$,

$$\tau_i(v) = \begin{cases} x_{j-1} & \text{if } v = x_j \text{ and } j \in [s] \text{ is even,} \\ x_{j+1} & \text{if } v = x_j \text{ and } j \in [s] \text{ is odd,} \\ v & \text{otherwise.} \end{cases}$$

Hence, τ_i acts on V_0 by reflecting the vertices of $\mathcal{L}^i$ across a diagonal and fixing the remaining vertices. The next lemma says that the set of τ_i for $i \in [t]$ generate a commutative subgroup of $\mathfrak{S}_{|V_0|}$.

Lemma 5.4. *For all $\tau_i, \tau_r \in \mathfrak{S}_{|V_0|}$, the following properties hold: (a) $\tau_i^2 = 1$ and (b) $\tau_i\tau_r = \tau_r\tau_i$.*

Definition 5.5. Let $\mathfrak{T}(\mathbf{w})$ denote the subgroup of $\mathfrak{S}_{|V_0|}$ generated by the set of the τ_i's. We call $\mathfrak{T}(\mathbf{w})$ *the twist group of* $Q_{\mathbf{w}}$. Elements of $\mathfrak{T}(\mathbf{w})$ are called *twists* and the elements τ_i are called *elementary twists*.

Note that by Lemma 5.4, $\mathfrak{T}(\mathbf{w}) = \langle \tau_i : i \in [t] \rangle$ is isomorphic to $\mathbb{Z}_2^t$. As the next theorem demonstrates, the twist group acts on the component of the flip graph of triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ containing the canonical triangulation, and flips are preserved by twists. A priori, a simplex σ in the triangulation T after twisting becomes a collection of vertices $\tau(\sigma)$ of $\mathcal{O}(Q_{\mathbf{w}})$ that may or may not also form a simplex. Hence, a twist $\tau(\mathcal{T})$ of a triangulation $\mathcal{T}$ is a collection of subsets of vertices obtained by applying the twist τ to every simplex in $\mathcal{T}$, so $\tau(\mathcal{T})$ is not necessarily a triangulation. However, in the case when twisting results in a triangulation, the following theorem says that twists and flips behave well with each other. Further note that twists act on the set of circuits, so $\tau(Z)$ is well-defined.

Recall that if Z is a circuit in $\mathcal{O}(Q_{\mathbf{w}})$ and $\mathcal{T}$ is a triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ that admits a flip using Z, then $\mathcal{T} = \mathcal{T}_Z^+$ and $\mathcal{T}_Z^-$ are the triangulations related by flips at Z. Below $\tau(\mathcal{T}_Z^+) = \tau(\mathcal{T}_Z^+)^+_{\tau(Z)}$ and $\tau(\mathcal{T}_Z^+)^-_{\tau(Z)} = \tau(\mathcal{T}_Z^-)$ denote the two triangulations obtained that related by a flip through $\tau(Z)$.

Theorem 5.6. *Let $\mathbf{w} \in \mathcal{V}$, $Q_{\mathbf{w}} = \mathrm{Irr}_\wedge(\widehat{P}(\mathbf{w}))$, and let $\mathcal{T}$ and $\tau(\mathcal{T})$ be two triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ where τ is a twist. If $\mathcal{T} = \mathcal{T}_Z^+$ can be flipped at circuit Z, then $\tau(\mathcal{T}_Z^+) = \tau(\mathcal{T}_Z^+)^+_{\tau(Z)}$ and $\tau(\mathcal{T}_Z^+)^-_{\tau(Z)} = \tau(\mathcal{T}_Z^-)$. In other words, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{T}_Z^+ \arrow[r, "\text{flip in } Z"] \arrow[d, "\text{twist}"'] & \mathcal{T}_Z^- \arrow[d, "\text{twist}"] \\
\tau(T_Z^+) = \tau(\mathcal{T}_Z^+)^+_{\tau(Z)} \arrow[r, "\text{flip in } \tau(Z)"] & \tau(\mathcal{T}_Z^+)^-_{\tau(Z)} = \tau(\mathcal{T}_Z^-)
\end{tikzcd}$$

Corollary 5.7. *Let $\mathbf{w} \in \mathcal{V}$, $Q_{\mathbf{w}} = \mathrm{Irr}_\wedge(\widehat{P}(\mathbf{w}))$, and let $\mathcal{T}$ and $\tau(\mathcal{T})$ be two triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ where τ is a twist. Then $\mathcal{T}$ and $\tau(\mathcal{T})$ admit the same number of flips.*

Lastly, we show that twists of regular triangulations lead to regular triangulations.

Definition 5.8. If x_i is the k-th element in the canonical order, let $\rho(x_i) = k - 1$. The *canonical height function* is the function $\omega\colon \mathbf{A} \to \mathbb{R}$ given by $\omega(x_i) = 2^{\rho(x_i)}$. Furthermore, we define the *twisted height function* $\omega_\tau\colon \mathbf{A} \to \mathbb{R}$ to be given by $\omega_\tau(x_i) = \omega(\tau(x_i))$. Note that taking $\tau = \mathrm{id}$ gives the canonical height function.

Theorem 5.9. *Let $\mathbf{w} \in \mathcal{V}$, $Q_{\mathbf{w}} = \mathrm{Irr}_\wedge(\widehat{P}(\mathbf{w}))$, and let $\mathcal{T}_{\mathbf{w}}$ be the canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$. Then $\mathcal{T}_{\mathbf{w}}$ is a regular triangulation with height function $\omega(x_i) = 2^{\rho(x_i)}$ defined in Definition 5.8. Furthermore, for any twist τ, $\tau(\mathcal{T}_{\mathbf{w}})$ is a regular triangulation with the corresponding twisted height function.*

When $\mathbf{w} \in \mathcal{V}$, a twist of a canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ again yields a regular triangulation, by Theorem 5.9. Therefore, if Conjecture 4.9 holds, we obtain an action of the twist group on the set of all (regular) triangulations. Hence, the number of triangulations would be divisible by the order of the twist group. In the special case when $Q_{\mathbf{w}} = S_n$ the twist group has order 2^{n+1}. We conjecture the following.

Conjecture 5.10. *The number of regular triangulations of $\mathcal{O}(S_n)$ is $2^{n+1} \cdot \mathrm{Cat}(2n+1)$.*

Acknowledgements

The authors thank Raman Sanyal for helpful discussions about triangulations of order polytopes, and Paco Santos for helpful comments on a preliminary version of this article.

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