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Tournaments and Slide Rules for Products of ψ and ω Classes on $\overline{M}_{0,n}$

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Abstract. We give two combinatorial rules for the intersections of ψ and ω classes on the Deligne–Mumford moduli space of stable rational curves with $n+3$ marked points. The first, via *lazy tournaments*, describes products of ω classes in dimension 0 using boundary points of the moduli space. We use this rule to give a simple proof that the total degree of $\overline{M}_{0,n+3}$ in the iterated Kapranov embedding is $(2n-1)!!$, and we give a bijection with the *column-restricted parking functions* known to enumerate each multidegree. The second rule, via *slides*, expresses products of ω or ψ classes in all dimensions as positive, multiplicity-free sums of boundary strata. We show that these strata can moreover be obtained as limits of complete intersections of $\overline{M}_{0,n+3}$ with explicitly-defined families of hyperplanes.

Résumé. On donne deux règles combinatoires pour les intersections des classes de cohomologie ψ et ω sur l'espace de modules des courbes stables (au sens de Deligne–Mumford) avec $n+3$ points marqués. Le premier, qu'on appelle un *tournoi des paresseux*, associe à un produit de classes ω en dimension zéro une collection de points limites de $\overline{M}_{0,n+3}$. Cela nous permet de démontrer de façon très simple que le degré total de $\overline{M}_{0,n+3}$ dans l'immersion itérée de Kapranov équivaut à $(2n-1)!!$. On donne aussi une bijection avec les *fonctions de stationnement restreintes par colonne* qui énumèrent les multidegrés. Le deuxième, qu'on appelle une *glissade*, exprime tout produit de classes ψ ou ω, en toute dimension, comme une somme, positive et sans multiples, de strates limites de $\overline{M}_{0,n+3}$. On démontre aussi que les strates énumérées ainsi peuvent être obtenues en tant que limites d'intersections complètes de $\overline{M}_{0,n+3}$ avec certaines familles explicites d'hyperplans.

Keywords: moduli space of curves, psi class, omega class, tournament, tree, intersection theory

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1 Introduction

We give two new combinatorial rules for certain intersections of divisor classes on the moduli space $\overline{M}_{0,n+3}$ of stable $(n+3)$-pointed rational complex curves ($n \geq 0$). Such a curve consists of a union of $\mathbb{P}^1$s with nodal intersections and $n+3$ labeled marked points $p_a, p_b, p_c, p_1, p_2, \ldots, p_n$, joined in a tree structure (see Figure 1 and Section 1.2).

The geometry and intersection theory of $\overline{M}_{0,n+3}$ are of intensive and ongoing interest, particularly questions of an enumerative nature (see [1, 2, 9, 12, 16, 17] for a range of enumerative work on $\overline{M}_{0,n}$ and related spaces). These problems also have applications connecting $\overline{M}_{0,n+3}$ to questions in birational geometry, mathematical physics, and graph theory (among others). A broad question is to understand to what extent $\overline{M}_{0,n+3}$ resembles more well-understood varieties such as toric varieties and flag varieties. Notably, $\overline{M}_{0,n+3}$ has a boundary stratification akin to the Schubert stratification, and maps to projective space similar to the Plücker embeddings.

Our combinatorial rules are for calculating divisor products coming from these fundamental maps to projective space. Just as the flag variety $\mathrm{Flags}(\mathbb{C}^n)$ has Plücker morphisms $\mathrm{Flags}(\mathbb{C}^n) \to \mathrm{Gr}(i, \mathbb{C}^n) \to \mathbb{P}^{\binom{n}{i}-1}$ for each i, describing the i-th part of the flag, the moduli space $\overline{M}_{0,n+3}$ has for each i a *Kapranov morphism*

$$|\psi_i| \colon \overline{M}_{0,n+3} \to \mathbb{P}^n, \tag{1.1}$$

which describes the part of the stable curve near the marked point p_i (see Section 3.1). The corresponding divisor class, given by a hyperplane section from this map, is called the i-th *psi class*. There is a closely-related "reduced" Kapranov morphism for each i,

$$|\omega_i| \colon \overline{M}_{0,n+3} \to \mathbb{P}^i, \tag{1.2}$$

a slight modification of ψ_i, whose corresponding divisor class is called the i-th *omega class* ω_i, following the notation in [4]. Just as combining all the Plücker morphisms gives an embedding of the flag variety in a product of Grassmannians (or projective spaces),

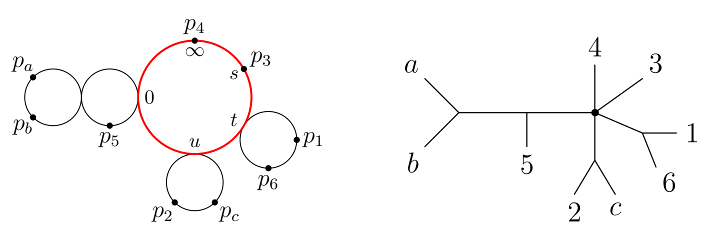


Figure 1: A stable curve (left) and its dual tree (right). Possible coordinate values are shown for the component with five special points (the large circle). Up to isomorphism, on each other component the special points are at $(0, 1, \infty)$.

the maps $|\psi_i|$ and $|\omega_i|$ for all i produce two basic maps to projective space, namely

$$\Psi_n\colon \overline{M}_{0,n+3} \to \mathbb{P}^n \times \mathbb{P}^n \times \cdots \times \mathbb{P}^n, \tag{1.3}$$

$$\Omega_n\colon \overline{M}_{0,n+3} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^n. \tag{1.4}$$

where the i-th factor is $|\psi_i|$ (for Ψ_n) and $|\omega_i|$ (for Ω_n). These are called the *total* and *iterated Kapranov morphisms*, respectively. The psi classes have been extensively studied, and the map Ψ_n is birational onto its image. The map Ω_n, meanwhile, is in fact a projective embedding [10, 13] and has been studied in more recent papers [4, 6, 15].

1.1 Intersection products and combinatorics

Let $\mathbf{k} = (k_1, \ldots, k_n)$ be a weak composition with n parts. We consider the intersection products in the Chow ring $A^\bullet(\overline{M}_{0,n+3})$ (equivalently, in the cohomology ring),

$$\psi^{\mathbf{k}} := \psi_1^{k_1} \cdots \psi_n^{k_n}, \quad \omega^{\mathbf{k}} := \omega_1^{k_1} \cdots \omega_n^{k_n} \in A^\bullet(\overline{M}_{0,n+3}). \tag{1.5}$$

When $\sum_{i=1}^n k_i = n$, these products give the *multidegrees* of the maps Ψ_n and Ω_n: they count the number of intersection points of $\overline{M}_{0,n+3}$ with the pullbacks of k_i general hyperplanes from the i-th projective space factor ($i = 1, \ldots, n$). The *string equation* recurrence for the ψ classes implies in this case that $\deg(\psi^{\mathbf{k}})$ is the multinomial coefficient $\binom{n}{\mathbf{k}} := \binom{n}{k_1,\ldots,k_n}$. The ω classes, meanwhile, satisfy an 'asymmetric' string recurrence, which was given a combinatorial interpretation via parking functions in [4]. There it was also shown, by an insertion algorithm on parking functions, that the *total degree* of Ω_n is

$$\deg(\Omega_n) := \sum_{\mathbf{k}} \deg(\omega^{\mathbf{k}}) = (2n-1)!!. \tag{1.6}$$

The odd double factorial is also the total number of trivalent trees with $n+2$ labeled leaves, that is, the total number of **boundary vertices** in the boundary stratification of $\overline{M}_{0,n+2}$ (not $n+3$). This is highly suggestive of the existence of a geometric realization of the multidegrees in terms of boundary points; we give such a realization (below).

When $\sum_{i=1}^n k_i < n$, the intersection products $\psi^{\mathbf{k}}$ and $\omega^{\mathbf{k}}$ can be expressed, using standard formulas (see, *e.g.*, [3]), as sums of positive-dimensional **boundary strata** X_T of $\overline{M}_{0,n+3}$, which are indexed by at-least-trivalent trees T with $n+3$ labeled leaves. However, the resulting expressions are not unique (unlike the Schubert classes on $\mathrm{Flags}(\mathbb{C}^n)$, the classes $[X_T]$ of boundary strata satisfy many linear relations in the Chow ring of $\overline{M}_{0,n+3}$). It is also unclear which such expressions, if any, are actually achievable as complete intersections of $\overline{M}_{0,n+3}$ by hyperplanes. Finally, many such expansions result in alternating sums, despite the fact that all these products are necessarily effective. (See the related work on intersections of tropical psi classes in [5, 8, 11, 14, 18].)

In this extended abstract we give two new combinatorial rules for the intersection products in Equation (1.5), via *tournaments* (Section 2, following [6]) and *slides* (Section 3, following [7]). The tournaments apply to the products $\omega^{\mathbf{k}}$ when $\sum k_i = n$ and result in a simple proof of Equation (1.6). We also give a bijection between tournaments and the *column-restricted parking functions* introduced in [4]. The slide rules express the products $\psi^{\mathbf{k}}$ and $\omega^{\mathbf{k}}$ for all $\mathbf{k}$ as sums of boundary strata. These combinatorial sums arise from geometry: for each $\mathbf{k}$ we construct an explicit parametrized family of hyperplane sections of $\overline{M}_{0,n+3}$, coming from the maps Ψ_n and Ω_n, whose intersection on $\overline{M}_{0,n+3}$ degenerates to the desired union of boundary strata.

1.2 Background and notation

A **stable** or **at least trivalent** tree is a tree T with labeled leaves, such that every non-leaf vertex has degree at least 3. If the degrees are exactly 3, T is **trivalent**. We always take leaf labels to be from the ordered set $S = \{a < b < c < 1 < \cdots < n\}$ for some $n \geq 0$.

An S**-marked stable curve** C **of genus** 0 is a union of $\mathbb{P}^1$'s with simple nodal intersections to form a tree structure, along with a tuple $(p_i \in C)_{i \in S}$ of distinct, smooth marked points, such that each $\mathbb{P}^1 \subseteq C$ has at least 3 nodes and/or marked points. The **dual tree** of C is the stable tree T with a vertex for each $i \in S$, a vertex v for each component $\mathbb{P}^1 \subseteq C$, a leaf edge i — v when p_i is on component v, and an internal edge v — v' when the components v and v' are connected by a nodal singularity. See Figure 1 for an example of a stable curve and its dual tree.

For each stable tree T, there is a closed **boundary stratum** $X_T \subseteq \overline{M}_{0,S}$: the closure of the locus of stable curves C with dual tree T. Its dimension of this stratum is $\dim(X_T) = \sum_{v \in T}(\deg(v) - 3)$, the sum ranging over the non-leaf vertices. In particular, T is trivalent if and only if X_T is a point.

For each $i \in S$, there is a *forgetful map* $\pi_i \colon \overline{M}_{0,S} \to \overline{M}_{0,S \setminus \{i\}}$. We obtain $\pi_i(C)$ by forgetting p_i, then if the $\mathbb{P}^1$ containing it is left with only 2 nodes and/or marked points, we contract that $\mathbb{P}^1$ to a point.

2 Lazy tournaments for $\omega^{\mathbf{k}}$ in dimension 0

Our first rule, called a *lazy tournament*, enumerates $\omega^{\mathbf{k}}$ in the special case when $\sum k_i = n$, and explains the odd double factorial phenomenon (Equation (1.6)). Throughout, we assume the ordering $a < b < c < 1 < \cdots < n$.

Definition 2.1. *The* ***lazy tournament*** *of a trivalent tree T is the edge labeling computed by first labeling each leaf edge by the label of the corresponding leaf, then iterating the following steps:*

1. ***Identify which pair 'face off'.*** *Among all pairs of labeled edges (i, j) (ordered so that $i < j$) that share a vertex and have a third unlabeled edge E attached to that vertex, choose the pair with the largest value of i.*

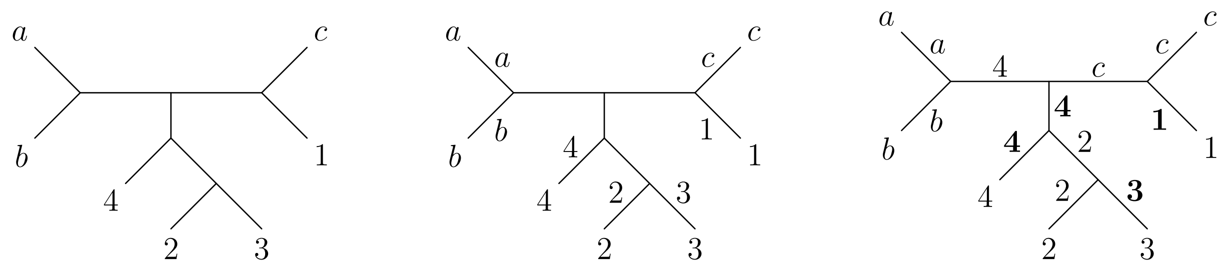


Figure 2: From left to right: A leaf-labeled trivalent tree T, its initial labeling of the leaf edges, and its full lazy tournament edge labeling. Winners of each round of the tournament are shown in boldface at right, indicating $T \in \mathrm{Tour}(1,0,1,2)$.

2. ***Determine the winner.*** *The larger number j is the* **winner**, *and the smaller number i is the* **loser** *of the round.*

3. ***Determine whether the winner or the loser advances.*** *Label E by i or j as follows:*

   (a) If E is adjacent to a labeled edge $u \neq j$ with $u > i$, label E by i. (We say i **advances**.*)*

   (b) Otherwise, label E by j. (We say j **advances**.*)*

We repeat steps 1-3 until all edges of the tree are labeled. We record the **winners composition** $\mathrm{win}(T) = (k_1, \ldots, k_n)$, *where k_i is the number of rounds won by the label i.*

We refer to Step 3(a) as the **laziness rule**, since j drops out of the tournament despite winning its match against i. This happens when j can "see" that its opponent i will be defeated, again, in its next round against u. An example of a lazy tournament is shown in Figure 2.

Definition 2.2. *For any weak composition $\mathbf{k} = (k_1, \ldots, k_n)$ of n, we let $\mathrm{Tour}(\mathbf{k})$ be the set of trivalent trees T such that the leaf edges of a and b share a vertex and $\mathrm{win}(T) = \mathbf{k}$.*

Our first main result is as follows. The proof can be found in [6], and relies on reproducing the recurrence of Definition 2.4 satisfied by the $\omega^{\mathbf{k}}$.

Theorem 2.3. *We have* $\deg(\omega^{\mathbf{k}}) = |\mathrm{Tour}(\mathbf{k})|$.

Definition 2.4 (Asymmetric string recurrence)**.** *Let $\mathbf{k} = (k_1, \ldots, k_n)$ be a composition of n. Let $i_0 = i_0(\mathbf{k})$ be the position of the rightmost zero in $\mathbf{k}$. For each $j > i_0$, let $\widetilde{\mathbf{k}}_j$ be the composition obtained by decrementing k_j by 1, then deleting the rightmost 0 (which is in position i_0 or j).*

The following recurrence, which is due to [4], is shown by using the forgetful morphism $\pi_{i_0} \colon \overline{M}_{0,S} \to \overline{M}_{0,S\setminus\{i_0\}}$, together with certain relabelings:

$$\deg(\omega^{\mathbf{k}}) = \sum_{j > i_0(\mathbf{k})} \deg(\omega^{\widetilde{\mathbf{k}}_j}). \tag{2.1}$$

Proof sketch of Theorem 2.3. We introduce an operation $\pi_{\mathbf{lazy}}$ on tournaments T, closely related to step 3 of Definition 2.1: we first delete the 'facing-off' pair i and j from T; in fact $i = i_0(\mathbf{k})$. We then relabel the shared vertex v to i and decrement the labels $i+1, \ldots, n$ by one (non-lazy case, when $k_j > 1$) or relabel v to j and decrement $j+1, \ldots, n$ by one (lazy case, when $k_j = 1$). We show that this restricts, for each $\mathbf{k}$, to a bijection

$$\pi_{\mathbf{lazy}} \colon \mathrm{Tour}(\mathbf{k}) \xrightarrow{\sim} \bigsqcup_{j > i_0(\mathbf{k})} \mathrm{Tour}(\widetilde{\mathbf{k}}_j).$$

We then identify these deletions and relabelings with the maps used in the asymmetric string recurrence. An example is shown below. □

Example 2.5. For $\mathbf{k} = (1,0,0,0,2,1,3)$, we have a bijection

$$\mathrm{Tour}(1,0,0,0,2,1,3) \xrightarrow{\pi_{\mathbf{lazy}}} \mathrm{Tour}(1,0,0,1,1,3) \sqcup \mathrm{Tour}(1,0,0,0,2,3) \sqcup \mathrm{Tour}(1,0,0,2,1,2).$$

Geometrically, viewing $\mathrm{Tour}(\mathbf{k})$ as a set of boundary points of $\overline{M}_{0,n+3}$, $\pi_{\mathbf{lazy}}$ maps those points to $\overline{M}_{0,n+2}$ as in the diagram

$$\begin{array}{ccccccc}
 & & & \overset{\cong}{=\!=} & \overline{M}_{0,abc123456} & \supset & \mathrm{Tour}(1,0,0,1,1,3) \\
\overline{M}_{0,abc1234567} & \xrightarrow{\pi_4} & \overline{M}_{0,abc123567} & \overset{(\cong)'}{=\!=} & \overline{M}_{0,abc123456} & \supset & \mathrm{Tour}(1,0,0,0,2,3) \\
\cup & & & \overset{\cong}{=\!=} & & & \\
\mathrm{Tour}(1,0,0,0,2,1,3) & & & & \overline{M}_{0,abc123456} & \supset & \mathrm{Tour}(1,0,0,2,1,2).
\end{array} \tag{2.2}$$

Here, the relabelings indicated by $\cong$ are *non-lazy* (decrementing 5, 6 and 7 by 1) and the middle relabeling $(\cong)'$ is *lazy*: it is given by first sending $6 \mapsto 4$, then decrementing only $7 \mapsto 6$. These are the maps used in the asymmetric string equation. The map $\pi_{\mathbf{lazy}}$ corresponds to these maps, applied to the appropriate subsets of $\mathrm{Tour}(1,0,0,0,2,1,3)$.

Finally, we give a simple calculation of the total degree $\deg(\Omega_n) = \sum_{\mathbf{k}} \deg(\omega^{\mathbf{k}})$.

Corollary 2.6. *The total degree of* Ω_n *is the odd double factorial* $(2n-1)!!$.

Proof. As $\mathbf{k}$ varies over all compositions, the sets $\mathrm{Tour}(\mathbf{k})$ partition the complete set of trivalent trees in which the leaves a, b share a vertex. These points are in bijection with *all* trivalent trees on $n+2$ labels, by deleting b and contracting. It is well known that there are exactly $(2n-1)!!$ of these. (Geometrically, these are the boundary points on the divisor $\delta_{a,b} \subset \overline{M}_{0,S}$, and the forgetful map π_b gives $\delta_{a,b} \cong \overline{M}_{0,\{a,c,1,\ldots,n\}} = \overline{M}_{0,n+2}$.) □

2.1 Bijection with parking functions

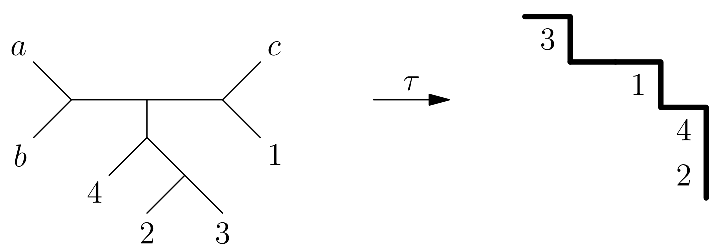
Finally, we give a bijection to the **column-restricted parking functions** defined in [4]. Recall that a *Dyck path* of height n is a down-right lattice walk from $(0, n)$ to $(n, 0)$ in the plane that stays weakly above the diagonal line $x + y = n$.

Definition 2.7. *A* ***parking function*** *P of size n is a Dyck path of height n along with a labeling of the unit squares just left of the up-steps by $1, 2, \ldots, n$ (called the* ***cars*** *of P) such that each column of cars is increasing from bottom to top.*

The ***dominance index*** *$d(i)$ of car i is the number of columns to its right in which all entries are $< i$ (including empty columns). We say P is* ***column-restricted*** *if $d(i) < i$ for all i. We let $\mathrm{CPF}(\mathbf{k})$ be the set of column-restricted parking functions with k_i cars in column i for all i.*

Theorem 2.8. *The following is a bijection $\tau\colon \mathrm{Tour}(\mathbf{k}) \xrightarrow{\sim} \mathrm{CPF}(\mathbf{k})$: for each round r of the tournament, place car r in column j, where j is the winner of the round.*

See [6, Section 4.2] for full details and proofs. As an example, the tree and parking function below correspond under the bijection τ.



Indeed, by Figure 2, the tree T shown above is in $\mathrm{Tour}(1, 0, 1, 2)$. Note that $i = 1$ wins round 3, $i = 2$ does not win any round, $i = 3$ wins round 1, and $i = 4$ wins rounds $2, 4$. Thus $\tau(T)$ is the parking function shown at right, and it is column-restricted because car 3 has dominance index 2 (it dominates two columns to its right, but not the final column) and the other cars have dominance index 0. In particular, $\tau(T) \in \mathrm{CPF}(1, 0, 1, 2)$.

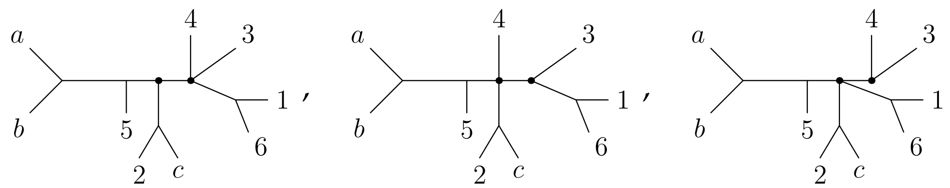
3 Slide rules

If T is a stable (at-least-trivalent) tree with leaves labeled by S and $i \in T$ is a leaf, let v_i denote the non-leaf vertex adjacent to i. The **branches of** T **at** i are the connected components of $T \setminus \{i, v_i\}$. We write Br_a for the branch containing the leaf a and e_a for the edge connecting v_i and Br_a. The i-**minimal branch** Br_m is the branch with the minimal leaf m of $T \setminus (\mathrm{Br}_a \cup \{i\})$.

Definition 3.1 (Slide at i)**.** *An i-**slide** on T is performed as follows: we add a vertex $\overline{v}$ in the middle of e_a, move Br_m to $\overline{v}$, and attach each other branch (excluding Br_a) to* either *v_i or $\overline{v}$.*

We write $\mathrm{slide}_i(T)$ for the set of stable trees obtained this way. Note that stability requires at least one branch to remain at v_i, so $|\mathrm{slide}_i(T)| = 2^{\deg(v_i)-3} - 1$.

Example 3.2. We calculate $\mathrm{slide}_3(T)$, where T is shown at right in Figure 1 with v_3 indicated by a dot. The $\mathbb{P}^1$ corresponding to v_3 is drawn in bold in the accompanying stable curve picture. Then Br_a is the subtree with leaves $\{a, b, 5\}$, and the other branches at 3 have leaves $\{4\}$, $\{1, 6\}$, and $\{c, 2\}$. So $m = c$ and Br_m is the branch with leaves $\{c, 2\}$ (the minimal leaf outside Br_a). Moving Br_m and distributing the other two branches gives the following set of trees:

$\mathrm{slide}_3(T) = \Big\{$  $\Big\}$.

The following lemma, while simple, is essential to the multiplicity-freeness and generic reducedness results.

Lemma 3.3 (Injectivity)**.** *For $T \neq T'$, the sets $\mathrm{slide}_i(T)$ and $\mathrm{slide}_i(T')$ are disjoint.*

Proof. Let $R \in \mathrm{slide}_i(T)$. Recall that $e_a \in R$ is the edge connecting the branch Br_a of R at i to the vertex v_i. Contracting e_a recovers T. □

Let $\mathbf{k} = (k_1, \ldots, k_n)$ be a composition. We write ⋎ (resp. ✳) for the unique tree with a single internal vertex and leaves a, b, c (resp. $a, b, c, 1, \ldots, n$).

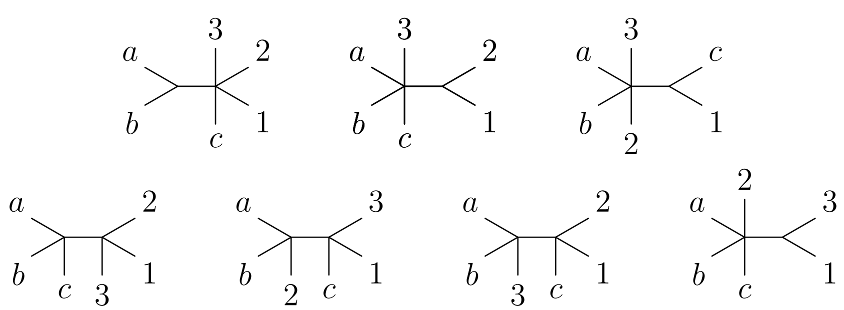
Definition 3.4 (Slide rule for $\psi^{\mathbf{k}}$)**.** *We define* $\mathrm{Slide}^{\psi}(\mathbf{k})$ *as the set of trees obtained as follows.*

1. *Start with ✳ as step $i = 0$.*
2. *For $i = 1, \ldots, n$, apply k_i successive i-slides all possible ways to all trees from step $i - 1$.*

Definition 3.5 (Slide rule for $\omega^{\mathbf{k}}$)**.** *We define* $\mathrm{Slide}^{\omega}(\mathbf{k})$ *as the set of trees obtained as follows.*

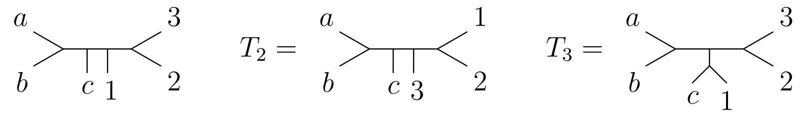
1. *Start with ⋎ as step $i = 0$.*
2. *For $i = 1, \ldots, n$:*
   a. *For each tree obtained in step $i - 1$, consider all trees formed by inserting i at any existing non-leaf vertex.*
   b. *For each tree obtained in step a, apply k_i successive i-slides in all possible ways.*

Example 3.6. We compute $\mathrm{Slide}^{\psi}(1, 0, 2)$. We start with the unique tree with a single internal vertex and six leaves labeled $a, b, c, 1, 2, 3$. The 1-slide produces seven trees:



We then apply two successive 3-slides. Notice that $\deg(v_3) < 5$ for the trees in the bottom row, so only the trees in the top row generate nonempty sets after two 3-slides. We obtain three trees:

$$\mathrm{Slide}^{\psi}(1,0,2) = \Big\{ T_1 = \quad T_2 = \quad T_3 = \Big\}.$$



Theorem 3.7. *For all* $\mathbf{k}$*, we have in* $A^{\bullet}(\overline{M}_{0,n+3})$ *that*

$$\psi^{\mathbf{k}} = \sum_{T \in \mathrm{Slide}^{\psi}(\mathbf{k})} [X_T], \quad \textit{and} \quad \omega^{\mathbf{k}} = \sum_{T \in \mathrm{Slide}^{\omega}(\mathbf{k})} [X_T].$$

By the Injectivity Lemma 3.3 (and the obvious injectivity of Step 2(a) in the ω-slide rule), the above sums are both multiplicity-free. We will deduce Theorem 3.7 from an explicit geometric construction. We first recall some background on the Kapranov maps.

3.1 Kapranov morphisms

We describe the Kapranov morphism $|\psi_i| \colon \overline{M}_{0,S} \to \mathbb{P}^n$, for each $i \in S \setminus \{a\}$, defined in [10]. We give $\mathbb{P}^n$ the projective coordinates $[Z_{i,b} : Z_{i,c} : Z_{i,1} : \cdots : \widehat{Z_{i,i}} : \cdots : Z_{i,n}]$; the notation $\widehat{Z_{i,i}}$ means there is no coordinate named $Z_{i,i}$.

Let C be an S-marked stable curve, and let T be its dual tree. Let $C' \subseteq C$ be the irreducible component containing p_i. Each node or marked point $q \in C'$ corresponds to a branch of T at i. For each marked point $j \neq i$, let $q(j) \in C'$ be the node or marked point corresponding to the branch containing j. (If j, j' are on the same branch of T at i, then $q(j) = q(j')$ is a node on C'. If p_j itself is on C', then $q(j) = p_j$.)

Then $|\psi_i|(C)$ is computed as follows: since $C' \cong \mathbb{P}^1$, we change coordinates on C' so that $p_i = \infty$ and $q(a) = 0$. The remaining special points have well-defined coordinates up to a common scalar, and we put (where the notation $\widehat{q(i)}$ means $q(i)$ is omitted)

$$|\psi_i|(C) = [q(b) : q(c) : q(1) : \cdots : \widehat{q(i)} : \cdots : q(n)] \in \mathbb{P}^n.$$

Example 3.8. For $i = 4$, the curve C in Figure 1 has $|\psi_4|(C) = [0 : u : t : u : s : 0 : t]$.

The 'reduced' Kapranov morphism $|\omega_i| \colon \overline{M}_{0,S} \to \mathbb{P}^i$ is the composition of the forgetful map

$$f_i = \pi_{i+1} \circ \cdots \circ \pi_n \colon \overline{M}_{0,abc1\cdots n} \to \overline{M}_{0,abc1\cdots i},$$

followed by the Kapranov morphism $|\psi_i|$ of the smaller moduli space. Thus $\omega_i = f_i^* \psi_i$. We write $[Y_{i,b} : Y_{i,c} : Y_{i,1} : \cdots : Y_{i,i-1}]$ for the projective coordinates on the target $\mathbb{P}^i$.

3.2 Hyperplanes

Definition 3.9. *Let t be a parameter. We define hyperplanes $H_i^\psi(t) \subseteq \mathbb{P}^n$ and $H_i^\omega(t) \subseteq \mathbb{P}^i$ by*

$$H_i^\psi(t) := \mathbb{V}(Z_{i,b} + tZ_{i,c} + \cdots + t^i Z_{i,i-1} + t^{i+1} Z_{i,i+1} + \cdots + t^n Z_{i,n}), \tag{3.1}$$
$$H_i^\omega(t) := \mathbb{V}(Y_{i,b} + tY_{i,c} + \cdots + t^i Y_{i,i-1}). \tag{3.2}$$

By abuse of notation, we also write $H_i^\psi(t)$ and $H_i^\omega(t)$ for the loci

$$\mathbb{P}^n \times \cdots \times \mathbb{P}^n \times H_i^\psi(t) \times \mathbb{P}^n \times \cdots \times \mathbb{P}^n \subseteq \mathbb{P}^n \times \mathbb{P}^n \times \cdots \times \mathbb{P}^n, \tag{3.3}$$
$$\mathbb{P}^1 \times \cdots \times \mathbb{P}^{i-1} \times H_i^\omega(t) \times \mathbb{P}^{i+1} \times \cdots \times \mathbb{P}^n \subseteq \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^n. \tag{3.4}$$

A key feature of $\Psi_n^{-1}(H_i^\psi(t))$ (similarly $\Omega_n^{-1}(H_i^\omega(t))$) is that its intersection with *any* boundary stratum is either transverse or empty. We will consider families of hyperplanes consisting of, for each i, k_i copies of $H_i^\psi(t)$ or $H_i^\omega(t)$ with independent parameters t.

Definition 3.10 (Family of hyperplanes)**.** *Let $\mathbf{k} = (k_1, \ldots, k_n)$ be a composition. Let $\vec{t} = (t_{i,j})$ for $1 \le i \le n$ and $1 \le j \le k_i$ be a tuple of independent complex parameters. We define*

$$V^\psi(\mathbf{k}; \vec{t}) := \bigcap_{i=1}^{n} \bigcap_{j=1}^{k_i} \Psi_n^{-1}\big(H_i^\psi(t_{i,j})\big), \tag{3.5}$$

$$V^\omega(\mathbf{k}; \vec{t}) := \bigcap_{i=1}^{n} \bigcap_{j=1}^{k_i} \Omega_n^{-1}\big(H_i^\omega(t_{i,j})\big), \tag{3.6}$$

where Ψ_n is the total Kapranov map and Ω_n is the iterated Kapranov embedding.

Example 3.11. Let $\mathbf{k} = (1, 0, 2)$. The equations defining $V^\psi(\mathbf{k}; \vec{t})$ pulled back from $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$ are

$$0 = Z_{1,b} + t_{1,1} Z_{1,c} + t_{1,1}^2 Z_{1,2} + t_{1,1}^3 Z_{1,3}, \tag{3.7}$$
$$0 = Z_{3,b} + t_{3,1} Z_{3,c} + t_{3,1}^2 Z_{3,1} + t_{3,1}^3 Z_{3,2}, \tag{3.8}$$
$$0 = Z_{3,b} + t_{3,2} Z_{3,c} + t_{3,2}^2 Z_{3,1} + t_{3,2}^3 Z_{3,2}, \tag{3.9}$$

whereas $V^\omega(\mathbf{k}; \vec{t})$ is pulled back from the following equations on $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$,

$$0 = Y_{1,b} + t_{1,1} Y_{1,c}, \tag{3.10}$$
$$0 = Y_{3,b} + t_{3,1} Y_{3,c} + t_{3,1}^2 Y_{3,1} + t_{3,1}^3 Y_{3,2}, \tag{3.11}$$
$$0 = Y_{3,b} + t_{3,2} Y_{3,c} + t_{3,2}^2 Y_{3,1} + t_{3,2}^3 Y_{3,2}. \tag{3.12}$$

The second and third equations are the same, but with independent parameters $t_{i,j}$.

Theorem 3.12. *Let* $\mathbf{k}$ *and* $\vec{t} = (t_{i,j})$ *be as in Definition 3.10. Let* $\lim_{\vec{t}\to\vec{0}}$ *denote the iterated limit*

$$\lim_{\vec{t}\to\vec{0}}(-) := \lim_{t_{n,k_n}\to 0} \cdots \lim_{t_{n,1}\to 0} \cdots\cdots \lim_{t_{2,k_2}\to 0} \cdots \lim_{t_{2,1}\to 0} \lim_{t_{1,k_1}\to 0} \cdots \lim_{t_{1,1}\to 0}(-).$$

(The i-th block is empty if $k_i = 0$*, and* $\lim$ *denotes the flat limit.) Then we have set-theoretically*

$$\lim_{\vec{t}\to\vec{0}} V^{\psi}(\mathbf{k};\vec{t}) = \bigcup_{T\in\mathrm{Slide}^{\psi}(\mathbf{k})} X_T \quad \text{and} \quad \lim_{\vec{t}\to\vec{0}} V^{\omega}(\mathbf{k};\vec{t}) = \bigcup_{T\in\mathrm{Slide}^{\omega}(\mathbf{k})} X_T. \tag{3.13}$$

Moreover, each boundary stratum X_T *appearing in the union is an irreducible component and is generically reduced in the limit.*

Proof. We take the limits sequentially. We show inductively that at each step, we have a generically reduced union of strata $\bigcup X_T$, and that taking the limit with one additional hyperplane of type ψ_i or ω_i cuts out, from each X_T, the sub-strata indexed by $\mathrm{slide}_i(T)$. By the Injectivity Lemma 3.3, no stratum ever arises twice, so the resulting union is multiplicity-free, *i.e.*, generically reduced. □

Equation (3.13) immediately implies Theorem 3.7.

3.2.1 Application to kappa classes

Theorem 3.7 also immediately yields boundary formulas for the *kappa classes*, answering a question of Cavalieri [3]. Let $0 \le i \le n$ and let $\mathbf{r} = (r_1, \dots, r_m)$ be a weak composition.

Definition 3.13. *Let* $\pi_{n+1} \colon \overline{M}_{0,abc1\ldots n+1} \to \overline{M}_{0,abc1\cdots n}$ *and* $\pi_{n+1,\ldots,n+m} \colon \overline{M}_{0,abc1\cdots n+m} \to \overline{M}_{0,abc1\cdots n}$ *be the forgetting maps. The* (generalized) kappa classes κ_i *and* $R_{n;\mathbf{r}}$ *are given by*

$$\kappa_i := (\pi_{n+1})_*(\psi_{n+1}^{i+1}), \tag{3.14}$$

$$R_{n;\mathbf{r}} := (\pi_{n+1,\ldots,n+m})_*(\psi_{n+1}^{r_1} \cdots \psi_{n+m}^{r_m}). \tag{3.15}$$

Definition 3.14 (Boundary strata for kappa classes)**.** *Let* $K(n;i) \subseteq \mathrm{Slide}^{\psi}(0^n, i+1)$ *be the subset of trees* T *such that* $\deg(v_{n+1}) = 3$*.*

Let $R(n;\mathbf{r}) \subseteq \mathrm{Slide}^{\psi}(0^n, r_1, \ldots, r_m)$ *be the subset of trees* T *such that, for all* $n+1 \le j \le n+m$*, the tree* $\pi_{j+1,\ldots,n+m}(T)$ *has* $\deg(v_j) = 3$*.*

Theorem 3.15. *We have* $\kappa_i = \sum_{T\in K(n;i)} [X_{\pi_{n+1}(T)}]$ *and* $R_{n;\mathbf{r}} = \sum_{T\in R(n;\mathbf{r})} [X_{\pi_{n+1,\ldots,n+m}(T)}]$ *on* $\overline{M}_{0,S}$*.*

Proof. We push forward the expression from Theorem 3.7 and check which terms $\pi_*[X_T]$ vanish. Note that these sums have repeated terms (in fact no multiplicity-free boundary formula can exist for $R_{n;\mathbf{r}}$ in general). □

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