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Subdivisions of Generalized Permutahedra

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Abstract. We study valuated matroids, their tropical incidence relations, flag matroids and total positivity. Our techniques employ the polyhedral geometry of the hypersimplices, the regular permutahedra and their subdivisions.

Keywords: flag matroids, total positivity, Bruhat interval polytopes

1 Introduction

A *generalized permutahedron* in $\mathbb{R}^n$ is a convex polytope such that all its edges are parallel to $e_i - e_j$ for $i \neq j$. Here $e_1, e_2, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. Generalized permutahedra were introduced by Postnikov [18], but they were known before under different names, most notably in the context of submodular function optimization [9] and discrete convex analysis [15]. Relevant examples of generalized permutahedra include the regular permutahedra, Bruhat interval polytopes, hypersimplices and more general matroid polytopes. In this way the topic is connected to group and representation theory, algebraic and tropical geometry, optimization and beyond.

Here we continue the combinatorial study of polyhedral subdivisions of generalized permutahedra into cells which are again generalized permutahedra [2, 1, 7]. We call these *permutahedral subdivisions*. An important special case are the tropical linear spaces, which amount to regular subdivisions of matroid polytopes with matroidal cells; see [14, Section 4.4] and [12, Section 10.5]. Our first main contribution (Theorem 10) is a description of regular permutahedral subdivisions of the regular permutahedra in terms of conditions on the 2-skeleton. This result is an adaptation of the characterization of the uniform tropical linear spaces via the 3-term Plücker relations. One motivation for research in this direction comes from the wish to understand flags of tropical linear spaces and flag matroids. As a new tool, we prove a characterization of pairs of valuated matroids in terms of tropical incidence relations (Theorem 7). Combined with [7], we gain new insights in the flag Dressian, the space of incident valuated matroids. Speyer and Williams [22] and, independently, Arkani-Hamed, Lam, and Spradlin [3] recently

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showed that the positive Dressian equals the positive tropical Grassmannian. Our second main result (Theorem 12) is related, as it shows that the valuated flags which can be realized by a totally positive flag of linear spaces correspond to the permutahedral subdivisions whose cells are Bruhat interval polytopes.

2 Geometry of the regular permutahedron

We denote the symmetric group of degree n as $\mathrm{Sym}(n)$. The (regular) permutahedron $\Pi_n \subseteq \mathbb{R}^n$ is defined as the convex hull of the points $(\sigma(1), \sigma(2), \ldots, \sigma(n))$, where σ ranges over $\mathrm{Sym}(n)$. These points form the $n!$ vertices of Π_n, and $\mathrm{Sym}(n)$ acts on this set, *e.g.*, by multiplication on the right. We have $\dim \Pi_n = n-1$. Throughout, we identify $\mathrm{Sym}(n)$ with the vertices of Π_n.

The face structure of Π_n is well-known, and we will use the following description. A *flag* $(F_1, \ldots, F_k)$ in $[n]$ with k *constituents* is a strictly increasing sequence $F_1 \subset \cdots \subset F_k$ of non-empty subsets of $[n]$; a flag is *full* if it has n constituents. We say that a flag *extends* a pair (U, V) of subsets $U \subset V \subseteq [n]$ if there are $i, j \in [k]$ with $U = F_i$ and $V = F_j$. We set $e_A := \sum_{i\in A} e_i$. For the flag $\mathcal{F} = (F_1, \ldots, F_k)$ we can pick real numbers λ_i with $\lambda_1 \gg \cdots \gg \lambda_k > 0$ to obtain the vector

$$\lambda_{\mathcal{F}} := \lambda_1 e_{F_1} + \lambda_2 e_{F_2\setminus F_1} + \cdots + \lambda_k e_{F_k \setminus F_{k-1}} .$$

The following summarizes results from [9, Section 3.3(d)] and [4, Proposition 1.3].

Proposition 1. *The map which sends the flag $\mathcal{F} = (F_1, \ldots, F_k)$ in $[n]$ to the non-empty proper face of Π_n which maximizes $\lambda_{\mathcal{F}}$ is a bijection. In particular,*

1. *the flag $\mathcal{F}$ with k constituents corresponds to a face of codimension k, which is affinely equivalent with $\Pi_{|F_1|} \times \Pi_{|F_2\setminus F_1|} \times \cdots \times \Pi_{|F_k\setminus F_{k-1}|} \times \Pi_{|[n]\setminus F_k|}$;*
2. *the facets correspond to non-empty proper subsets of $[n]$;*
3. *the vertices correspond to the full flags;*
4. *the edges correspond to pairs of full flags which differ in exactly one constituent;*
5. *each 2-face, where $k = n-3$, is either a hexagon (if there exists i with $|F_{i+1} \setminus F_i| = 3$), or it is a square (if there exist distinct i, j with $|F_{i+1} \setminus F_i| = 2$ and $|F_{j+1} \setminus F_j| = 2$).*
6. *for two sets $A, B \subset [n]$ with $|A| = |B|$ and $|A \triangle B| = 2$, the edges defined by the flags extending $(A \cap B, A \cup B)$ span a face isomorphic with $\Pi_{|A\cap B|} \times \Pi_2 \times \Pi_{n-|A\cup B|}$.*

Here, '$\triangle$' denotes symmetric difference. Notice that Π_n also arises as a secondary polytope of the prism over an $(n-1)$-simplex; see [8, Theorem 6.2.6]. This leads to another way to describe the face lattice.

3 Flag matroids and subpermutahedra

Matroid polytopes and valuated matroids A *subpolytope* of a polytope is the convex hull of a subset of the vertices. Each face is a subpolytope, but the converse is false. Let P be a generalized permutahedron. A *subpermutahedron* of P is a subpolytope of P which itself is a generalized permutahedron. A *matroid (base) polytope* M of *rank* d is a subpermutahedron of the hypersimplex $\Delta(d,n)$. The bases $\mathcal{B}(M) \subseteq \binom{[n]}{d}$ of a matroid consist of all subsets whose indicator vector is a vertex of M. In the sequel, we will identify a matroid with its matroid base polytope. The *uniform matroid* $U_{d,n}$, whose bases are exactly $\binom{[n]}{d}$, corresponds to $\Delta(d,n)$.

For a lattice polytope P, we abbreviate $P_{\mathbb{Z}} = P \cap \mathbb{Z}^n$. A function $f \colon P_{\mathbb{Z}} \to \mathbb{R}$ is *M-convex* if the regular subdivision of the point configuration $P_{\mathbb{Z}}$ by the height function f is permutahedral. We use the convention that regular subdivisions are induced by lower convex hulls; see [8] or [12, Section 1.2] A *valuated matroid* μ over a matroid M is an M-convex function on $\mathcal{B}(M)$. Equivalently, a function $\mu \colon \mathcal{B}(M) \to \mathbb{R}$ is a valuated matroid if the 3-term Plücker relations hold: for each $S \in \binom{[n]}{d-2}$ and $i,j,k,l \notin S$, the minimum in

$$\min(\mu(Sij) + \mu(Skl),\ \mu(Sik) + \mu(Sjl),\ \mu(Sil) + \mu(Sjk))$$

is attained at least twice; see [14, Section 4.4] and [12, Section 10.4].

Flag matroids Let M and N be matroids of ranks d and $d+1$, respectively. Then the pair (M,N) forms a *quotient* if the convex hull of $M \times \{1\} \cup N \times \{0\}$ is a matroid. We denote this by $M \twoheadleftarrow N$. Two valuated matroids μ and ν, with respective underlying matroids M and N, are a *(valuated matroid) quotient* if $M \twoheadleftarrow N$ and the *3-term tropical incidence relations* are fulfilled; that is, for all $S \in \binom{[n]}{d-1}$ and $i,j,k \notin S$,

$$\min(\mu(Si) + \nu(Sjk),\ \mu(Sj) + \nu(Sik),\ \mu(Sk) + \nu(Sij)) \tag{3TIR}$$

attains the minimum at least twice. Similarly, we denote this by $\mu \twoheadleftarrow \nu$.

We remark that (valuated) matroid quotients are often defined differently, but it is always equivalent to inclusion of (tropical) linear spaces. For consecutive ranks, this is equivalent to the existence of a (valuated) matroid ξ over $[n+1]$ such that $\mu = \xi/(n+1)$ and $\nu = \xi\backslash(n+1)$. Both equivalences can be found in [7]. As the 3-term Plücker relations suffice to define valuated matroids, if its support is a matroid [19, Corollary 5.5], our definition coincides with this.

A sequence of matroids $\mathcal{M} = (M_1, \ldots, M_n)$ is a *(full) flag matroid* if the rank of M_i is i and $M_i \twoheadleftarrow M_j$ for every $1 \leq i < j \leq n$. The *flag matroid (base) polytope* of $\mathcal{M}$ is

$$M_1 + \cdots + M_n = \operatorname{conv}\left\{ \sum_{i=1}^{n} e_{B_i} \,\middle|\, (B_1, \ldots, B_n) \in (\mathcal{B}(M_1), \ldots, \mathcal{B}(M_n)) \right\}.$$

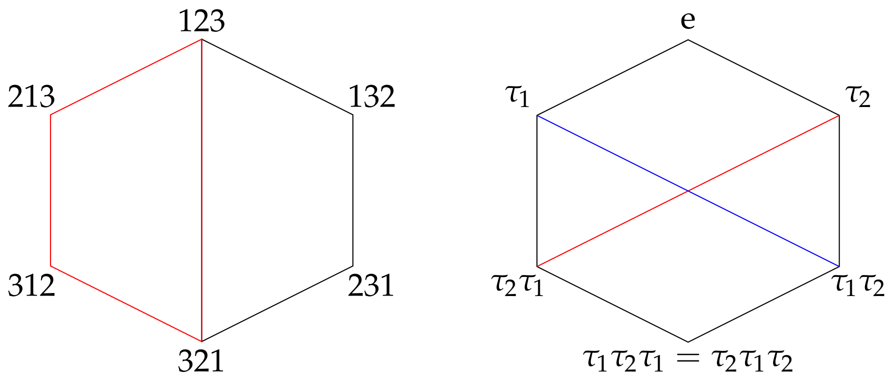

Figure 1: Left: regular hexagon Π_3 subdivided into two flag polytopes, neither of which are Bruhat interval polytopes. Right: Bruhat order of $\mathrm{Sym}(3)$.

The equivalence of (i) and (ii) in [6, Theorem 1.11.1] characterizes base polytopes of flag matroids as those generalized permutahedra which arise as the convex hull over sums of characteristic vectors of flags of bases of a sequence of matroids. We use a reformulation adapted from [7, Theorem 4.1.5].

Theorem 2 ([6]). *A polytope is the base polytope of a full flag matroid if and only if it is a subpermutahedron of the regular permutahedron.*

Similarly, a *(full) valuated flag matroid* is a sequence of valuated matroids $(\mu_1, \ldots, \mu_n)$ such that $\mu_i \twoheadleftarrow \mu_j$ for every $1 \leq i < j \leq n$.

Bruhat (interval) polytopes. We recall the *(strong) Bruhat order* of $\mathrm{Sym}(n)$. Consider the simple transpositions $\tau_1, \ldots, \tau_{i-1}$, with $\tau_i = (i, i+1)$, which generate the group $\mathrm{Sym}(n)$. A sequence $\tau_{i_1}, \ldots, \tau_{i_\ell}$ of minimal length with $\sigma := \tau_{i_1} \cdots \tau_{i_\ell}$, is a *reduced word* of σ. Now $\sigma_1 \leq \sigma_2$ if any (equivalently every) reduced word of σ_2 contains a subsequence (not necessarily consecutive) which is a reduced word of σ_1. This imposes a lattice structure on $\mathrm{Sym}(n)$ with rank function given by the lengths of reduced words. For $\sigma_1 \leq \sigma_2$ the convex hull of the interval $[\sigma_1, \sigma_2]$ in the Bruhat order is a *Bruhat (interval) polytope*.

Proposition 3. *A subpolytope of a permutahedron is a Bruhat interval polytope if and only if it is the base polytope of a flag matroid which can be realized in the totally non-negative flag variety.*

Proof. Any Bruhat interval $[\sigma_1, \sigma_2]$ defines a stratum $G^{>0}_{\sigma_1,\sigma_2}$ in the totally non-negative (TNN) flag variety and by [13, Proposition 6.7 and Theorem 6.10] the polytope of any flag associated to a point of $G^{>0}_{\sigma_1,\sigma_2}$ is the Bruhat polytope of $[\sigma_1, \sigma_2]$. Since the strata of the form $G^{>0}_{\sigma_1,\sigma_2}$ cover the TNN flag variety the converse also follows. □

This implies that every constituent of a Bruhat interval polytope is a positroid [13, Corollary 6.11]. However, the converse is not true:

Example 4. Consider the quadrangle $\operatorname{conv}(123, 213, 312, 321)$, which is a subpermutahedron of the hexagon Π_3; see Figure 1 (left). This is not a Bruhat interval polytope since the inclusion of 123 and 321 would imply that it is the entire permutahedron. Yet it is a flag matroid base polytope, with constituents $(\{1,3\}, \{12,13,23\}, \{123\})$.

Compression of valuated flag matroids. A height function $w\colon \operatorname{Sym}(n) \to \mathbb{R}$ is a *valuated permutahedron* if each cell in the induced subdivision of Π_n is a subpermutahedron.

Remark 5. Alternatively, a height function $w\colon \operatorname{Sym}(n) \to \mathbb{R}$ is a valuated permutahedron if and only if its piecewise-linear extension (*i.e.* linearly extending over each cell of the subdivision) on the lattice points $\Pi_n \cap \mathbb{Z}^n$ is an M-convex function. In the latter case, it is the supremum over all M-convex functions that agree with w on $\operatorname{Sym}(n)$.

It turns out that these are essentially the same as full valuated flags of uniform matroids. Let $(\mu_1, \dots, \mu_k)$ be a sequence of valuated matroids with $\operatorname{rk}(\mu_1) < \cdots < \operatorname{rk}(\mu_k)$ and underlying matroid $M_1, \dots, M_k$. In [10], the *compression* $u\colon \sum_{i=1}^k M_i \to \mathbb{R}$ of $(\mu_1, \dots, \mu_k)$ is defined as the function

$$u(x) = \min\left\{ \sum_{i\in[k]} \mu_i(Y_i) \;\middle|\; x = \sum_{i\in[k]} e_{Y_i}, \text{ for all } i \in [k] : Y_i \in \mathcal{B}(M_i) \right\} .$$

When $(\mu_1, \dots, \mu_k)$ in the former definition is a valuated flag matroid, by [7, Theorem 4.4.2], the cells in the subdivision of $\Pi_n \cap \mathbb{Z}^n$ induced by u are themselves base polytopes of flag matroids. Therefore, using Theorem 2, the subdivision is composed of subpermutahedra of Π_n. On the other hand, [7, Corollay 4.4.5] provides us with the converse, namely that such subdivisions arise from the compression of a valuated flag matroid.

Theorem 6 ([7])**.** *A height function $w\colon \operatorname{Sym}(n) \to \mathbb{R}$ is a valuated permutahedron if and only if it is the restriction $u|_{\operatorname{Sym}(n)}$ of the compression of a full flag of valuated uniform matroids.*

Incidence relations imply Plücker relations Now, we deal with the interplay of the valuated matroids in a flag. For (non-tropical) Plücker vectors, the implication of the Plücker relation from the incidence relation occurred in [11]. On the combinatorial level, this was studied in the context of "M$^\natural$-convex set functions"; cf. [17]. Indeed, imposing supermodularity among the constituents of a valuated flag matroid gives rise to an M$^\natural$-convex set function.

Theorem 7. *Let $\nu\colon \binom{[n]}{d} \to \mathbb{R}$ and $\mu\colon \binom{[n]}{d+1} \to \mathbb{R}$ be any two functions satisfying the tropical incidence relations* (3TIR)*. Then μ and ν are valuated matroids.*

Proof. We show that μ satisfies the 3-term Plücker relations: for all $S \in \binom{[n]}{d-1}$ and $i, j, k, l \in [n]\backslash S$ the minimum in $\min(\mu(Sij) + \mu(Skl), \mu(Sik) + \mu(Sjl), \mu(Sil) + \mu(Sjk))$ is attained at least twice. For a contradiction, suppose there is a set S and $i, j, k, l \in [n]\backslash S$ with

$$\mu(Sij) + \mu(Skl) < \mu(Sik) + \mu(Sjl) \quad \text{and} \quad \mu(Sij) + \mu(Skl) < \mu(Sil) + \mu(Sjk) . \tag{3.1}$$

Let $\xi = \nu(Si)e_i + \nu(Sj)e_j + \nu(Sk)e_k + \nu(Sl)e_l$. Defining μ' with $\mu'(T) = \mu(T) - \langle \xi, e_T \rangle$ for all $T \in \binom{[n]}{d+1}$ and defining ν' by the same translation from ν, the pair (μ, ν) is a valuated matroid quotient if and only if (μ', ν') is a quotient. Hence, we can assume that $\nu(Si) = \nu(Sj) = \nu(Sk) = \nu(Sl) = 0$. With this, (3TIR) yields that the minima in $\min(\mu(Sij), \mu(Sik), \mu(Sjk))$, $\min(\mu(Sik), \mu(Sil), \mu(Skl))$, $\min(\mu(Sik), \mu(Sil), \mu(Skl))$, are attained twice. By (3.1), $\min(\mu(Sij), \mu(Skl)) \leq \max(\mu(Sik), \mu(Sjl), \mu(Sil), \mu(Sjk))$, so we can assume that $\mu(Sij) < \mu(Sik)$.

Combining the two observations yields $\mu(Sij) = \mu(Sjk)$. From the second inequality in (3.1), we get $\mu(Skl) < \mu(Sil)$. By the minimum condition, we have $\mu(Skl) = \mu(Sik)$. From the first inequality in (3.1), we get $\mu(Sij) < \mu(Sjl)$. Again by the minimum condition, $\mu(Sij) = \mu(Sil)$. With the above, this yields $\mu(Sij) = \mu(Sil) > \mu(Skl) = \mu(Sik)$, which contradicts the original assumption. Hence μ satisfies all 3-term Plücker relations and therefore it is a valuated matroid. By duality, ν must also be a valuated matroid. □

4 Regular permutahedral subdivisions

Based on the structure of the permutahedron, we derive conditions for a regular subdivision to be permutahedral: a height function induces a permutahedral subdivision of Π_n if and only if it does so in the 2-skeleton of Π_n.

To this end consider an arbitrary polytope P with vertex-edge graph $\Gamma = (V, E)$. We duplicate each edge, equipped with two opposite orientations; this turns Γ into a directed graph, which we denote $\Gamma^{\pm} = (V, E^{\pm})$. Now any function $f \colon V \to \mathbb{R}$ defined on the vertices yields a function $g \colon E^{\pm} \to \mathbb{R}$ on the directed edges by letting

$$g(v, w) = f(v) - f(w) \qquad \text{for distinct } v, w \in V, \tag{4.1}$$

where $g(w, v) = -g(v, w)$. We assume $n := \dim P \geq 2$, whence $\partial P \approx \mathbb{S}^{n-1}$ is simply connected. The key observation is that f can be recovered from g under the conditions:

Proposition 8. *Let $g \colon E^{\pm} \to \mathbb{R}$ be a function on the directed edges of P which satisfies $\sum_{i=1}^{k} g(v_i, v_{i+1}) = 0$ for every 2-face of P with vertices $v_1, v_2, \ldots, v_{k+1} = v_1$ (labeled cyclically). Then there is a unique function $f \colon V \to \mathbb{R}$ on the vertices with* (4.1) *and $f(s) = f_0$ for any fixed $s \in V$ and $f_0 \in \mathbb{R}$.*

Proof. Pick a directed spanning tree T of $\Gamma^{\pm}$ rooted at s, and define a function $f \colon V \to \mathbb{R}$ by inductively setting $f(s) = f_0$ and $f(v) = g(v, w) + f(w)$ along the directed edges of T. We need to show that f satisfies $f(v) - f(w) = g(v, w)$ for distinct vertices v, w. To this end it suffices to prove that $h(c) := \sum_{i=1}^{\ell} g(w_i, w_{i+1}) = 0$ holds for any directed cycle $c = (w_1, w_2, \ldots, w_\ell, w_{\ell+1} = w_1)$ in Γ. The boundary complex ∂P is a polytopal complex homeomorphic to $\mathbb{S}^{n-1}$. A combinatorial procedure for computing the fundamental group of a polytopal complex is given in [20, Section 44] (where this is proved for simplicial complexes). This has the following consequences. First, any path in ∂P is homotopic to a path in Γ. Second, due to $n \geq 2$, the boundary $\partial P \approx \mathbb{S}^{n-1}$ is simply connected, and thus every closed path in Γ can be contracted to a constant path within the 2-skeleton of P. In this way, up to homotopy, the cycle c can be contracted combinatorially in the following sense: there is a sequence of cycles $c_1, c_2, \ldots, c_m$ in Γ such that $c_1 = c$, the cycle c_m is trivial (without any edges), and the symmetric difference between the edges in c_i and c_{i+1} forms a 2-face. Now the assumption on g gives $0 = h(c_m) = h(c_{m-1}) = \cdots = h(c_0) = h(c)$, as we wanted to show. A similar argument yields the uniqueness of f. □

Before we will apply this statement to the hypersimplex, we need to relate functions on the permutahedron and on the hypersimplex. To this end let $G^{\pm}_{\Pi}(n)$ be the directed vertex-edge graph of Π_n as above. Similarly, let $G^{\pm}_{\Delta}(d,n)$ be a directed version of the vertex-edge graph of the hypersimplex $\Delta(d,n)$. Recall that each edge of Π_n gives rise to a pair (A,B) of equicardinal subsets of $[n]$ with $|A \triangle B| = 2$, and that each edge of $\Delta(d,n)$ corresponds to a pair (A,B) of d-subsets of $[n]$ with $|A \triangle B| = 2$.

Lemma 9. *Let g be a function on the directed edges of $G^{\pm}_{\Pi}(n)$ such that, on each 2-face of Π_n which is a square, parallel directed edges attain the same g-value. Then this yields a function g' on the directed edges of $\bigcup_{d=0}^{n} G^{\pm}_{\Delta}(d,n)$ such that $g'(A,B) = g(e)$ for all equicardinal subsets of $[n]$ with $|A \triangle B| = 2$ and all edges e of Π_n corresponding to (A,B).*

Proof. Let (A,B) be a pair of equicardinal subsets of $[n]$ with $|A \triangle B| = 2$. By Proposition 1, the vertices given by the full flags extending $(A \cap B, A \cup B)$ form a face isomorphic with $\Pi_{|A \cap B|} \times \Pi_2 \times \Pi_{n-|A \cup B|}$. In particular, any two of these edges are connected by a sequence of squares on this face, where each square is composed by a pair of parallel edges corresponding to (A,B) and are adjacent to each other in this sequence along these edges. By the condition on the squares, all the edges in $G^{\pm}_{\Pi}(n)$ corresponding to (A,B) have the same g-value. So we take that number to define g' on the arc of $G^{\pm}_{\Delta}(d,n)$ corresponding to (A,B), where $d = |A| = |B|$. □

Theorem 10. *A height function $w\colon \mathrm{Sym}(n) \to \mathbb{R}$ induces a permutahedral subdivision if and only if it induces a permutahedral subdivision of the 2-skeleton of Π_n. That is:*

(HEX) for every hexagon $abcdef$ (labeled cyclically) in the 2-skeleton of Π_n, we have

- *(HXE) $w(a) + w(c) + w(e) = w(b) + w(d) + w(f)$,*
- *(HXM) the maximum in $\max(w(a) + w(d), w(b) + w(e), w(c) + w(f))$ is attained twice;*

(SQR) for every square face $abcd$ of Π_n (labeled cyclically): $w(a) + w(c) = w(b) + w(d)$.

Proof. A subdivision of a polytope defines a subdivision on each face. Therefore, as each cell in the subdivision is a subpermutahedron, the 2-faces are also subdivided into subpermutahedra. Since squares cannot be subdivided any further, the second condition follows. To show the first condition, observe that at least four cyclically consecutive vertices have to lie in the same hyperplane, the other two on or above. By relabeling and subtracting a linear form, we can assume that $w(a) = w(b) = w(c) = w(d) = 0$ and that $w(e) = w(f) \geq 0$. This implies that the conditions on the 2-skeleton are necessary.

For the converse, suppose that w satisfies the conditions (HEX) and (SQR). We will show that w can be decomposed into a flag of valuated matroids $(g_1, \ldots, g_n)$ such that w is the result of their compression. Then, by Theorem 6, w is a valuated permutahedron.

On each directed edge (a, b) in $G^{\pm}_{\Pi}(n)$, we let $h'(a, b) = w(a) - w(b)$ as above. Using (SQR), by Lemma 9, this defines a function h_d on the directed edges of $G^{\pm}_{\Delta}(d, n)$ for any d. We fix an arbitrary vertex u of Π_n, which corresponds to a full flag $\mathcal{G} = (G_1, \ldots, G_n)$ in $[n]$. Recall that a hexagon corresponds to the flags extending $(S, Sijk)$ for some S and $i, j, k \in [n] \setminus S$. With this notation the condition (HXE) amounts to

$$h(Si, Sj) + h(Sj, Sk) + h(Sk, Si) = h(Sij, Sjk) + h(Sjk, Sik) + h(Sik, Sij) = 0.$$

It follows that h sums to zero along each oriented 2-face of $\Delta(d, n)$ (*i.e.*, a triangle). Thus Proposition 8 yields a function g on the vertices of $\Delta(d, n)$ with $\sum_{d=1}^{n} g(G_d) = w(u)$.

Let v be an arbitrary vertex of Π_n, which corresponds to a full flag $\mathcal{F} = (F_1, \ldots, F_n)$ in $[n]$. As a path from u to v decomposes into paths from G_d to F_d on $\Delta(d, n)$, we have $w(v) = \sum_{d=1}^{n} g_d(F_d)$. For a hexagon face described by the flags extending $(S, Sijk)$ with some $S \subset [n]$ and $i, j, k \in [n] \setminus S$, the sum representation of w agrees in all terms with $d \leq |S|$ and $d \geq |S| + 3$. Hence, by (HXM), the maximum is attained at least twice in

$$\max\big(g(Si) + g(Sij) + g(Sk) + g(Sjk), g(Sj) + g(Sjk) + g(Si) + g(Sik), \\ g(Sk) + g(Sik) + g(Sj) + g(Sij) \big).$$

Subtracting $g(Si) + g(Sj) + g(Sk) + g(Sij) + g(Sik) + g(Sjk)$ and multiplying by -1 yields that $\min(g(Sj) + g(Sik), g(Sk) + g(Sij), g(Si) + g(Sjk))$ attains the minimum twice. This is the 3-term incidence relation. Summarizing, we have proven that $(g_1, \ldots, g_n)$ satisfy the 1-step Plücker relations. By Theorem 7 all of the g_d are valuated matroids and so $(g_1, \ldots, g_n)$ is a valuated flag matroid. $\square$

5 Total positivity

The positive tropical Grassmannian $\mathrm{TGr}^{+}(d, n)$ is the tropicalization of the positive part of the Grassmannian $\mathrm{Gr}^{+}(d, n)$ consisting of linear spaces over the reals with all Plücker coordinates positive; see [21, 22, 3]. Following [3, Equation (1.1)], a valuated matroid v is *positive* if it fulfills the three-term positive tropical Plücker relations; *i.e.*, for every S and $i < j < k < l$ not in S, we have

$$v(Sik) + v(Sjl) = \min(v(Sij) + v(Skl), v(Sil) + v(Sjk)) \ . \tag{3TPR+}$$

Similarly, the positive part of the (full) flag variety can be defined as the space of full flags $\mathcal{L} = (L_1, \ldots, L_n)$ where $L_d \in \mathrm{Gr}^+(d,n)$. Let us formulate an "untropicalized" analog of one of the directions of Theorem 10. The Plücker coordinates of a linear space $L_d \subseteq \mathbb{K}^n$ over some field $\mathbb{K}$ can be organized into a polynomial:

$$f_{L_d}(x) = \sum_{B \in \binom{[n]}{d}} p_B \prod_{i \in B} x_i \in \mathbb{K}[x_1, \ldots, x_n],$$

whose support is the base polytope of the matroid $M(L_d)$. Given the flag $\mathcal{L}$, the coefficients of these polynomials satisfy

$$p_{Si} p_{Sjk} - p_{Sj} p_{Sik} + p_{Sk} p_{Sij} = 0 \,. \tag{5.1}$$

for all $S \subseteq [n]$ and $i < j < k$ not in S. To give an analog of (3TPR+), we work our way from the equation above to reach the condition (HXM).

The product $f_{L_1} \cdots f_{L_n}$ is a polynomial whose support is the flag matroid base polytope $M(L_1) + \cdots + M(L_n)$. Suppose this polytope is the regular permutahedron Π_n and let q_σ be the coefficient of $x_1^{\sigma(1)} \cdots x_n^{\sigma(n)}$ in $f_{L_1} \cdots f_{L_n}$. Hence, for $n = 3$, we have that $q_\sigma = p_{\sigma^{-1}(3)} p_{\sigma^{-1}(2)\sigma^{-1}(3)} p_{\sigma^{-1}(1)\sigma^{-1}(2)\sigma^{-1}(3)}$ yielding with (5.1) then

$$\begin{aligned} q_{(321)} q_{(123)} q_{(231)} q_{(213)} - q_{(231)} q_{(213)} q_{(312)} q_{(132)} + q_{(312)} q_{(132)} q_{(321)} q_{(123)} \\ = p_1 p_2 p_3 p_{12} p_{13} p_{23} p_{123}^4 \left(p_{12} p_3 - p_{13} p_2 + p_{23} p_1 \right) = 0 \,. \end{aligned}$$

Notice as well that $p_1 p_2 p_3 p_{12} p_{13} p_{23} = q_{(123)} q_{(231)} q_{(312)} = q_{(321)} q_{(213)} q_{(132)}$. From here we can deduce the following relations that must be satisfied for any n:

- For every hexagonal face $abcdef$ (labeled cyclically) of Π_n we have $q_a q_c q_e = q_b q_d q_f$ and $q_b q_c q_e q_f - q_a q_c q_b q_f + q_a q_b q_d q_e = 0$. By [23, Section 4.3] hexagons are Bruhat interval polytopes, and the negative term $q_a q_c q_b q_f$ is the one not containing the lowest and largest elements of this interval.

- For every square face $abcd$ (labeled cyclically) of Π_n we have $q_a q_c = q_b q_d$.

This suggests the following positivity condition for the tropicalization of the positive part of the flag variety; see Example 4:

(HXM+) For every (cyclically labeled) hexagon $abcdef$ of Π_n, where b is the lowest permutation in the Bruhat order, $w(b) + w(e) = \max(w(a) + w(d), w(c) + w(f))$.

We will need the following positivity adaptation of Theorem 7:

Lemma 11. *Let $\nu \colon \binom{[n]}{d} \to \mathbb{R}$ and $\mu \colon \binom{[n]}{d+1} \to \mathbb{R}$ be any two functions such that for every $S \in \binom{[n]}{d-1}$ and $i < j < k \notin S$,*

$$\nu(Sj) + \mu(Sik) = \min(\nu(Si) + \mu(Sjk), \nu(Sk) + \mu(Sij)) \,.$$

Then $\nu \in \mathrm{TGr}^+(d,n)$ and $\mu \in \mathrm{TGr}^+(d+1,n)$.

Proof. By Theorem 7 we know already that ν and μ are valuated matroids. Again we can do a translation to assume $\nu(Si) = \nu(Sj) = \nu(Sk) = \nu(Sl) = 0$. Suppose μ does not satisfy (3TPR+). Then without loss of generality $\mu(Sik) + \mu(Sjl) > \mu(Sij) + \mu(Skl)$. However, by the assumption of the lemma, $\mu(Sik) \leq \mu(Sij)$ and $\mu(Sjl) \leq \mu(Skl)$, a contradiction. □

Theorem 12. *Let $u\colon \mathrm{Sym}(n) \to \mathbb{R}$. Then the following are equivalent:*

1. *The function u is the compression of a valuated flag $(w_1, \ldots, w_n)$ which can be realized by a totally positive flag of linear spaces.*
2. *All polytopes in the subdivision induced by w are Bruhat interval polytopes.*
3. *The function u satisfies conditions (HXE) and (SQR) from Theorem 10 as well as (HXM+).*

Proof. "(1) $\to$ (2)". Consider a totally positive flag $\mathcal{L} = (L_1, \ldots, L_n)$ that realizes $(w_1, \ldots, w_n)$, *e.g.*, over the field of Puiseux series, or a suitable extension for irrational coefficients; see [12, Section 2.6]. For any polytope P in the subdivision induced by w, we can obtain a flag $\widetilde{\mathcal{L}} = (\widetilde{L}_1, \ldots, \widetilde{L}_n)$ that realizes the flag matroid corresponding to P by taking a suitable rescaling of the flag $\mathcal{L}$ and taking its quotient to the residue field, which is a subfield of $\mathbb{R}$. By construction, $\widetilde{\mathcal{L}}$ is in the totally non-negative part; yet some Plücker coordinates may vanish in the quotient. Due to Proposition 3, P is a Bruhat polytope.

"(2) $\to$ (3)". By Theorem 10 and Proposition 3, if the subdivision consists of Bruhat polytopes then (HXE) and (SQR) are satisfied. Suppose (HXM+) fails, so there is a hexagon where the maximum is not attained by the diagonal connecting the lowest and largest terms in the Bruhat order. Then this diagonal appears in the subdivision and it is not a Bruhat interval polytope; see Figure 1.

"(3) $\to$ (1)". First notice that by Theorem 10 the function u is the compression of the valuated flag matroid $(w_1, \ldots, w_n)$. Now consider the valuated matroid μ on the uniform matroid $U_{n,2n}$ given by $\mu(B) = w_{|B\cap[n]|}(B \cap [n])$; this construction appears, *e.g.*, in [16]. That μ is actually a valuated matroid depends on the right choice of w_d, which vary up to adding a constant. To see this and that, moreover, $\mu \in \mathrm{TGr}^+(n, 2n)$ we look at the 3-term Plücker relations (3TPR+). These are given by a set $S \in \binom{[2n]}{d-2}$ and $i, j, k, l \notin S$ and suppose $i < j < k < l$. We have the following cases:

- $|\{i, j, k, l\} \cap [n]| \leq 1$. In this case all terms in the Plücker relation are equal.
- $|\{i, j, k, l\} \cap [n]| = 2$. Assuming $i, j \in [n]$, we have that $\mu(Sik) + \mu(Sjl) = \mu(Sil) + \mu(Sjk) = w_{m+1}(Ti) + w_{m+1}(Tj)$ and $\mu(Sij) + \mu(Skl) = w_{m+2}(Tij) + w_m(T)$, for $T = S \cap [n]$ and $m = |T|$. Here is where we need to ensure supermodularity, *i.e.* $w_{m+2}(Tij) + w_m(T) \geq w_{m+1}(Ti) + w_{m+1}(Tj)$. Since we have the freedom to choose each w_d up to addition of a constant, we can achieve this by adding to each w_d

a suitably scaled convex function on d. As such a transformation preserves the compression u, we can ensure that $\mu(Sik) + \mu(Sjl)$ does attain the minimum.

- $|\{i,j,k,l\} \cap [n]| = 3$. The positive Plücker relation here is equivalent to the positive 3-term incidence relation between $w_{|S\cap[n]|+1}$ and $w_{|S\cap[n]|+2}$. The proof of Theorem 10 shows that the 3-term incidence relations are already implied by (HXM). Further, (HXM+) strengthens this to imply (3TPR+), since the terms of different sign agree under the correspondence of this implication, as seen in the discussion preceding the formulation of (HXM+).
- $|\{i,j,k,l\} \cap [n]| = 4$. In this case the Plücker relation is equivalent to a Plücker relation in $w_{|S\cap[n]|+2}$. By Lemma 11, the positive variation of this relation follows from the last case.

It was recently shown in [22, 3] that the positive tropical Grassmannian equals the positive Dressian. Hence, μ is realizable by some totally positive subspace $L \in \mathrm{Gr}^+(n, 2n)$. Consider π to be the projection to the first n coordinates and let $L_d = \pi(L \cap \{x_{n+d+1} = \cdots = x_{2n} = 0\})$. The tropicalization of L_d corresponds to the valuated matroid

$$\mu \backslash \{n+1, \dots, n+d\} / \{n+d+1, \dots, 2n\} = w_d \, .$$

We have $L_1 \subset \cdots \subset L_n$, all of which are positive. So $(w_1, \dots, w_d)$ is in the positive tropical flag variety. □

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