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Biclosed Sets in Affine Root Systems

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Abstract. The extended weak order is a partial order associated to a Coxeter system (W, S). It is the containment order on "biclosed" sets of positive roots in the (real) root system associated to W. When W is finite, this order coincides with the weak order on W, and is a lattice; when W is infinite, the weak order on W is a proper order ideal in the extended weak order. It is a longstanding conjecture of Matthew Dyer that the extended weak order is a lattice for any W. We prove this conjecture for the affine Coxeter groups. Furthermore, we give a parametrization of the extended weak order for these groups in terms of the face lattice of the Coxeter arrangement for the associated finite group.

Keywords: Coxeter groups, root systems, affine Coxeter groups, lattice theory

1 Key examples: The finite and affine symmetric groups

This is a paper about Coxeter groups, root systems, and the weak order. Discussing these topics in general requires a lot of notation, so we begin with two examples: The symmetric group S_n and the affine symmetric group $\widetilde{S}_n$. Some of the notation in this section will be replaced with more general notation in the following sections.

The symmetric group S_n is the group of permutations of $[n] := \{1, 2, \ldots, n\}$. Put $T = \{(i, j) : 1 \leq i < j \leq n\}$. For $\sigma \in S_n$, the set of ***inversions*** of σ is

$$I(\sigma) := \{(i, j) \in T : \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$

The (right) weak order on S_n is the partial order on S_n defined by $\sigma \leq \tau$ if and only if $I(\sigma) \subseteq I(\tau)$. Surprisingly, weak order is a lattice, meaning that every subset of S_n has a unique least upper bound and a unique greatest lower bound.

The group $\widetilde{S}_n$ is the group of bijections $\widetilde{\sigma} \colon \mathbb{Z} \to \mathbb{Z}$, obeying the conditions:

$$\widetilde{\sigma}(i + n) = \widetilde{\sigma}(i) + n, \qquad \sum_{k=1}^{n} (f(k) - k) = 0.$$

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Put $\widetilde{T} = \{(i,j) \in \mathbb{Z}^2/\big(\mathbb{Z}\cdot(n,n)\big) : i<j,\ i \not\equiv j \bmod n\}$. To be clear, $\mathbb{Z}^2/\big(\mathbb{Z}\cdot(n,n)\big)$ is the quotient of $\mathbb{Z}^2$ by the subgroup $\mathbb{Z}\cdot(n,n)$. For $\widetilde{\sigma} \in \widetilde{S}_n$, the set of ***inversions*** of $\widetilde{\sigma}$ is

$$I(\widetilde{\sigma}) := \{(i,j) \in \widetilde{T} : \widetilde{\sigma}^{-1}(i) > \widetilde{\sigma}^{-1}(j)\}.$$

We once again define weak order on $\widetilde{S}_n$ by $\widetilde{\sigma} \leq \widetilde{\tau}$ if and only if $I(\widetilde{\sigma}) \subseteq I(\widetilde{\tau})$.

This time, weak order is only a semilattice. This means that, if a subset $\mathcal{X}$ of $\widetilde{S}_n$ has an upper bound, then it has a least upper bound, but no upper bound need exist at all. (Every non-empty subset of $\widetilde{S}_n$ continues to have a greatest lower bound.)

Weak order has many applications, for example in the representation theory of quivers [5] and in the structure of cluster algebras [8]. These applications are well-understood for S_n. When applied to $\widetilde{S}_n$, however, the weak order fails to capture all of the structure we would like. For instance, in using Cambrian (semi-)lattices to model cluster algebra exchange graphs, the framework in [8] gives only "half" of the exchange graph for $\widetilde{S}_n$. Since quotients of the S_n weak order as a lattice are important for these constructions, we are motivated to find an extension of the weak order on $\widetilde{S}_n$ to some larger poset which is a (complete) lattice.

It is natural to imagine doing this by defining a collection of subsets of $\widetilde{T}$ that would play the role of "generalized inversion sets". Such a definition was proposed and studied by Matthew Dyer (see, *e.g.*, [2]); we will describe it here for the case of S_n and $\widetilde{S}_n$, and for any Coxeter group in the next section. **The main result of this paper will be that Dyer's construction (the *extended weak order*) is a lattice for all affine Coxeter groups.**

We define a subset J of T to be ***closed*** if, whenever $1 \leq i < j < k \leq n$ and (i,j) and $(j,k) \in J$, then $(i,k) \in J$.

We define a subset J of $\widetilde{T}$ to be ***closed*** if it obeys the three conditions

1. If $i<j<k$ with (i,j) and $(j,k) \in J$, and $i \not\equiv k \bmod n$, then $(i,k) \in J$.
2. If $i<j<k$ with (i,j) and $(j,k) \in J$, and $i \equiv k \bmod n$, then $(i, j+\ell n)$ and $(j, k+\ell n) \in J$ for all $\ell \geq 0$.
3. If $i<j<k$ with (i,j) and $(i,k) \in J$, and $j \equiv k \bmod n$, then $(i, j+\ell n) \in J$ for all $\ell \geq 0$ such that $j + \ell n \leq k$.

We define J to be ***coclosed*** if the complement of J is closed, and we define J to be ***biclosed*** if J is both closed and coclosed. We note that the intersection of any collection of closed sets is closed, and, for an arbitrary subset K of T or $\widetilde{T}$, we define the ***closure*** $\overline{K}$, to be the intersection of all closed sets containing K. Similarly, the union of any collection of coclosed sets is coclosed; we define the ***interior*** K° to be the union of all coclosed sets contained in K. The following is a special case of a result for all Coxeter groups:

Proposition 1.1. *The finite biclosed subsets of T (respectively $\widetilde{T}$) are exactly the sets of inversions of the permutations in S_n (respectively $\widetilde{S}_n$).*

The set T is finite, so all subsets of T are finite. But $\widetilde{T}$ is infinite, so it is natural to consider all biclosed subsets of $\widetilde{T}$, without the assumption of finiteness. The following is a special case of Dyer's conjectures and Dyer tells us that he already checked this case; we will provide the first published proof here:

Theorem 1.2. *The collection of all biclosed subsets of $\widetilde{T}$, ordered by containment, is a complete lattice. For any collection of biclosed sets $\mathcal{X}$, we have $\bigvee \mathcal{X} = \overline{\bigcup_{J \in \mathcal{X}} J}$ and $\bigwedge \mathcal{X} = \left(\bigcap_{J \in \mathcal{X}} J\right)^{\circ}$.*

Furthermore, we give a combinatorial model for the biclosed subsets of $\widetilde{T}$. Define a ***translationally invariant total order*** on $\mathbb{Z}$ to be a total order $\prec$ obeying the condition that $i \prec j$ if and only if $i + n \prec j + n$. Each translationally invariant total order $\prec$ defines a biclosed subset $I(\prec)$ of $\widetilde{T}$ by $I(\prec) = \{(i, j) \in \widetilde{T} : i \succ j\}$.

It turns out that any biclosed subset of $\widetilde{T}$ is of this form, but not in a unique way. To see the issue, take $n = 2$, and consider the following total orders on $\mathbb{Z}$:

$$\cdots \prec_1 -5 \prec_1 -3 \prec_1 -1 \prec_1 1 \prec_1 3 \prec_1 5 \prec_1 \cdots \prec_1 -4 \prec_1 -2 \prec_1 0 \prec_1 2 \prec_1 4 \prec_1 \cdots,$$
$$\cdots \prec_2 -5 \prec_2 -3 \prec_2 -1 \prec_2 1 \prec_2 3 \prec_2 5 \prec_2 \cdots \prec_2 4 \prec_2 2 \prec_2 0 \prec_2 -2 \prec_2 -4 \prec_2 \cdots.$$

In other words, both $\prec_1$ and $\prec_2$ put all the odd numbers before all the even numbers, but $\prec_1$ preserves the standard ordering within each parity class whereas $\prec_2$ reverses the even numbers. Then $I(\prec_1) = I(\prec_2) = \{(i, j) : i < j,\ i \equiv 0 \bmod 2,\ j \equiv 1 \bmod 2\}$.

Theorem 1.3. *Biclosed subsets of $\widetilde{T}$ are in bijection with translationally invariant total orders, modulo reversing the order on intervals of the form*

$$\cdots \prec k - 3n \prec k - 2n \prec k - n \prec k \prec k + n \prec k + 2n \prec k + 3n \prec \cdots.$$

The main results of this paper are not simply Theorems 1.2 and 1.3, but generalizations of these results to all affine Coxeter types, and a reduction of Dyer's conjectures in general to a more precise conjecture for rank three Coxeter groups. In the next section, we will introduce the vocabulary necessary to state our results.

2 Background

2.1 Coxeter groups and root systems

Let $(m_{ij})_{i,j=1}^{n}$ be a symmetric matrix of nonnegative integers such that $m_{ii} = 1$ for all i, and such that $m_{ij} \geq 2$ for $i \neq j$. We further allow ∞ to appear in the matrix. Such a matrix is a ***Coxeter matrix***. A ***Coxeter group*** is a group with a presentation of the form

$$W = \langle s_1, \ldots, s_n : (s_i s_j)^{m_{ij}} = 1,\ 1 \leq i, j \leq n \rangle$$

for some Coxeter matrix (m_{ij}). In this case we say (W, S) is a ***rank n Coxeter system***, where $S = \{s_1, \ldots, s_n\}$ is the set of ***simple reflections***. The ***length*** $\ell(w)$ of an element w is the minimal number k of simple reflections in any word $s_1 \cdots s_k = w$. We further write

$$T = \{wsw^{-1} : w \in W, s \in S\}$$

for the set of ***reflections*** in W.

Associated to each Coxeter group is a faithful n-dimensional representation V called the ***geometric*** or ***reflection representation***. Fix a ***Cartan matrix*** $(A_{ij})_{i,j=1}^n$ such that $A_{ii} = 2$, such that $A_{ij} = 0$ if $m_{ij} = 2$, such that A_{ij}, $A_{ji} < 0$ with $A_{ij}A_{ji} = 4\cos^2 \frac{\pi}{m_{ij}}$ if $3 \leq m_{ij} < \infty$, and such that A_{ij}, $A_{ji} < 0$ with $A_{ij}A_{ji} \geq 4$ if $m_{ij} = \infty$.

The geometric representation has a basis $\alpha_1, \ldots, \alpha_n$. We will also want to work with the dual representation $V^\vee$, which has dual basis $\omega_1, \omega_2, \ldots, \omega_n$. We put $\alpha_j^\vee = \sum_i a_{ij}\omega_i$, so $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$. (Note that $\alpha_i^\vee$ need not be a basis of $V^\vee$.) Then W acts on V by $s_i(v) = v - \langle \alpha_i^\vee, v\rangle \alpha_i$ and acts on $V^\vee$ by the dual formula $s_i(f) = f - \langle f, \alpha_i \rangle \alpha_i^\vee$. It is a standard result that V and $V^\vee$ are faithful representations of W.

The (real) ***roots*** are the vectors of the form $w\alpha_i$ for $w \in W$, $1 \leq i \leq n$. Each root is either ***positive***, meaning in the positive linear span of the α_i, or ***negative***, meaning the negation of a positive root. We write Φ and Φ_+ for the sets of roots and positive roots respectively; these are called the ***root system*** and the ***positive root system***. Similarly, we define the co-root system $\Phi^\vee = \{w\alpha_i^\vee : w \in W,\ i \in I\}$.

The reflections in T are precisely those elements of W which act on V and $V^\vee$ by involutions fixing a codimension 1 subspace. Specifically, for each $t \in T$, there is a unique positive root β_t and co-root $\beta_t^\vee$ such that $t(v) = v - \langle \beta_t^\vee, v\rangle \beta_t$.

We are interested in the weak order on W. To this end, define the set of (left) ***inversions*** of a group element $w \in W$ to be

$$N(w) = \{t \in T : \ell(tw) < \ell(w)\}.$$

The inversion set also admits a geometric description: Take a point f in $V^\vee$ such that $\langle f, \alpha_i \rangle > 0$ for each i, then t is an inversion of w if and only if $\langle wf, \beta_t \rangle < 0$. We say $w \leq v$ in ***weak order*** if $N(w) \subseteq N(v)$.

A subset Φ' of Φ is called a ***subsystem*** of Φ if Φ' is preserved by reflections over any $\beta \in \Phi'$. We define $\Phi'_+ = \Phi' \cap \Phi_+$ and $\Phi'_- = \Phi' \cap \Phi_-$. The following is a collection of results from [3] and [4].

Proposition 2.1. *Let Φ' be a subsystem of Φ. There is a unique minimal set $\Pi' \subseteq \Phi'_+$ such that*

$$\mathbb{R}_{\geq 0}\Pi' \cap \Phi' = \Phi'_+ = -\Phi'_-.$$

Let W' be the reflection subgroup of W generated by reflections over the roots in Φ', and let S' be the reflections over elements of Π'. Then (W', S') is a Coxeter system, and the bijection between T and Φ_+ restricts to a bijection between Φ'_+ and reflections in W'. Furthermore,

$$S' = \{t \in T : N(t) \cap W' = \{t\}\}.$$

If additionally the span of Φ' is 2-dimensional, then $|\Pi'| = 2$.

The elements of Π' are called the ***simple roots*** of Φ'. The cardinality of Π' is called the ***rank*** of Φ'. The group W' is called the ***reflection subgroup*** generated by Φ'. We say a subsystem Φ' is ***full*** if whenever α and $\beta \in \Phi'$, it follows that any γ in the span $\mathbb{R}\alpha + \mathbb{R}\beta$ is also in Φ'. The ***type*** of Φ' is a description of the Coxeter matrix of (W', S'). If X_n is a Coxeter type, then a *full* subsystem of Φ with type X_n is called an X_n***-subsystem***. (*E.g.*, an A_2-subsystem consists of roots $\alpha, \beta, \alpha + \beta$ and their negations, such that no other elements of Φ are in their span.)

A root system is ***indecomposable*** when it cannot be written as the disjoint union of two nonempty full subsystems. A Coxeter group with an indecomposable root system is called ***irreducible***. Every root system can be written uniquely as a disjoint union of indecomposable subsystems, and each Coxeter group can be written uniquely as a product of irreducible factors.

2.2 Finite and affine Coxeter groups

Let (A_{ij}) be a Cartan matrix with integer entries such that the associated root system Φ_0 is finite and indecomposable. Such matrices are classified by ***Dynkin diagrams***. There is a unique root $\theta \in (\Phi_0)_+$ such that $\theta - \alpha$ is in the positive span of $(\Phi_0)_+$ for any $\alpha \in (\Phi_0)_+$. The root θ is called the ***highest root*** of Φ_0.

We now construct a new root system Φ. If Φ_0 is of type X_n, then we say Φ is of type $\widetilde{X}_n$, and call it the ***affine root system*** associated to Φ_0. If V_0 is the ambient n-dimensional vector space of Φ_0, then let $V := V_0 \oplus \mathbb{R}\delta$ to be an $(n+1)$-dimensional vector space, with new basis vector δ. We define $\langle \alpha_i^\vee, \delta \rangle = 0$ for all simple coroots α_i. Then Φ is the set

$$\Phi = \{\alpha + k\delta : \alpha \in \Phi_0,\ k \in \mathbb{Z}\}.$$

We endow Φ with the set of simple roots $\alpha_1, \dots, \alpha_n, \alpha_0$, where $\alpha_1, \dots, \alpha_n$ are the simple roots of Φ_0 and $\alpha_0 = \delta - \theta$. The corresponding new coroot is $\alpha_0^\vee := -\theta^\vee$. Then Φ is a root system [6]. The associated Coxeter group is called the ***affine Coxeter group*** of type $\widetilde{X}$. Up to isomorphism, the irreducible affine Coxeter groups are classified into the families $\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n$, or one of the exceptional types $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8, \widetilde{F}_4, \widetilde{G}_2$. The affine symmetric group $\widetilde{S}_n$ from the introduction is the affine Coxeter group of type $\widetilde{A}_{n-1}$.

For each pair $\pm\beta$ of roots in Φ_0, there is a type $\widetilde{A}_1$ root subsystem of Φ, whose elements are $\{\pm\beta + k\delta : k \in \mathbb{Z}\}$. Each root of Φ lies in exactly one such $\widetilde{A}_1$ subsystem.

To give a concrete model for biclosed sets in affine type in Section 3, we recall the definition of the ***Coxeter fan*** of a finite Coxeter group W_0. This is a complete fan living in the coroot representation $V_0^\vee$ of W_0. The set F_0 of $f \in V_0^\vee$ such that $\langle f, \alpha \rangle \geq 0$ for all $\alpha \in (\Phi_0)_+$ is called the ***dominant chamber***. A ***chamber*** is any cone of the form wF_0 for $w \in W_0$. The Coxeter fan consists of all cones formed by intersections of chambers. These cones are called the ***faces*** of the fan. Let $\mathrm{Faces}(W_0)$ denote the set of faces of the Coxeter fan of W_0. The action of W_0 on $V_0^\vee$ permutes the elements of $\mathrm{Faces}(W_0)$.

There is also an action of the affine Coxeter group W associated to W_0 on $\mathrm{Faces}(W_0)$, described as follows. If $V^\vee$ is the coroot representation for W, then notice that W preserves the subspace consisting of functionals which annihilate the vector $\delta \in V$. This subspace has a basis given by $\alpha_1^\vee, \dots, \alpha_n^\vee$ and thus can be canonically identified with $V_0^\vee$; we do so from here on. This identification respects the action of $W_0 \subseteq W$. Given a face $F \in \mathrm{Faces}(W_0)$, let W_F denote the stabilizer of F under the W action, and let Φ_F be the set of roots in Φ which are annihilated by all functionals in F. Then W_F is the reflection subgroup corresponding to the root subsystem Φ_F, called a ***parahoric subgroup*** of W.

2.3 Closed and biclosed sets

A subset J of Φ_+ is called ***closed*** if, for any three roots α, β and γ with $\gamma \in \mathbb{R}_{>0}\alpha + \mathbb{R}_{>0}\beta$, if α and $\beta \in J$ then $\gamma \in J$. A subset J of Φ_+ is called ***coclosed*** if $\Phi_+ \setminus J$ is closed. The set J is ***biclosed*** if it is both closed and coclosed. As in the introduction, for a subset A of Φ_+, we define the ***closure*** $\overline{A}$ to be the intersection of all closed sets containing A and we define the ***interior*** A° to be the union of all coclosed sets contained in A.

We pause to connect this notion to the closed subsets from the introduction. Let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{R}^n$, then $\{e_i - e_j : 1 \le i < j \le n\}$ form a root system of type A_{n-1}, which spans an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. The rank two subsystems of Φ_+ come in two forms: The type A_2 subsystems are $\{e_i - e_j,\ e_i - e_k,\ e_j - e_k\}$ for $i < j < k$, and the type $A_1 \times A_1$ subsystems are $\{e_i - e_j,\ e_k - e_\ell\}$ for $i < j$ and $k < \ell$ distinct. The closure conditions for the A_2-subsystems are the conditions from the introduction; the closure conditions for the $A_1 \times A_1$-subsystems are trivial, since they only have two positive roots.

Similarly, define $\widehat{V}$ to be the $n+1$-dimensional vector space spanned by a collection of vectors $\widetilde{e}_i$, for $i \in \mathbb{Z}$, modulo the relation $\widetilde{e}_{i+n} - \widetilde{e}_i = \widetilde{e}_{j+n} - \widetilde{e}_j$ for all i, j. Then the vectors $\{\widetilde{e}_i - \widetilde{e}_j : i < j,\ i \not\equiv j \bmod n\}$ form a copy of the positive $\widetilde{A}_{n-1}$-system, inside an n-dimensional subspace of $\widehat{V}$. There are three types of full rank two subsystems: For $i < j < k$ all distinct modulo n, we have an A_2-subsystem $\{\widetilde{e}_i - \widetilde{e}_j,\ \widetilde{e}_i - \widetilde{e}_k,\ \widetilde{e}_j - \widetilde{e}_k\}$; for $i < j < i+n$ we have an $\widetilde{A}_1$-subsystem $\{\widetilde{e}_{i+pn} - \widetilde{e}_{j+qn} : i+pn < j+qn\} \cup \{\widetilde{e}_{j+pn} - \widetilde{e}_{i+qn} : j+pn < i+qn\}$ and, for $i < j$, $k < \ell$ distinct modulo n, we have an $A_1 \times A_1$-subsystem $\{\widetilde{e}_i - \widetilde{e}_j,\ \widetilde{e}_k - \widetilde{e}_\ell\}$. The closure conditions for the A_2 and $\widetilde{A}_1$-subsystems are the conditions from the introduction; the closure conditions for the $A_1 \times A_1$-subsystems are trivial.

Recall that the set of inversions of a Coxeter group element can be described geometrically: $N(w) = \{t \in T : \langle wf, \beta_t \rangle < 0\}$, where $f \in V^\vee$ obeys $\langle f, \alpha_i \rangle > 0$. From this geometric description, it is clear that $\{\beta_t : t \in N(w)\}$ is biclosed for any $w \in W$. This characterizes the finite biclosed sets:

Proposition 2.2 ([2]). *A set B of positive roots is finite and biclosed if and only if there exists $w \in W$ such that $B = \{\beta_t : t \in N(w)\}$. In this case, w is uniquely defined.*

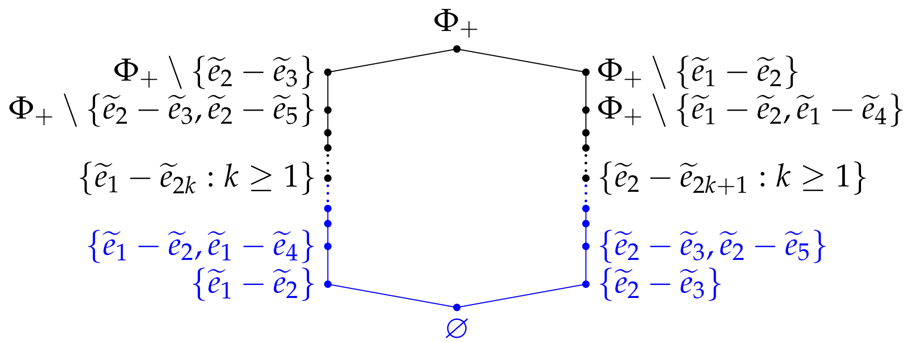

Figure 1: The extended weak order for $\widetilde{A}_1$. The finite biclosed sets are shaded in blue. The subposet consisting of the finite biclosed sets is the usual weak order for $\widetilde{A}_1$; note the usual weak order is not a lattice, but the extended weak order is a complete lattice.

Define the ***extended weak order*** to be the poset of biclosed sets ordered by inclusion. This paper is motivated by Dyer's conjecture, first discussed in [1, Remark 2.14]:

Conjecture 2.3. *The extended weak order is a complete lattice. For X a collection of biclosed sets, we have $\bigvee X = \overline{\bigcup_{J \in X} J}$ and $\bigwedge X = \left(\bigcup_{J \in X} J\right)^{\circ}$.*

This conjecture is known in finite type [2, Theorem 1.5] and rank two, and was recently shown by Weijia Wang to hold for rank three affine Coxeter groups [10]. In Section 4, we give the first proof of Dyer's conjecture for all affine Coxeter groups. In Section 3, we give a useful parametrization of biclosed sets for these groups (which is independent of Section 4).

3 A combinatorial model for biclosed sets in affine type

We now use the material from Section 2.2 to give a description of the biclosed sets for affine Coxeter groups generalizing Theorem 1.3. We omit proofs in this section.

We define two biclosed sets, B and C, to be ***commensurable*** if the symmetric difference $B \oplus C := (B \setminus C) \sqcup (C \setminus B)$ is finite. We will begin by describing biclosed sets up to commensurability. For a biclosed subset B of Φ_+, we define

$$B_\infty = \{\beta \in \Phi_0 : \beta + k\delta \in B \text{ for } k \gg 0\}.$$

It is easy to see that $B_\infty = C_\infty$ if and only if B and C are commensurable. To understand the sets which can appear as B_∞, we will use the Coxeter fan described in Section 2.2.

Let F be a face of the W_0-Coxeter fan and let f lie in the relative interior of F. Then $\{\beta \in \Phi_+ : \langle f, \beta \rangle < 0\}$ and $\{\beta \in \Phi_+ : \langle f, \beta \rangle \leq 0\}$ are both biclosed sets. More generally, let $\Phi_F = \{\beta \in \Phi : \langle f, \beta \rangle = 0\}$, then Φ_F is the root system associated to the parahoric subgroup W_F. If Φ' is any union of indecomposable components of Φ_F, then $B(F, \Phi') := \{\beta \in \Phi_+ : \langle f, \beta \rangle < 0\} \cup (\Phi' \cap \Phi_+)$ is biclosed.

Proposition 3.1. *Each biclosed set is commensurable to a unique biclosed set of the form $B(F, \Phi')$ for $F \in \mathrm{Faces}(W_0)$ and Φ' a (possibly empty) union of indecomposable components of Φ_F.*

We note that $B(\mathbf{0}, \varnothing) = \varnothing$ and that the commensurability class of $B(\mathbf{0}, \varnothing)$ consists of the finite biclosed sets which, by Proposition 2.2, are in bijection with the elements of W. Following [2, Section 4.1], we define an action of W on the set of biclosed classes by

$$w \cdot B = (wB \cap \Phi_+) \cup \big(\{\beta_t : t \in N(w)\} \setminus (-wB)\big).$$

Then we can generalize our description of $B(\mathbf{0}, \varnothing)$ to

Proposition 3.2. *The set of biclosed sets commensurable to $B(F, \Phi')$ is the W_F-orbit of $B(F, \Phi')$, and W_F acts freely on this orbit.*

Combining our results, biclosed sets are in bijection with triples (F, Φ', w) where F is a face of $\mathrm{Faces}(W_0)$, where Φ' is a union of indecomposable components of Φ_F, and where $w \in W_F$.

Remark 3.3. The limit weak order of Lam and Pylyavskyy [7] corresponds to triples $(F, \varnothing, w)$.

Let us work out what this looks like for $W = \widetilde{S}_n$, the affine Coxeter group of type $\widetilde{A}_{n-1}$. We will reuse the notation $\widehat{V}$ from Section 2.3 and we let V be the subspace of $\widehat{V}$ spanned by $\widetilde{e}_i - \widetilde{e}_j$. Similarly, we let V_0 be the subspace of $\mathbb{R}^n$ spanned by $e_i - e_j$. The projection $V \to V_0$ is induced by the map $\widehat{V} \to \mathbb{R}^n$ sending $\widetilde{e}_i$ to $e_{i \bmod n}$. So the dual vector space, $(V_0)^\vee$, is $\mathbb{R}^n / \mathbb{R}(1, 1, \ldots, 1)$, and we will write elements of $(V_0)^\vee$ as $(f_1, f_2, \ldots, f_n)$.

The faces of the S_n Coxeter fan are in bijection with ordered set partitions (equivalently, total preorders) of $[n]$. For example, with $n = 4$, the ordered set partition $(\{1, 3\}, \{2, 4\})$ corresponds to the face $\{f_1 = f_3 \leq f_2 = f_4\}$. Given an ordered set partition $(B_1, B_2, \ldots, B_r)$ of $[n]$ with corresponding face F, a permutation $w \in S_n$ stabilizes F if and only if w sends each block B_i to itself. For example, the stabilizer of $\{f_1 = f_3 \leq f_2 = f_4\}$ is $\langle (13),\ (24) \rangle$.

The parahoric subgroup W_F is the preimage of this stabilizer in $\widetilde{S}_n$. To describe this, put $\widetilde{B}_a = \{i \in \mathbb{Z} : i \bmod n \in B_a\}$. Then Φ_F is $\bigcup_{a=1}^{r} \{\widetilde{e}_i - \widetilde{e}_j : i, j \in \widetilde{B}_q,\ i \not\equiv j \bmod n\}$. The irreducible components of Φ_F correspond to the non-singleton blocks of the set partition. So, to give a pair (F, Φ') is to give an ordered set partition $(B_1, B_2, \ldots, B_r)$ of $[n]$ and a subset R of the non-singleton blocks of this partition.

The biclosed set $B(F, \Phi')$ corresponds to the inversions of a translation invariant total order defined as follows: If $a < b$, $i \in \widetilde{B}_a$ and $j \in \widetilde{B}_b$, then we put $i \prec j$. If B_a is a non-singleton block, and i and $j \in \widetilde{B}_a$ with $i < j$, then we put $i \prec j$ if $B_a \notin R$ and $i \succ j$ if $B_a \in R$. If B_a is a singleton block, the order on $\widetilde{B}_a$ is undetermined, this will give the indeterminacy in Theorem 1.3.

More generally, an element w of W_F is a permutation in $\widetilde{S}_n$ which takes each $\widetilde{B}_a$ to itself. Applying such permutations to the translation invariant total orders described above gives the whole of Theorem 1.3.

4 Dyer's conjecture and the CU-property

In this section we prove Conjecture 2.3 for affine Coxeter groups. Conjecture 2.3 would follow from the following conjecture (also due to Dyer, see [2, Conjecture 2.6]).

Conjecture 4.1. *Let U be a coclosed subset of Φ_+, then $\overline{U}$ is also coclosed. Dually, let K be a closed subset of Φ_+, then K° is also closed.*

The statements about $\overline{U}$ and about K° are equivalent, by putting $K = \Phi_+ \setminus U$.

Proof that Conjecture 4.1 implies Conjecture 2.3. Let $\mathcal{X}$ be any collection of biclosed subsets of Φ_+ and put $V := \overline{\bigcup_{J\in\mathcal{X}} J}$. We will show that V is biclosed and is the join of $\mathcal{X}$. Indeed, V is closed since it is defined as a closure. The set $\bigcup_{J\in\mathcal{X}} J$ is coclosed, since it is a union of coclosed sets, so Conjecture 4.1 implies that V is coclosed; we have now checked that Y is biclosed.

For any $J_0 \in \mathcal{X}$, we have $J_0 \subseteq \bigcup_{J\in\mathcal{X}} J \subseteq V$, so V is an upper bound for $\mathcal{X}$. If Y is any other biclosed upper bound for $\mathcal{X}$ then, by the definition of an upper bound, we have $\bigcup_{J\in\mathcal{X}} J \subseteq Y$; since Y is closed, we then have $V \subseteq Y$ as well. We have thus shown that Y is the unique least upper bound of $\mathcal{X}$. The case of lower bounds is analogous. □

To prove Conjecture 4.1, we will need the notion of a ***unipodal*** set of roots. A set $U \subseteq \Phi_+$ is unipodal if, for any full rank two subsystem Φ' of Φ, whenever $\Phi' \cap U$ is nonempty it follows that some simple root α of Φ' is in U. All coclosed sets are unipodal, and unipodal sets are preserved by arbitrary unions. Furthermore, any closed set which is unipodal is in fact biclosed.

We now introduce a new property, which is designed to prove Conjecture 4.1. Let A be a subset of Φ_+. We will say that a positive root γ is ***nocked*** with respect to A there are positive roots α and β, with γ a positive linear combination of α and β, such that $\alpha, \beta \in A$ and $\gamma \notin A$. We write $\mathrm{Nock}(A)$ for the set of γ which are nocked with respect to A. To motivate the definition of $\mathrm{Nock}(A)$, note that the closure of A is the ascending union of the chain $A \subseteq A \cup \mathrm{Nock}(A) \subseteq A \cup \mathrm{Nock}(A) \cup \mathrm{Nock}(A \cup \mathrm{Nock}(A)) \subseteq \cdots$, so the nocked[1] roots are the ones which get added to the closure "first".

We partially order Φ_+ by ***cutting order***. If α and β are positive roots, then β is over α in cutting order if, in the full rank two system containing them, α is simple and β is not. To show that (the transitive closure of) these relations in fact define a partial order,

[1]The word "nocked" comes from archery and describes an arrow ready to be fired, as the nocked roots are ready to be added.

rather than a preorder, it is enough to exhibit a partial order which extends the cutting order. Indeed, Proposition 4.3.18(3) of [9] shows that if α is below β in cutting order, then α has a smaller depth than β.

Definition 4.2. We say that Φ_+ has the *CU**-property*** if, whenever U is a unipodal set and γ is minimal in $\mathrm{Nock}(U)$, then $U \cup \{\gamma\}$ is unipodal.

It is immediate to see that the CU-property holds if Φ_+ has rank ≤ 2.

Proposition 4.3. *If Φ_+ has the CU-property then Conjecture 4.1, and hence Conjecture 2.3, hold with respect to Φ_+.*

Proof. Let U be a unipodal set, let K be the closure $\overline{U}$ and let V be the interior K°. We want to show that $K = V$.

Since V is the union of all coclosed sets contained in K, we have $U \subseteq V$ and thus K is the closure of V. If $K \neq V$ then there must be some element of K which is nocked with respect to V. Choose a γ in K which is nocked with respect to V and is minimal with respect to this property. Then, by the CU-property, $V \cup \{\gamma\}$ is unipodal. But then γ would be in the interior of K, contradicting our choice of γ not in V. □

The advantage of the CU-property is the next result:

Proposition 4.4. *Suppose that every rank three subsystem of Φ_+ has the CU-property. Then Φ_+ has the CU-property.*

Proof. Suppose that Φ_+ does not obey the CU-property. Then there is some unipodal set U and some root γ which is minimal in $\mathrm{Nock}(U)$, such that $U \cup \{\gamma\}$ is not unipodal.

The condition that $\gamma \in \mathrm{Nock}(U)$ means that there are positive roots α and β, with γ in the positive span of α and β, such that α, $\beta \in U$ and $\gamma \notin U$. The condition that $U \cup \{\gamma\}$ is not unipodal means that there must be some positive roots α', β' which are simple in the rank two subsystem they span, and some root γ' in the positive span of α and β. These satisfy $\alpha', \beta' \notin U \cup \{\gamma\}$ and $\gamma' \in U \cup \{\gamma\}$. But, if $\gamma' \in U$ then this would contradict that U is unipodal, so we must have $\gamma = \gamma'$.

Let R and R' be the full rank two subsystems containing $\{\alpha, \gamma, \beta\}$ and $\{\alpha', \gamma, \beta'\}$ respectively. Let X be the minimal full subsystem containing R and R' and let Φ_+^X be its positive roots; X is the full subsystem generated by α, γ, α', and hence has rank three by the following lemma.

Lemma 4.5. *The minimal full subsystem containing a given set of r roots has rank at most r.*

Then $U \cap \Phi_+^X$ is unipodal in Φ_+^X, and the root system R shows that γ is still nocked with respect to $U \cap \Phi_+^X$. The partial order on Φ_+^X is the restriction of the one on Φ^+ so γ is still minimal in $\mathrm{Nock}(U \cap \Phi_+^X)$. But then R' shows that $(U \cap \Phi_+^X) \cup \{\gamma\}$ is not unipodal in Φ_+^X, contradicting the CU-property in the rank three subsystem Φ_X^+. □

It is a short verification that the CU-property holds for the finite root systems of rank three with integral Cartan matrix: A_3, B_3, and $A_1 \times X_2$ for X_2 a finite rank 2 root system. Thus, we immediately deduce that the CU-property, Conjecture 4.1 and Conjecture 2.3 hold for all finite crystallographic root systems. The fact that biclosed sets form a complete lattice is classical in this setting, but the formula for the join is already interesting. We note that the CU-property fails in type $\widetilde{C}_2$, so we cannot expect every root system to have this property. However, one of the main results of this extended abstract is:

Theorem 4.6. *Conjectures 2.3 and 4.1 hold for any affine root system. Furthermore, the* **simply-laced** *affine root systems (types $\widetilde{A}_n$, $\widetilde{D}_n$, and $\widetilde{E}_n$) have the CU-property.*

The proof of the theorem reduces to the case of simply-laced types by the technique of ***folding*** root systems (which for brevity we shall not discuss here). To show the CU-property in that case (and thus Conjectures 2.3 and 4.1), we use Proposition 4.4 to reduce to checking the rank three subsystems, which in a simply-laced system are all of type $\widetilde{A}_2$, A_3, or $A_1 \times X_2$ for X_2 of rank two. The finite and decomposable systems are a quick check; we will take on the affine system $\widetilde{A}_2$ in the following.

Theorem 4.7. *Let Φ_+ be the positive root system of type $\widetilde{A}_2$, and let $U \subseteq \Phi_+$ be a unipodal subset. Let γ be a minimal element of $\mathrm{Nock}(U)$, with respect to cutting order. Then $U \cup \{\gamma\}$ is also unipodal.*

We will denote the simple roots of Φ_+ as α_0, α_1 and α_2, so the imaginary root is $\delta = \alpha_0 + \alpha_1 + \alpha_2$. We write β_j for $\delta - \alpha_j$ and, for $t \geq 0$, we set $\alpha_j^t = \alpha_j + t\delta$ and $\beta_j^t = \beta_j + t\delta$. So each element of Φ_+ is denoted as one of α_0^t, α_1^t, α_2^t, β_0^t, β_1^t, β_2^t for $t \geq 0$. Note that we have $\alpha_0^t + \alpha_1^u = \beta_2^{t+u}$ and $\beta_0^t + \beta_1^u = \alpha_2^{t+u+1}$, and similarly for all permutations of the subscripts.

The condition that γ is minimal in $\mathrm{Nock}(U)$ means that, anytime that we have η' and $\eta'' \in U$ and some root η is in the positive span of η' and η'', with η less than γ, we can deduce that η is in U. In this case, we will say that η' and η'' ***force*** η.

We first outline the cases we will consider: The condition that γ is nocked means that $\gamma \notin U$ but γ is in the positive span of some other roots γ' and γ'' which are in U; let P be the rank two subsystem containing γ' and γ''. The only way that $U \cup \{\gamma\}$ could fail to be unipodal is if γ is contained in some rank two subsystem Q whose simple roots are not in U. We break into cases according to the types of P and Q:

Lemma 4.8. *If P and Q are both type $\widetilde{A}_1$, then $U \cup \{\gamma\}$ is unipodal with respect to Q.*

Proof. In this case, since $\gamma \in P \cap Q$, we must have $P = Q$. But then $\gamma' \in Q \cap U$, and U is unipodal, so U contains one of the simple roots of Q, as desired. □

Lemma 4.9. *If P is type A_2, and Q is type $\widetilde{A}_1$, then $U \cup \{\gamma\}$ is unipodal with respect to Q.*

Proof. Without loss of generality, Q is the subsystem spanned by α_0 and β_0, so either $\gamma = \alpha_0^t$ or $\gamma = \beta_0^t$ for some t. The statement is trivial for $t = 0$, so we assume $t > 0$. We also note that, as soon as we show that any α_0^u or β_0^u is in U, we are done by the unipodality of U.

Case 1: $\gamma = \alpha_0^t$, $\gamma' = \beta_1^u$ and $\gamma'' = \beta_2^{t-u-1}$:

Now, $\gamma' = \beta_1^u = \alpha_0^u + \alpha_2$ so, by unipodality, either α_0^u or α_2 is in U. If $\alpha_0^u \in U$, we are done, so we may assume $\alpha_2 \in U$. Similarly, $\gamma'' = \beta_2^{t-u-1} = \alpha_0^{t-u-1} + \alpha_1$ show that we may assume $\alpha_1 \in U$. But then α_1 and α_2 force β_0 into U, and we are done.

Case 2: We have $\gamma = \beta_0^t$, $\gamma' = \alpha_1^u$ and $\gamma'' = \alpha_2^{t-u}$:

Since $t > 0$, either $u > 0$ or $t - u > 0$; without loss of generality, we assume $u > 0$. Now, $\gamma' = \alpha_1^u = \beta_0^{u-1} + \beta_2$ so, by unipodality, either β_0^{u-1} or β_2 is in U. If $\beta_0^{u-1} \in U$, we are done. If $\beta_2 \in U$, then unipodality implies that either α_0 or $\alpha_1 \in U$. If $\alpha_0 \in U$, we are again done. We have now shown that both α_1 and α_2^{t-u} are in U. But then α_1 and α_2^{t-u} force β_0^{t-u} are we are done again. □

The other cases are similar, but longer; they are available in the paper accompanying this extended abstract.

References

[1] M. Dyer. "Quotients of twisted Bruhat orders". *J. Algebra* **163**.3 (1994), pp. 861–879. DOI.

[2] M. Dyer. "On the Weak Order of Coxeter Groups". *Canadian Journal of Mathematics* **71**.2 (2019), pp. 299–336. DOI.

[3] M. Dyer. "Reflection subgroups of Coxeter systems". *J. Algebra* **135**.1 (1990), pp. 57–73. DOI.

[4] M. Dyer. "On the âĂIJBruhat graphâĂİ of a Coxeter system". *Compos. Math.* **2**.2 (1991), pp. 185–191. Link.

[5] O. Iyama, N. Reading, I. Reiten, and H. Thomas. "Lattice structure of Weyl groups via representation theory of preprojective algebras". *Compos. Math.* **154**.6 (2018), 1269âĂŞ1305.

[6] V. G. Kac. *Infinite-Dimensional Lie Algebras*. Boston, MA: Birkhäuser Boston, 1983. DOI.

[7] T. Lam and P. Pylyavskyy. "Total positivity for loop groups II: Chevalley generators". *Transform. Groups* **18**.1 (2013), pp. 179–231. DOI.

[8] N. Reading and D. E. Speyer. "Combinatorial Frameworks for Cluster Algebras". *Int. Math. Res. Not. IMRN* **2016**.1 (May 2015), pp. 109–173. DOI.

[9] F. Viard. "Des graphes orientés aux treillis complets : une nouvelle approche de l'ordre faible sur les goupes de Coxeter". Theses. Université Claude Bernard - Lyon I, Nov. 2015.

[10] W. Wang. "Infinite reduced words, lattice property and braid graph of affine Weyl groups". *J. Algebra* **536** (2019), pp. 170–214. DOI.