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A Web Basis of Invariant Polynomials From Noncrossing Partitions

Rebecca Patrias*[1], Oliver Pechenik†[2] and Jessica Striker‡[3]

[1] *Department of Mathematics, University of St. Thomas, St. Paul, MN 55105, USA*
[2] *Department of Combinatorics & Optimization, University of Waterloo, ON N2L 3G1, Canada*
[3] *Department of Mathematics, North Dakota State University, Fargo, ND 58102, USA*

Abstract. The irreducible representations of symmetric groups can be realized as certain graded pieces of invariant rings, equivalently as global sections of line bundles on partial flag varieties. There are various ways to choose useful bases of such Specht modules S^λ. Particularly powerful are web bases, which make important connections with cluster algebras and quantum link invariants. Unfortunately, web bases are only known in very special cases — essentially, only the cases $\lambda = (d, d)$ and $\lambda = (d, d, d)$. Building on work of B. Rhoades (2017), we construct an apparent web basis of invariant polynomials for the 2-parameter family of Specht modules with λ of the form $(d, d, 1^\ell)$. The planar diagrams that appear are noncrossing set partitions, and we thereby obtain geometric interpretations of earlier enumerative results in combinatorial dynamics.

Keywords: noncrossing partitions, web invariants, Specht modules

1 Introduction

Specht modules S^λ, indexed by integer partitions, are the irreducible complex representations of symmetric groups S_n. Unsurprisingly, there are many different ways to construct and describe Specht modules. These various constructions yield distinct linear bases of S^λ that are variously well adapted to one task or another. Understanding the resulting basis changes often leads to deep and hard problems.

An important construction of the Specht module is as a space of invariant polynomials for a Lie group of block lower-triangular matrices, or equivalently as global sections of a certain line bundle on a *partial flag variety*. In this avatar, *standard monomial theory* endows each S^λ with a natural basis of polynomials, encoded by *standard Young tableaux* of shape λ.

*rebecca.patrias@stthomas.edu
†oliver.pechenik@uwaterloo.ca. Partially supported by NSERC Discovery Grant RGPIN-2021-02391 and Launch Supplement DGECR-2021-00010.
‡jessica.striker@ndsu.edu. Partially supported by Simons Foundation/SFARI grant (527204, JS).

In special cases, this realization of S^λ has another remarkable basis, consisting of a different set of invariant polynomials, encoded by planar diagrams called *webs*. Webs were introduced by G. Kuperberg [5], building on the diagrammatics of the Temperley–Lieb algebra. The web bases have applications to representations of *quantum groups*, *cluster algebras*, the geometry of *Springer fibres*, and *quantum link invariants*. As a combinatorial application, web bases were used in [9] to elucidate the orbit structure of certain standard Young tableaux under *promotion*, a instance of the *cyclic sieving phenomenon* [10] previously established through more difficult tools in [11].

While definitions have been proposed for $\mathfrak{sl}_r$ webs with $r \geq 3$ (*e.g.* [2]), the resulting constructions are not entirely satisfactory from our perspective, as they lack important combinatorial properties. In particular, web bases have only been successfully applied to tableau combinatorics in the 2- and 3-row rectangular cases $\lambda = (d, d)$ and $\lambda = (d, d, d)$.

The goal of this paper is to develop a web basis for the two-parameter family of Specht modules S^λ with λ of the form $(d, d, 1^\ell)$. (Here, 1^ℓ is shorthand for ℓ parts each of size 1.) We call these partitions ***pennants*** after the visual appearance of their Young diagrams. We were guided to the study of pennant Specht modules by considerations in the combinatorics of tableau dynamics; however, this family of partitions has attracted combinatorial interest since work of R. Stanley [13] in the mid-1990s. We find that pennant Specht modules have a useful web basis, directly extending the Temperley–Lieb basis for the case $\ell = 0$.

Standard Young tableaux of this shape $(d, d, 1^\ell)$ are in bijection with *increasing tableaux* of shape $(d + \ell, d + \ell)$ with entries at most $2d + \ell$ [8]. Increasing tableaux are analogues of standard Young tableaux, useful in K-theoretic Schubert calculus. In [14], H. Thomas and A. Yong introduced a *K-jeu de taquin* through which increasing tableaux calculate the K-theory structure coefficients of *Grassmannians* and other *minuscule varieties*.

Analogous to the cyclic sieving theorems of [9, 11] for standard Young tableaux, O. Pechenik [8] gave a cyclic sieving theorem to describe the orbit structure of 2-row increasing tableaux under a K-jeu de taquin analogue of promotion. The $\mathfrak{sl}_2$ webs relevant to promotion of 2-row *standard Young tableaux* may be combinatorially identified with *noncrossing matchings*, special cases of *noncrossing set partitions*. The cyclic sieving theorem of [8] relied on the combinatorics of noncrossing set partitions (without singleton blocks), leading Pechenik to write "it is tempting to think of noncrossing partitions without singletons as "K-webs" for $\mathfrak{sl}_2$, although their representation-theoretic significance is unknown." However, the proof in [8] was by explicit calculation, rather than representation-theoretic arguments.

In [12], B. Rhoades provided representation-theoretic meaning to noncrossing partitions without singletons by combinatorially reconstructing the Specht module $S^{(d,d,1^\ell)}$ with these combinatorial diagrams as an apparently new "skein" basis. Hence, Rhoades established the first algebraic proofs and interpretations of some theorems from [8, 10]. Nonetheless, the results remained somewhat mysterious, since Rhoades' definitions were highly non-obvious and the verifications that they gave a module structure involved many pages of laborious calculations. Moreover, [12, Section 7] noted an incompatibility of signs between different constructions in the paper; specifically, the signs appearing in the S_n action on general set partitions are slightly different from those used for *almost noncrossing set partitions*.

Our main result is to realize Rhoades' skein basis in a geometrically-natural setting. More precisely, we associate to each noncrossing set partition without singletons a global section $[\pi]$ of a line bundle on the 2-step partial flag variety $F\ell(2, \ell+2; n)$, yielding a basis of the pennant Specht module $S^{(d,d,1^\ell)}$ that is equivalent to Rhoades' skein basis up to signs. The polynomial $[\pi]$ will be defined in Section 3 in terms of *jellyfish tableaux*, which we introduce. Let $\mathcal{W}(n,d)$ denote the set of noncrossing partitions of n with d blocks and no singletons.

Theorem 1.1. *The set* $\{[\pi] : \pi \in \mathcal{W}(n,d)\}$ *forms a basis for* S^λ *where* $\lambda = (d, d, 1^{n-2d-2})$. *Moreover, up to signs, the long cycle* $c = n12\cdots(n-1) \in S_n$ *acts by rotation of diagrams and the long element* $w_0 = n(n-1)\cdots 21 \in S_n$ *acts by reflection. Specifically, we have*

$$w_0 \cdot [\pi] = (-1)^{\binom{n}{2}}[\mathtt{refl}(\pi)] \text{ and } c \cdot [\pi] = (-1)^{n-1}[\mathtt{rot}(\pi)].$$

Our construction directly extends that of the $\mathfrak{sl}_2$ web basis for the Specht module $S^{(d,d)}$ and partially realizes the dream from [8] of interpreting noncrossing partitions without singletons as "K-webs" for $\mathfrak{sl}_2$. However, they turn out to be webs, not for $\mathfrak{sl}_2$, but rather for a block lower-triangular Lie subalgebra of $\mathfrak{sl}_{\ell+2}$, and any connection to K-theoretic geometry is not yet apparent. An asset of our construction is that we actually obtain an invariant polynomial $[\pi] \in S^\lambda$ for each set partition π (not necessarily noncrossing) with the actions of the permutations c and w_0 still given as in Theorem 1.1. With this construction, it moreover becomes straightforward to obtain "uncrossing rules," describing the expansion of $[\pi]$ for crossing π in the noncrossing basis.

A different algebraic realization of the skein basis in a space of *fermionic diagonal harmonics* appeared recently in work of Rhoades with J. Kim [4]. While our work was carried out independently, our construction as a space of invariant polynomials appears to be a concretization of their more abstract theory. In particular, their *block operators* appear to correspond to our determinants and they satisfy a 5-term relation analogous to our Theorem 3.4. It would likely be valuable to work out the details of this correspondence.

To our knowledge, Theorem 1.1 is the first development of a web basis for any family of Lie algebras of unbounded dimensions. We have yet to fully explore the consequences of this idea. Our first applications are to understanding enumerative questions in combinatorial dynamics. However, it seems reasonable to expect that there is a quantum group whose representation theory is also governed by this diagrammatic basis. We do not expect this theory to yield a new quantum link invariant; however, it could perhaps be applied to obtain quantum invariants for spatial embeddings of hypergraphs. Similarly, we hope that our constructions may yield insights into the topology of the Springer fibre for shape $(d, d, 1^\ell)$. The dynamics of K-promotion on arbitrary rectangular increasing tableaux are mysterious but known to be very complicated. However, the 3-row rectangular case is believed to be tractable [1, Conjecture 4.12]. Preliminary investigations with Julianna Tymoczko suggest that this case is also amenable to study via a web basis of planar hypergraphs.

This is an extended abstract; see [7] for the full version, which includes further proofs and citations.

2 Specht modules, flag varieties, noncrossing partitions

The complex ***Grassmannian*** $\mathrm{Gr}_k(n)$ is the parameter space for k-dimensional linear subspaces of $\mathbb{C}^n$. Our focus in this paper is on generalizing constructions for $\mathrm{Gr}_2(n)$, so we begin by reviewing these. Then we will describe more general constructions for partial flag varieties, after first fixing ideas and notation in the Grassmannian case.

Let M_n denote the matrix $\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}$ of $2n$ distinct indeterminates and let $\mathrm{SL}_2(\mathbb{C})$ act by left multiplication. This gives an action of $\mathrm{SL}_2(\mathbb{C})$ on the ring $\mathbb{C}[M_n]$ of polynomials in these $2n$ variables, which we think of as the coordinate ring of a $2n$-dimensional affine space $\mathbb{C}^{2n}$. A classical task in invariant theory is to characterize those polynomials in the invariant subspace $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ of this action. Classically, the answer is that $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ is generated as an algebra by the 2×2 minors of M_n.

By definition, $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ is the coordinate ring of the GIT (geometry invariant theory) quotient $\mathbb{C}^{2n} /\!/ \mathrm{SL}_2$ (the affine cone over $\mathrm{Gr}_2(n)$). A 2×2 minor of M_n is specified by a pair of column indices $1 \leq i < j \leq n$. Commonly, we identify the $(i < j)$–minor of M_n with the ***Plücker variable*** p_{ij} on the complex projective space $\mathbb{P}^{\binom{n}{2}-1}$. Realizing a 2-dimensional linear subspace of $\mathbb{C}^n$ as a rank 2 complex matrix of shape $2 \times n$, we may evaluate all of its 2×2 minors and map it to the point in $\mathbb{P}^{\binom{n}{2}-1}$ with those Plücker coordinates. This construction is well-defined, since different matrix representations of the same 2-plane will yield the same set of 2×2 minors, up to global scaling by a nonzero constant. We thereby realize the Grassmannian $\mathrm{Gr}_2(n)$ as a smooth projective subvariety of $\mathbb{P}^{\binom{n}{2}-1}$, embedded via this ***Plücker embedding***. The Grassmannian, in its Plücker embeddding, is cut out scheme-theoretically by quadratic relations in the Plücker variables, called the ***Plücker relations***. Hence, the homogeneous coordinate ring of $\mathrm{Gr}_2(n)$ is a polynomial ring in the Plücker variables modulo these Plücker relations.

We then have an isomorphism between $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ and the homogeneous coordinate ring of $\mathrm{Gr}_2(n)$, except that the gradings differ by a factor of 2. The degree $2d$ homogeneous part of $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ corresponds to degree d polynomials in the Plücker variables, since 2×2 minors are quadratic. In particular, $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ is supported only in even degrees. We will move back and forth between these gradings as convenient.

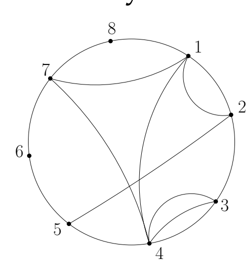
The polynomial ring $R = \mathbb{C}[p_{ij} : 1 \leq i < j \leq n]$ is the homogeneous coordinate ring of $\mathbb{P}^{\binom{n}{2}-1}$. Such polynomials of homogeneous degree d form the global sections of the line bundle $\mathcal{O}(d)$ on $\mathbb{P}^{\binom{n}{2}-1}$. Similarly, if we quotient R by the Plücker relations, the resulting functions of homogeneous degree d are the global sections of the pullback line bundle $\mathcal{O}_{\mathrm{Gr}_2(n)}(d)$. Of course, we may then identify these functions with the degree $2d$ part of the invariant ring $\mathbb{C}[M_n]^{\mathrm{SL}_2}$.

For each d, the degree $2d$ homogeneous part of $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ is clearly finite-dimensional, since it is spanned by d-fold products of Plücker variables and there are finitely many such variables. There are two convenient graphical ways to encode such a d-fold product.

In the tableau encoding, we fill a $2 \times d$ grid T with positive integers weakly between 1 and n. For each Plücker variable p_{ij}, we fill a column of T with i and j. Note that this grid is exactly the Young diagram of the partition (d, d). We can choose to insist that, whenever $i < j$, we write i in the top row and j in the bottom row. The order of the columns is also inconsequential, since a product of Plücker variables is independent of the order of the factors. Hence, we may assume that we have sorted the columns of T so that the entries of the first row weakly increase from left to right. Thus, a d-fold product of Plücker variables is encoded by an array such as $\begin{array}{|c|c|c|c|c|c|c|} \hline 1&1&1&2&3&3&4\\ \hline 2&4&7&5&4&4&7\\ \hline \end{array}$ where in this example we are taking $d = 7$ and $n \geq 7$. Note that the bottom row of the array satisfies no particular increasingness condition. We say that such a filling of a Young diagram is a ***semistandard tableau*** if the entries of each row weakly increase from left to right, just as we insisted for the entries of the top row.

A fundamental result of *standard monomial theory* is that, for each d, a linear basis for the degree $2d$ homogeneous part of $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ is given by those products of Plücker variables corresponding to semistandard tableaux of shape (d, d). This fact allows one, for example, to combinatorially determine the dimensions of these spaces.

In the web encoding, we cyclically place n labeled points around the boundary of a disk, and then draw d arcs through the interior of this disk joining boundary points. Precisely, for each Plücker variable p_{ij}, we draw an arc from vertex i to vertex j. Note that some arcs may necessarily cross other arcs and some pairs of arcs may share the same endpoints. Letting $d = 7$ and $n = 8$, the d-fold product encoded by our example array above may be alternatively encoded by the diagram below.



Let T_n denote the rank n torus of diagonal invertible $n \times n$ matrices. Then T_n acts on

M_n by right multiplication, scaling the columns. We thereby also have an action of T_n on the invariant ring $\mathbb{C}[M_n]^{\mathrm{SL}_2}$. The T_n action breaks the degree $2d$ homogeneous part of $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ into a direct sum of *weight spaces*. Specifically, we obtain an extra $\mathbb{Z}^n$-grading on invariant polynomials. Let e_i denote the ith standard basis vector of $\mathbb{Z}^n$. Then we treat each Plücker variable p_{ij} as having $\mathbb{Z}^n$-degree $e_i + e_j$. For each $v \in \mathbb{Z}^n$ with coordinates summing to $2d$, we may then consider the part of $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ with $\mathbb{Z}^n$-degree v, a subspace of the space of degree $2d$ invariants. For $v = (v_1, \ldots, v_n)$, the homogeneous degree v part of $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ corresponds to those tableaux where each value i appears exactly v_i times, or alternatively, to those arc diagrams where boundary vertex i has degree v_i.

We are particularly interested in the case $n = 2d$ and the degree $(1, 1, \ldots, 1)$ homogeneous part of $\mathbb{C}[M_n]^{\mathrm{SL}_2}$. In this case, we have a basis given by semistandard tableaux where each number from 1 to $n = 2d$ appears exactly once. Such tableaux are called ***standard***. However, the theory of $\mathfrak{sl}_2$ webs gives us another, completely different basis for this space. Clearly, the degree $(1, 1, \ldots, 1)$ homogeneous part is spanned by products of Plücker variables corresponding to ***matchings***, *i.e.*, arc diagrams where each boundary vertex is paired to exactly one other boundary vertex. It turns out that a linear basis is given by those matchings that are ***noncrossing***, *i.e.*, such that the arcs can be drawn so as to be pairwise nonintersecting. Noncrossing matchings are also known as $\mathfrak{sl}_2$ ***webs***, and the corresponding basis is called the $\mathfrak{sl}_2$ ***web basis*** or ***Temperley–Lieb basis***.

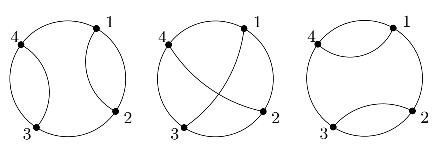
The tableau and web bases are genuinely different from each other. For example, if $d = 2$ and $n = 4$, there are 3 products of pairs of distinct Plücker variables to consider:

$p_{12}p_{34}$ $\quad$ $p_{13}p_{24}$ $\quad$ $p_{14}p_{23}$

1	3
2	4

1	2
3	4

1	2
4	3



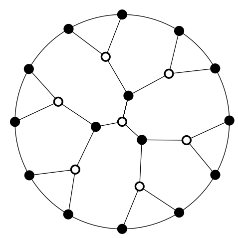
The standard tableau basis is given by the first two of these products, whereas the $\mathfrak{sl}_2$ web basis is given by the first and the third. Famously, standard Young tableaux of shape (d, d) and noncrossing matchings of $2d$ are both counted by the *Catalan numbers*. There is a well-known bijection between these sets.

The space $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ carries an action of the symmetric group S_n by right multiplication, permuting the columns of M_n. For μ a partition of $2d$, we can consider the union of the homogeneous degree α parts of $\mathbb{C}[M_n]^{\mathrm{SL}_2}$ over all α that are permutations of μ. The S_n action restricts to this union, making it an S_n-module; however, the isomorphism type of this representation is not easy to determine in general. Nonetheless, in the case

$n = 2d$, restricting to the homogeneous degree $(1, 1, \ldots, 1)$ part yields an irreducible S_n-representation, called the ***Specht module*** $S^{(d,d)}$.

If instead we let M_n be a $3 \times n$ matrix of distinct indeterminants and consider the ring $\mathbb{C}[M_n]^{\mathrm{SL}_3}$, the story is similar. This invariant ring is generated by the 3×3 minors, which we identify with the Plücker coordinates p_{ijk} on $\mathrm{Gr}_3(n)$. The homogeneous degree $3d$ part of $\mathbb{C}[M_n]^{\mathrm{SL}_3}$ is the space of global sections of the line bundle $\mathcal{O}_{\mathrm{Gr}_3(n)}(d)$ and has a basis given by semistandard tableaux of shape (d, d, d).

Again, we have an action of T_n and may consider the weight spaces that this action induces. For $n = 3d$, the homogeneous degree $(1, 1, \ldots, 1)$ part has a basis given by standard tableaux of shape (d, d, d). The symmetric group S_n acts on this space by right multiplication, making it an S_n-module. Indeed, it is an irreducible representation and is a geometric realization of the Specht module $S^{(d,d,d)}$. However, there is again a genuinely different basis of this space encoded by planar diagrams. These diagrams are called $\mathfrak{sl}_3$ ***webs*** and look, for example, like:



We now extend the previous constructions to general partial flag varieties of type A, including Grassmannians of k-planes with $k > 3$. The main difference from the previous constructions (besides some added complexity) is that we will not have web bases is these settings. Although we consider arbitrary Specht modules, our primary interest will be in those of the form $S^{(d,d,1^\ell)}$; the main result of this paper is to describe an explicit basis of invariant polynomials for those cases, realizing Rhoades' skein basis geometrically.

Let λ be any partition and let μ be the partition whose Young diagram is the transpose of λ. In particular, if $\lambda = (d, d, 1^\ell)$, then $\mu = (\ell + 2, 2^{d-1})$. Choose $n \geq \mu_1$ and let M denote the $\mu_1 \times n$ matrix

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{\mu_1 1} & x_{\mu_1 2} & \cdots & x_{\mu_1 n} \end{pmatrix} \tag{2.1}$$

of $\mu_1 n$ distinct indeterminates. We may have $\mu_i = \mu_{i+1}$; if so, delete one of them until we obtain a ***strict partition*** ν in which all parts are distinct. For example, if $\mu = (\ell + 2, 2^{d-1})$, then $\nu = (\ell + 2, 2)$. Consider the set of complex block-lower-triangular matrices of determinant 1 with block sizes $\nu_n, \nu_{n-1} - \nu_n, \nu_{n-2} - \nu_{n-1}, \ldots, \nu_1 - \nu_2$. Note that since ν

is a strict partition, all of these block sizes are positive. (We could have used μ in place of ν, treating blocks of size 0 as nonexistent, but this indexing would be less convenient later.) These matrices form a subgroup P of the Lie group $\mathrm{SL}_{\mu_1}(\mathbb{C})$, which acts on $\mathbb{C}[M]$. We consider again the invariant subring $\mathbb{C}[M]^P$.

On the one hand, $\mathbb{C}[M]^P$ is the coordinate ring of the GIT quotient $\mathbb{C}^{\mu_1 n} /\!/ P$, which is the affine cone over the ***partial flag variety*** $F\ell(\nu_n, \nu_{n-1}, \ldots, \nu_1; n)$. This partial flag variety is a parameter space for ***partial flags*** $0 \leq V_n \leq V_{n-1} \leq \cdots \leq V_1 \leq \mathbb{C}^n$ of nested vector subspaces with V_i of dimension ν_i. In particular, these spaces include the Grassmannians, as $\mathrm{Gr}_k(n) = F\ell(k; n)$. Our main interest is in the 2-step flag variety $F\ell(2, \ell + 2; n)$.

On the other hand, classical invariant theory tells us algebraic generators for $\mathbb{C}[M]^P$. For k any of the parts of ν, let $p_{i_1 i_2 \ldots i_k}$ denote the $k \times k$ minor of M that uses the top k rows and the columns indexed $i_1, i_2, \ldots, i_k$. Then $\mathbb{C}[M]^P$ is algebraically generated by the set $\{p_{i_1 i_2 \ldots i_k} : k \in \nu, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$. We call each $p_{i_1 i_2 \ldots i_k}$ a ***k-Plücker variable***.

Representing a partial flag as a $\mu_1 \times n$ complex matrix such that V_i is the span of the first ν_i rows, computing all of the Plücker variables embeds $F\ell(\nu_n, \nu_{n-1}, \ldots, \nu_1; n)$ in a product of projective spaces $\mathbb{P}^{\binom{n}{\nu_n}-1} \times \mathbb{P}^{\binom{n}{\nu_{n-1}}-1} \times \cdots \times \mathbb{P}^{\binom{n}{\nu_1}-1}$ as a smooth subvariety. We also call this map the ***Plücker embedding***.

Now, consider the invariants in $\mathbb{C}[M]^P$ of the form $\prod_{i=1}^{\lambda_1} p^{(\mu_i)}$, where each $p^{(\mu_i)}$ is a μ_i-Plücker variable. For example, with $\lambda = (d, d, 1^\ell)$, we are looking at products of one minor of size $\ell + 2$ with $d - 1$ top-justified minors of size 2. We consider this set to be the invariants of homogeneous multidegree μ.

We may record such a product of Plücker variables by filling the Young diagram of λ such that the ith column contains the μ_i subscripts of the ith Plücker variable. The key result of standard monomial theory is that this space of invariants has a basis given by such fillings that are semistandard tableaux.

Again, there is an action of the torus T_n by right multiplication on M, breaking up our invariants into weight spaces. We now restrict to the case $n = |\lambda|$ and consider the degree $(1, 1, \ldots, 1)$ homogeneous part. This subspace has a basis given by standard tableaux of shape λ. This subspace also carries an action of S_n by right multiplication on M. It is, in fact, an irreducible module, the ***Specht module*** S^λ.

A ***set partition*** of size n is a collection of nonempty pairwise disjoint subsets of $\{1, 2, \ldots, n\}$ whose union is $\{1, 2, \ldots, n\}$. Each of the subsets is called a ***block*** of the set partition. We draw a set partition by placing n cyclically labeled points around the boundary of a disk, and then, for each block, drawing the convex hull of the corresponding boundary points on the disk. Observe that a set partition with all blocks of size 2 is exactly a matching. A ***noncrossing partition*** is a set partition with the property that the convex hulls (in the graphical depiction above) are pairwise disjoint. In particular, a matching is a noncrossing partition if and only if it is a noncrossing matching, *i.e.*, an $\mathfrak{sl}_2$ web.

A ***singleton*** in a set partition is a block of size 1. For $n \in \mathbb{Z}^+$, let $\Pi(n)$ be the set of all set partitions of $\{1, 2, \ldots, n\}$ with no singletons and let $\Pi(n, d)$ denote the set of all $\pi \in \Pi(n)$ with d blocks. If the blocks of $\pi \in \Pi(n)$ are denoted $\pi_1, \pi_2, \ldots, \pi_d$, we say $\pi = \{\pi_1, \pi_2, \ldots, \pi_d\}$. Let $\mathcal{W}(n)$ denote the set of all $\pi \in \Pi(n)$ that are noncrossing. Let $\mathcal{W}(n, d)$ be the set of $\pi \in \mathcal{W}(n)$ with d blocks.

An ***increasing tableau*** is a semistandard tableau in which rows (like columns) are strictly increasing. All increasing tableaux in this paper are assumed to be ***packed***, meaning that the set of numbers appearing is an initial segment of the set of positive integers. We write $\mathrm{Inc}(\lambda)$ for the set of all increasing tableaux of shape λ and $\mathrm{Inc}^q(\lambda)$ for the subset whose maximum entry is q. We are interested in an invertible operator called ***K-promotion*** on increasing tableaux.

A useful bijection appeared in [8] between $\mathrm{Inc}^q(m, m)$ and noncrossing partitions of size q with $q - m$ blocks and no singletons. This bijection restricts to a classical bijection between standard tableaux of shape (m, m) and $\mathfrak{sl}_2$ webs with $2m$ boundary vertices. There is also an explicit bijection [3] between standard tableaux of shape (m, m, m) and $\mathfrak{sl}_3$ webs with $3m$ black boundary vertices. These bijections all have the property that they carry K-promotion of tableaux to rotation of the corresponding planar diagram (either a web or a noncrossing partition) [8, 9]. Similarly, they all carry K-evacuation to reflection of planar diagrams [6, 8]. Finally, K-promotion of these tableaux corresponds to *rowmotion* of order ideals in certain posets [1]; thus this equivariant bijection to planar diagrams that rotate provides a good explanation for the order of rowmotion as well.

3 Main results

Choose and fix a convention for ordering the blocks of a set partition. We show in the full version that our results are independent of this convention choice. We begin by defining a set of tableaux we will use to construct polynomials associated to set partitions. Because the shape consists of two full rows of boxes followed by one box in each additional row, we call these *jellyfish tableaux* (cf. Example 3.3).

Definition 3.1. Given a set partition $\pi = \{\pi_1, \pi_2, \ldots, \pi_d\} \in \Pi(n, d)$, let $\mathcal{J}(\pi)$ be the set of generalized tableaux T_{ij} (in English notation with matrix indexing) with d columns (so $1 \leq j \leq d$) and $n - 2d + 2$ rows ($1 \leq i \leq n - 2d + 2$) obeying the following constraints:

1. $T_{ij} \in [n]$ or T_{ij} is empty.
2. If $i \in \{1, 2\}$, T_{ij} is nonempty.
3. If $i > 2$, there exists exactly one j such that T_{ij} is nonempty.

4. The nonempty entries in column j are exactly the elements of π_j, in increasing order.

Call $\mathcal{J}(\pi)$ the set of ***jellyfish tableaux*** of π.

Given $T \in \mathcal{J}(\pi)$, define the ***inversion number*** $\mathrm{inv}(T)$ as the number of inversions in the row reading word (left to right, top to bottom). Define the ***sign*** of T as $\mathrm{sgn}(T) = (-1)^{\mathrm{inv}(T)}$. Note that $i < j$ form an inversion of T if and only if either j appears in a higher row than i or else j appears left of i in the same row.

Recall the matrix M from (2.1). Let I and J be finite subsets of $\mathbb{N}$ and let M_I^J denote the determinant of the submatrix of M with rows indexed by I and columns indexed by J. For convenience, we sometimes write elements separated by commas in the subscript and superscript rather than formal sets.

Given $\pi \in \Pi(n,d)$ and $T \in \mathcal{J}(\pi)$, define the product of determinants $\mathrm{J}(T) = \prod_{i=1}^{d} M_{R_i(T)}^{\pi_i}$, where $R_i(T)$ is the set of rows of T containing an entry in π_i. Note this reduces to the Plücker case when π is a matching.

We now define a polynomial web invariant for each set partition π.

Definition 3.2. Given a set partition π with no singletons, let $[\pi]$ denote the polynomial

$$[\pi] = \sum_{T \in \mathcal{J}(\pi)} \mathrm{sgn}(T)\,\mathrm{J}(T).$$

If π has a singleton block, we set $[\pi] = 0$.

Example 3.3. Suppose $\pi = \{\{2,3,6,10\},\{5,7,8,9\},\{1,4\}\}$ (where we assume the blocks are ordered as written). Then $\mathcal{J}(\pi)$ consists of the tableaux below.

2	5	1
3	7	4
6		
10		
	8	
	9	

2	5	1
3	7	4
6		
	8	
10		
	9	

2	5	1
3	7	4
6		
	8	
	9	
10		

2	5	1
3	7	4
	8	
6		
10		
	9	

2	5	1
3	7	4
	8	
6		
	9	
10		

2	5	1
3	7	4
	8	
	9	
6		
10		

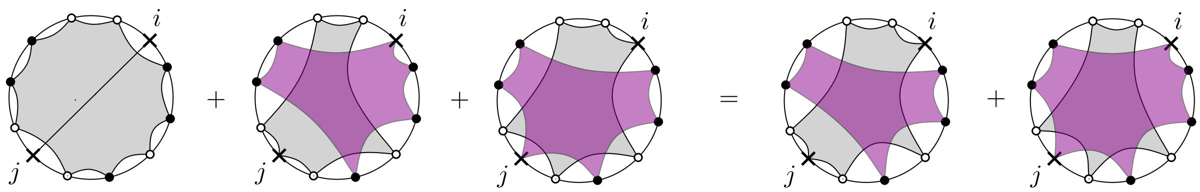
The leftmost tableau has row reading word 2,5,1,3,7,4,6,10,8,9 and thus has 8 inversions. Reading the list of tableaux from left to right, the tableaux have 8, 7, 6, 8, 7, and 8 inversions, respectively. Finally, we have that

$$\begin{aligned}[\pi] = {} & M_{1,2,3,4}^{2,3,6,10} \cdot M_{1,2,5,6}^{5,7,8,9} \cdot M_{1,2}^{1,4} - M_{1,2,3,5}^{2,3,6,10} \cdot M_{1,2,4,6}^{5,7,8,9} \cdot M_{1,2}^{1,4} + M_{1,2,3,6}^{2,3,6,10} \cdot M_{1,2,4,5}^{5,7,8,9} \cdot M_{1,2}^{1,4} \\ & + M_{1,2,4,5}^{2,3,6,10} \cdot M_{1,2,3,6}^{5,7,8,9} \cdot M_{1,2}^{1,4} - M_{1,2,4,6}^{2,3,6,10} \cdot M_{1,2,3,5}^{5,7,8,9} \cdot M_{1,2}^{1,4} + M_{1,2,5,6}^{2,3,6,10} \cdot M_{1,2,3,4}^{5,7,8,9} \cdot M_{1,2}^{1,4}.\end{aligned}$$

All the following statements are used in the proof of Theorem 1.1; see [7] for proofs. We only need the statements in the case of noncrossing partitions. However, we give many in greater generality for future use. The image below illustrates Theorem 3.4, the technical heart of our investigation.

Theorem 3.4. *Partition* $\{1, \ldots, n\}$ *into four nonempty sets:* A, B, I, *and* J, *where* $|I| = |J| = 1$.

$$\Big[\{A \cup B, I \cup J\}\Big] + \Big[\{A \cup I, B \cup J\}\Big] + \Big[\{A \cup J, B \cup I\}\Big] = \Big[\{A \cup I \cup J, B\}\Big] + \Big[\{A, B \cup I \cup J\}\Big].$$



Lemma 3.5. *The set* $\{[\pi] : \pi \in \mathcal{W}(n, d)\}$ *of invariants of noncrossing partitions of* n *with* d *blocks and no singletons is linearly independent.*

Note $[\pi]$ is a sum over jellyfish tableaux, where each summand is not generally in the pennant Specht module. Nonetheless, we have the following lemma, which may be derived inductively from Theorem 3.4.

Lemma 3.6. *The set* $\{[\pi] : \pi \in \Pi(n, d)\}$ *of invariants of partitions of* n *with* d *blocks and no singletons is a subset of the pennant Specht module* $S^{(d,d,1^{n-2d})}$.

Theorem 3.7. *The set* $\{[\pi] : \pi \in \mathcal{W}(n, d)\}$ *of invariants of noncrossing partitions of* n *with* d *blocks and no singletons is basis of the pennant Specht module* $S^{(d,d,1^{n-2d})}$.

Proof. By Lemma 3.6, each $[\pi]$ is in the pennant Specht module $S^{(d,d,1^{n-2d})}$. By Lemma 3.5, they are linearly independent. By [8, Propositions 2.1 and 2.3], there is a bijection between $\mathcal{W}(n, d)$ and the set of standard Young tableaux of pennant shape $(d, d, 1^{n-2d})$, the number of which is the dimension of $S^{(d,d,1^{n-2d})}$. Hence, the $[\pi]$ form a basis. □

The following result is then relatively straightforward to verify.

Proposition 3.8. *Up to signs, the long cycle* $c_n = n12 \ldots (n-1)$ *acts by rotation and the long element* w_0 *acts by reflection. For any set partition* $\pi \in \Pi_n$ *(not necessarily noncrossing), we have* $c_n \cdot [\pi] = (-1)^{n-1}[\mathtt{rot}(\pi)]$, *where* `rot` *denotes counterclockwise rotation by* $(360/n)^\circ$, *and* $w_0 \cdot [\pi] = (-1)^{\binom{n}{2}}[\mathtt{refl}(\pi)]$, *where* `refl` *denotes reflection across the diameter with endpoint halfway between vertices* n *and* 1.

Using our polynomial representatives, one may easily obtain formulas for resolving crossing diagrams as linear combinations of noncrossing ones; in particular, we recover the uncrossing rules of [12]. With this representation theory in hand, it is also not hard to recover the cyclic sieving theorems of [8, 10] by similar arguments to [12, Section 8].

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