Séminaire Lotharingien de Combinatoire **86B** (2022)
Article #58, 12 pp.

Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)

Chromatic Quasisymmetric Class Functions of Linearized Combinatorial Hopf Monoids

Jacob A. White*[1]

[1]*School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, TX, USA*

Abstract. We study the chromatic quasisymmetric class function of a linearized combinatorial Hopf monoid. Given a linearized combinatorial Hopf monoid H, and an H-structure h on a set N, there are proper colorings of h, generalizing graph colorings and poset partitions. We show that the automorphism group of h acts on the set of proper colorings, which gives rise to the chromatic quasisymmetric class function. For the Hopf monoid of graphs this invariant generalizes Stanley's chromatic symmetric function and the orbital chromatic polynomial studied by Cameron and Kayibi.

We show that, under certain conditions, the chromatic quasisymmetric class function of h is the flag quasisymmetric class function of the coloring complex of h. We use this result to deduce various positivity results, and inequalities for the associated orbital polynomial invariants.

Keywords: Hopf monoids, quasisymmetric functions, class functions, balanced simplicial complexes

1 Introduction

Given a graph G, let $\mathfrak{G}$ be a subgroup of the automorphism group of G. Two colorings f and g are equivalent if $g = f \circ \mathfrak{g}^{-1}$ for some $\mathfrak{g} \in \mathfrak{G}$. Let $\chi(G, \mathfrak{G}, k)$ denote the number of equivalence classes of k-colorings of G. Then $\chi(G, \mathfrak{G}, k)$ is a polynomial, called the orbital chromatic polynomial studied by Cameron and Kayibi [6]. An orbital version of the order polynomial of a poset was studied by Jochemko [9], and a quasisymmetric function generalization for double posets was studied by Grinberg [8]. We generalized further to a quasisymmetric class function associated to a double poset [12].

The goal of this work is to provide a general method for creating new quasisymmetric class functions associated to combinatorial objects. These general invariants also specialize to orbital polynomials. Previously, Aguiar, Bergeron, and Sottile [2] have shown that the Hopf algebra of quasisymmetric functions QSYM is the terminal object in the category of combinatorial Hopf algebras. We showed [11] that, for Hopf monoids in species,

*jacob.white@utrgv.edu

we can describe the fundamental homomorphism of Aguiar, Bergeron and Sottile as a generating function over φ-proper colorings. We review this construction in Subsection 3.1. Given a linearized Hopf monoid $\mathbf{H}$ and a $\mathbf{H}$-structure $\mathbf{h}$, we let $F_\varphi(\mathbf{h})$ denote the set of φ-proper colorings of $\mathbf{h}$. Since combinatorial species involve labeled combinatorial objects, they come equipped with symmetric group actions. In particular, there is a notion of automorphism group $\mathrm{Aut}(\mathbf{h})$ for a $\mathbf{H}$-structure $\mathbf{h}$. Given a subgroup $\mathfrak{G} \subseteq \mathrm{Aut}(\mathbf{h})$, we show that $\mathfrak{G}$ acts on $F_\varphi(\mathbf{h})$.

Let $x_1, x_2, \ldots$ be commuting indeterminates. For $\mathfrak{g} \in \mathfrak{G}$, we define

$$\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x}; \mathfrak{g}) = \sum_{f \in F_\varphi(\mathbf{h}) : \mathfrak{g}f = f} \prod_{v \in N} x_{f(v)}.$$

The resulting power series is a quasisymmetric function, and as we vary $\mathfrak{g}$, we obtain a QSYM-valued class function on $\mathfrak{G}$. We call this the φ-chromatic quasisymmetric class function of $(\mathbf{h}, \mathfrak{G})$. There is also an orbital chromatic polynomial $\Psi^O_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, x)$, which counts the number of orbits of φ-proper colorings with largest color at most x. These two invariants generalize the orbital chromatic polynomial, the orbital order polynomial, the chromatic symmetric function, the orbital quasisymmetric functions of Grinberg, the Billera–Jia–Reiner quasisymmetric function of a matroid [5], and the quasisymmetric class functions introduced in [12]. There are many other examples that can be discussed related to the Hopf monoid of generalized permutohedra [1].

We review the definition of balanced convex character in Subsection 4.1. Most characters studied in the literature are balanced convex characters. We are able to prove the following results.

Theorem 1. *Let $\mathbf{H}$ be a linearized Hopf monoid in species with linearized character φ. Let N be a finite set, $\mathbf{h} \in \mathbf{H}_N$, and $\mathfrak{G} \subseteq \mathrm{Aut}(\mathbf{h})$. Suppose that φ is a balanced convex character. Then we have the following:*

1. *Write*

$$\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x}) = \sum_{S \subseteq [|N|-1]} \sum_{i=1}^{k} \Psi_{\mathfrak{G},S,i} \chi_i M_{S,|N|},$$

where $M_{S,|N|}$ are the monomial quasisymmetric functions and $\chi_1, \ldots, \chi_k$ are the irreducible characters of $\mathfrak{G}$. For $S \subseteq T \subseteq [|N|-1]$, and $i \in [k]$, we have $\Psi_{\mathfrak{G},S,i} \leq \Psi_{\mathfrak{G},T,i}$.

2. *If we write $\Psi^O_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, x) = \sum_{i=0}^{|N|} f_i \binom{x}{i}$, then we have the following inequalities:*

   (a) For $0 \leq i \leq \frac{|N|+1}{2}$, we have $f_i \leq f_{i+1}$.

   (b) For $1 \leq i \leq \frac{|N|+1}{2}$, we have $f_i \leq f_{|N|+1-i}$.

A quasisymmetric class function which satisfies the first result is M-increasing, while a polynomial which satisfies the second result is strongly flawless. The proofs of our results involve a mix of algebraic and geometric techniques. In Section 2, we discuss quasisymmetric class functions. We show that whenever a quasisymmetric class function is M-increasing, then the corresponding orbital polynomial is strongly flawless.

In [11], we studied a generalization of Steingrímsson's coloring complex of a graph. Given a balanced convex character φ, there exists a balanced relative simplicial complex $\Sigma_\varphi(\mathbf{h})$ such that $\Psi_{\mathbf{H},\varphi}(\mathbf{h},\mathbf{x}) = \sum_{S\subseteq[d]} F_S(\Phi) M_{S,d}$ where $F_S(\Phi)$ is the flag f-vector of Φ, and $M_{S,d}$ are the monomial quasisymmetric functions. In Section 4, we discuss balanced relative simplicial complexes Φ and define the flag quasisymmetric *class* function $F(\Phi, \mathfrak{G}, \mathbf{x})$ for Φ with respect to a group action. We show in Theorem 12 that the corresponding flag quasisymmetric class function is always M-increasing. We also show that $\Sigma_\varphi(\mathbf{h})$ comes with an action by $\mathfrak{G}$, and that $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x}; \mathfrak{g}) = F(\Sigma_\varphi(\mathbf{h}), \mathfrak{G}, \mathbf{x})$.

These results show some of the power of working with Hopf monoids in species: we define very general quasisymmetric functions whose coefficients are characters. We replace the combinatorial objects with geometric objects in a $\mathfrak{G}$-invariant manner, and then use simple combinatorial arguments to deduce results about those characters. As a result, we obtain new orbital polynomials for many combinatorial polynomials in the literature, *and* quasisymmetric class function generalizations, as well as inequalities for these invariants.

2 Preliminaries

Given a basis B for a vector space V over $\mathbb{C}$, and $\vec{\beta} \in B, \vec{v} \in V$, we let $[\vec{\beta}]\vec{v}$ denote the coefficient of $\vec{\beta}$ when we expand $\vec{v}$ in the basis B.

Often, we will define quasisymmetric functions that are generating functions over collections of functions. Given a function $w\colon S \to \mathbb{N}$, we define

$$\mathbf{x}^w = \prod_{v\in S} x_{w(v)}. \tag{2.1}$$

For example, the chromatic symmetric function of a graph G is defined as $\sum_{f\colon V\to\mathbb{N}} \mathbf{x}^f$ where the sum is over all proper colorings of G.

Let $\mathbf{x} = x_1, x_2, \ldots$ be a sequence of commuting indeterminates. Given $S \subseteq [d-1]$, with $S = \{s_1, \ldots, s_k\}$, where $s_1 < s_2 < \cdots < s_k$, we define

$$M_{S,d} = \sum_{i_0<\cdots<i_k} \prod_{j=0}^{k} x_{i_j}^{s_{j+1}-s_j},$$

where $s_0 = 0$ and $s_{k+1} = d$. These are the *monomial quasisymmetric functions*, which form a basis for the ring of quasisymmetric functions.

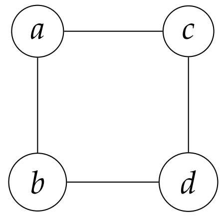

Figure 1: A graph.

We assume familiarity with the theory of complex representations of finite groups — see [7] for basic definitions. Recall that, given any group action of $\mathfrak{G}$ on a finite set X, there is a group action on $\mathbb{C}^X$ as well, which gives rise to a *permutation representation*.

The characters of the irreducible representations of $\mathfrak{G}$ form an orthonormal basis of $C(\mathfrak{G}, \mathbb{C})$, the vector space of class functions from $\mathfrak{G}$ to $\mathbb{C}$. We refer to elements $\chi \in C(\mathfrak{G}, \mathbb{C})$ that are nonnegative integer combinations of irreducible characters as *effective characters*, and say χ is a *permutation character* if it is the character of a permutation representation. We partially order effective characters by saying $\chi \leq_{\mathfrak{G}} \psi$ if $\psi - \chi$ is an effective character.

A *quasisymmetric class function* of degree e is a function $F\colon \mathfrak{G} \to \mathrm{QSYM}_e$, where QSYM_e is the space of quasisymmetric functions of degree e, that satisfies $F(\mathfrak{h}\mathfrak{g}\mathfrak{h}^{-1}) = F(\mathfrak{g})$ for all $\mathfrak{g}, \mathfrak{h} \in \mathfrak{G}$. We often write $F(\mathfrak{G}, \mathbf{x})$ to denote a quasisymmetric class function. Given such a function, and given $S \subseteq [e-1]$, we let $F_{\mathfrak{G},S}(\mathfrak{g}) = [M_{S,e}]F(\mathfrak{G}, \mathbf{x}; \mathfrak{g})$. Then $F_{\mathfrak{G},S}$ is a class function, and $F(\mathfrak{G}, \mathbf{x}) = \sum_{S \subseteq [e-1]} F_{\mathfrak{G},S} M_{S,e}$. Thus, quasisymmetric class functions may also be viewed as quasisymmetric functions whose coefficients are class functions.

As an example, given a graph $\mathbf{g}$ on vertex set N, the automorphism group $\mathrm{Aut}(\mathbf{g})$ acts on the set of colorings of $\mathbf{g}$. Thus, given $\mathfrak{G} \subseteq \mathrm{Aut}(\mathbf{g})$, and $\mathfrak{g} \in \mathfrak{G}$, we let

$$\chi(\mathbf{g}, \mathfrak{G}, \mathbf{x}; \mathfrak{g}) = \sum_{f : \mathfrak{g} f = f} \mathbf{x}^f,$$

where the sum is over proper colorings f of $\mathbf{g}$ fixed by $\mathfrak{g}$. Then $\chi(\mathbf{g}, \mathfrak{G}, \mathbf{x}; \mathfrak{g})$ is a quasisymmetric function, and as we vary $\mathfrak{g}$ we obtain a quasisymmetric class function. This is the chromatic quasisymmetric class function of $(\mathbf{g}, \mathfrak{G})$.

Let $\mathbf{g}$ be the graph in Figure 1. Let $\mathbb{Z}/4\mathbb{Z}$ act on $\mathbf{g}$ by cyclic rotation. We let ρ denote the character of the regular representation, and sgn denote the character of the sign representation. Then

$$\chi(\mathbf{g}, \mathbb{Z}/4\mathbb{Z}, \mathbf{x}) = (1 + \mathrm{sgn})M_{\{2\},4} + \rho(M_{\{1,2\},4} + M_{\{1,3\},4} + M_{\{2,3\},4} + 6M_{[3],4}).$$

We say that a quasisymmetric class function $F(\mathfrak{G}, \mathbf{x})$ is *M-increasing* if $F_{\mathfrak{G},S} \leq_{\mathfrak{G}} F_{\mathfrak{G},T}$ whenever $S \subseteq T$.

Let $F(\mathfrak{G}, \mathbf{x})$ be a quasisymmetric class function of degree e for $\mathfrak{G}$. Then we define the *orbital polynomial* to be

$$F^O(\mathfrak{G}, x) = \frac{1}{|\mathfrak{G}|} \sum_{\mathfrak{g} \in \mathfrak{G}} \sum_{S \subseteq [e-1]} F_{\mathfrak{G},S}(\mathfrak{g}) \binom{x}{|S|+1}.$$

We proved the following result in [12]:

Proposition 2. *Let $F(\mathfrak{G}, \mathbf{x})$ be a quasisymmetric class function of degree e. Write $F^O(\mathfrak{G}, x) = \sum_{i=0}^{e} f_i \binom{x}{i}$. If $F(\mathfrak{G}, \mathbf{x})$ is M-increasing, then we have the following inequalities:*

1. *For $0 \leq i \leq \frac{e+1}{2}$, we have $f_i \leq f_{i+1}$.*
2. *For $1 \leq i \leq \frac{e+1}{2}$, we have $f_i \leq f_{e+1-i}$.*

The proof involved using symmetric chain decompositions of the Boolean lattice in order to construct appropriate $\mathfrak{G}$-invariant injections.

3 Linearized combinatorial Hopf monoids

In this section, we review the definition of linearized combinatorial Hopf monoids. A *set species* is an endofunctor $\mathbf{F}\colon$ *Set* $\to$ *Set* on the category of finite sets with bijections. A *linear species* is a functor $\mathcal{F}\colon$ *Set* $\to$ *Vec* to the category of finite dimensional vector spaces over a field $\mathbb{K}$ and linear transformations. Given a set species $\mathbf{F}$, there is an associated linear species $\mathbb{K}\mathbf{F}$ called the *linearization*: we define $(\mathbb{K}\mathbf{F})_N$ to be the vector space with basis $\mathbf{F}_N$. We refer to $\mathbf{f}$ as an $\mathbf{F}$-structure if there exists a finite set N such that $\mathbf{f} \in \mathbf{F}_N$.

A *Hopf monoid* $\mathcal{H}$ is a Hopf monoid object in the category of linear species [3]. We refer to [4, 1] for more details. For every pair of disjoint finite sets M, N, there are multiplication maps $\mu_{M,N}\mathcal{H}_M \otimes \mathcal{H}_N \to \mathcal{H}_{M \sqcup N}$ and comultiplication maps $\Delta_{M,N}\colon \mathcal{H}_{M \sqcup N} \to \mathcal{H}_M \otimes \mathcal{H}_N$. We focus only on *connected* species, where $\dim \mathcal{H}_\emptyset = 1$. We let $\mathbf{x} \cdot \mathbf{y} = \mu_{M,N}(\mathbf{x} \otimes \mathbf{y})$. Let $\mathbf{H}$ be a set species. We say that $\mathbf{H}$ forms a *linearized* Hopf monoid if $\mathbb{K}\mathbf{H}$ is a Hopf monoid, and:

1. For every pair of disjoint finite sets M, N, and every $\mathbf{x} \in \mathbf{H}_M, \mathbf{y} \in \mathbf{H}_N$, we have $\mathbf{x} \cdot \mathbf{y} \in \mathbf{H}_{M \sqcup N}$.
2. For every pair of disjoint finite sets M, N, and every $\mathbf{h} \in \mathbf{H}_{M \sqcup N}$, if $\Delta_{M,N}(\mathbf{h}) \neq 0$ then there exists $\mathbf{h}|_M \in \mathbf{H}_M$ and $\mathbf{h}/M \in \mathbf{H}_N$ such that $\Delta_{M,N}(\mathbf{h}) = \mathbf{h}|_M \otimes \mathbf{h}/M$.

Our notion of linearized Hopf monoid is slightly more general than other notions in the literature, because we allow 0 as a coproduct.

Example 3. Given a finite set N, let $\mathbf{E}_N = \{1\}$. This gives rise to the *exponential species*. The product is given by $1 \cdot 1 = 1$, and the coproduct is $\Delta_{S,N \setminus S}(1) = 1 \otimes 1$.

Example 4. Given a finite set N, let $\mathbf{G}_N$ denote the collection of graphs with vertex set N. Then this gives rise to the species of graphs $\mathbf{G}$, which we turn into a linearized Hopf monoid. The product is given by the disjoint union of graphs. Given a graph $\mathbf{g}$, and $S \subseteq N$, $\mathbf{g}|_S$ is the induced subgraph on S, and $\mathbf{g}/S$ is the induced subgraph on $N - S$.

Example 5. Given a finite set N, let $\mathbf{DP}_N$ denote the collection of all double posets on N, where a double poset is a pair $(\mathbf{p}_1, \mathbf{p}_2)$ of partial orders. This gives rise to the set species of double posets $\mathbf{DP}$, which we turn into a linearized Hopf monoid. Let $\mathbf{p} \sqcup \mathbf{q}$ denote the disjoint union of posets, and $\mathbf{p} \oplus \mathbf{q}$ denote the ordinal sum, which is obtained from the disjoint union by requiring that every element of $\mathbf{p}$ be less than every element of $\mathbf{q}$. The product is $(\mathbf{p}_1, \mathbf{p}_2) \cdot (\mathbf{q}_1, \mathbf{q}_2) = (\mathbf{p}_1 \sqcup \mathbf{q}_1, \mathbf{p}_2 \oplus \mathbf{q}_2)$. Given a partial order $\mathbf{p}$, and $S \subseteq N$, let $\mathbf{p}|_S$ be the induced subposet on S. We define

$$\Delta_{S,N\setminus S}(\mathbf{p}_1, \mathbf{p}_2) = \begin{cases} (\mathbf{p}_1|_S, \mathbf{p}_2|_S) \otimes (\mathbf{p}_1|_{N\setminus S}, \mathbf{p}|_2|_{N\setminus S}) & \text{if } S \text{ is an order ideal of } \mathbf{p}_1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 6. Given a finite set N, let $\mathbf{M}_N$ denote the collection of matroids with ground set N. This gives rise to the species of matroids $\mathbf{M}$, which forms a linearized Hopf monoid. The product is given by the direct sum operation. Given a matroid $\mathbf{m}$, and $S \subset N$, we define $\mathbf{m}|_S$ to be the restriction, and $\mathbf{m}/S$ to be the contraction of matroids.

A natural transformation $\varphi \colon \mathcal{H} \to \mathbb{K}\mathbf{E}$, where $\mathcal{H}$ is a Hopf monoid in linear species, is a *character* if for all disjoint finite sets M and N, and all $x \in \mathcal{H}_M$ and all $y \in \mathcal{H}_N$, we have $\varphi_M(x) \cdot \varphi_N(y) = \varphi_{M \sqcup N}(x \cdot y)$. By an abuse of notation, we will write $\varphi(\mathbf{h})$ in place of $\varphi_N(\mathbf{h})$, when no confusion will arise. Given a linearized Hopf monoid $\mathbf{H}$, we say φ is *linearized* if $\varphi(\mathbf{h}) \in \{0, 1\}$ for all $\mathbf{H}$-structures $\mathbf{h}$. A *linearized combinatorial Hopf monoid* is a linearized Hopf monoid $\mathbf{H}$ with a *linearized* character φ. As an example, let $\mathbf{DP}$ denote the Hopf monoid of double posets. Given a double poset $(\mathbf{p}_1, \mathbf{p}_2)$ on a set N we define

$$\psi(\mathbf{p}_1, \mathbf{p}_2) = \begin{cases} 0 & \text{if there exists } x, y \in N \text{ such that } x \leq y \text{ in } \mathbf{p}_1 \text{ and } y < x \text{ in } \mathbf{p}_2, \\ 1 & \text{otherwise.} \end{cases}$$

Then ψ is a linearized character.

Example 7. Let $\mathbf{H}$ be a linearized Hopf monoid. We say a $\mathbf{H}$-structure $\mathbf{h} \in \mathbf{H}_N$ is *totally reducible* if $|N| = 1$, or there exists a nontrivial decomposition $N = S \sqcup T$, and totally reducible elements $\mathbf{x} \in \mathbf{H}_S$ and $\mathbf{y} \in \mathbf{H}_T$ such that $\mathbf{h} = \mathbf{x} \cdot \mathbf{y}$. We define

$$\chi(\mathbf{h}) = \begin{cases} 1 & \text{if } \mathbf{h} \text{ is totally reducible,} \\ 0 & \text{otherwise.} \end{cases}$$

We call χ the *chromatic* character, and $(\mathbf{H}, \chi)$ is always a linearized combinatorial Hopf monoid. For instance, if we let $\mathbf{H} = \mathbf{G}$, then a graph $\mathbf{g}$ is totally reducible if and only if it is edgeless. Finally, if $\mathbf{H} = \mathbf{M}$, then a matroid $\mathbf{m}$ is totally reducible if and only if is a direct sum of loops and coloops, which means $\mathbf{m}$ has a unique basis. These special cases of χ were studied in context of Hopf algebras in [2], and in the context of Hopf monoids in [1].

3.1 Chromatic quasisymmetric class functions

We introduce the φ-chromatic quasisymmetric class function. For trivial group actions, our invariant is a special case of the quaisymmetric function invariant that was defined for combinatorial Hopf algebras by Aguiar, Bergeron, and Sottile [2]. Let $\mathbf{H}$ be a linearized combinatorial Hopf monoid with character φ. Let N be a finite set, and for $\mathfrak{g} \in \mathfrak{S}_N$ and $\mathbf{h} \in \mathbf{H}_N$, let $\mathfrak{g} \cdot \mathbf{h} = \mathbf{H}_{\mathfrak{g}}(\mathbf{h})$. This defines a group action of $\mathfrak{S}_N$ on $\mathbf{H}_N$. Given $\mathbf{h} \in \mathbf{H}_N$, we say $\mathfrak{g}$ is an *automorphism* of $\mathbf{h}$ if $\mathfrak{g} \cdot \mathbf{h} = \mathbf{h}$. Let $\mathrm{Aut}(\mathbf{h})$ denote the set of automorphisms of $\mathbf{h}$, which is a subgroup of $\mathfrak{S}_N$.

Let $f\colon N \to \mathbb{N}$ be a function, and let $\{i : f^{-1}(i) \neq \emptyset\} = \{i_1, \dots, i_k\}$, with $i_1 < \cdots < i_k$. We let N_i be the set of vertices v such that $f(v) \leq i$. We call f a φ-proper coloring of $\mathbf{h}$ if $\varphi(\mathbf{h}|_{N_{i+1}}/N_i) = 1$ for all i. Let $F_\varphi(\mathbf{h})$ be the set of φ-proper colorings. Then $\mathrm{Aut}(\mathbf{h})$ acts on $F_\varphi(\mathbf{h})$ via $\mathfrak{g}f = f \circ \mathfrak{g}^{-1}$, where $\mathfrak{g} \in \mathrm{Aut}(\mathbf{h})$ and $f \in F_\varphi(\mathbf{h})$. Moreover we have $\mathbf{x}^f = \mathbf{x}^{\mathfrak{g}f}$.

Now we introduce our new quasisymmetric class function invariants. Fix a finite set N. Let $\mathbf{h} \in \mathbf{H}_N$ and let $\mathfrak{G}$ be a subgroup of $\mathrm{Aut}(\mathbf{h})$. For $\mathfrak{g} \in \mathfrak{G}$, define

$$\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x}; \mathfrak{g}) = \sum_{f \in F_\varphi(\mathbf{h}) : \mathfrak{g}f = f} \mathbf{x}^f.$$

This is the *chromatic quasisymmetric class function* associated to $\mathbf{H}$ with respect to $\mathbf{h}$. The fact that it is a quasisymmetric class function is proven in the full version (although for many of the examples considered in this abstract, the result follows from Theorem 14.)

As discussed in section 2, to every quasisymmetric class function $F(\mathfrak{G}, \mathbf{x})$ there is an orbital polynomial $F^O(\mathfrak{G}, x)$. We refer to $\Psi^O_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, x)$ as the *orbital chromatic polynomial*, and give a combinatorial interpretation of it. Fix $x \in \mathbb{N}$. Then $\mathfrak{G}$ also acts on the set of φ-proper colorings f with $f(N) \subseteq [x]$, and $\Psi^O_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, x)$ is the number of orbits of this action.

Example 8. Consider the linearized combinatorial Hopf monoid $(\mathbf{G}, \chi)$, and let $\mathbf{g}$ be a graph. Then a coloring is χ-proper for a graph $\mathbf{g}$ if it is a proper coloring in the usual sense. We obtain the chromatic quasisymmetric class function of a graph.

Example 9. Consider the linearized combinatorial Hopf monoid $(\mathbf{DP}, \psi)$, and let $(\mathbf{p}_1, \mathbf{p}_2)$ be a double poset on a finite set N. Then a function $f\colon N \to \mathbb{N}$. is ψ-proper if it satisfies the following conditions:

1. For $x, y \in N$ such that $x \leq y$ in $\mathbf{p}_1$, we have $f(x) \leq f(y)$.

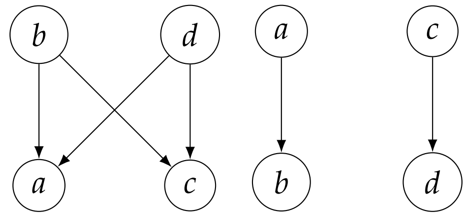

Figure 2: A double poset, with $\mathbf{p}_1$ on the left, and $\mathbf{p}_2$ on the right.

2. For $x, y \in N$ such that $x \leq y$ in $\mathbf{p}_1$ and $y < x$ in $\mathbf{p}_2$, we have $f(x) < f(y)$.

Grinberg refers to such functions as $(\mathbf{p}_1, \mathbf{p}_2)$-partitions. When $\mathbf{p}_2$ is a linear order this generalizes the notion of labeled P-partitions. Let $F_\psi(\mathbf{p}_1, \mathbf{p}_2)$ denote the set of double poset partitions, then $\mathrm{Aut}(\mathbf{p}_1, \mathbf{p}_2)$ acts on $F_\psi(\mathbf{p}_1, \mathbf{p}_2)$. Let $\mathfrak{G} \subseteq \mathrm{Aut}(\mathbf{p}_1, \mathbf{p}_2)$. Given $\mathfrak{g} \in \mathfrak{G}$, we see that $\Psi_{\mathbf{DP},\psi}(\mathbf{p}_1, \mathbf{p}_2, \mathfrak{G}, \mathbf{x}; \mathfrak{g})$ counts the double poset partitions f of $(\mathbf{p}_1, \mathbf{p}_2)$ that are fixed by $\mathfrak{g}$, weighted by $\mathbf{x}^f$.

Let $(\mathbf{p}_1, \mathbf{p}_2)$ be the double poset in Figure 2. Let $\mathbb{Z}/2\mathbb{Z}$ act on $\mathbf{p}$ by swapping a with c and b with d. We let ρ denote the regular representation. Then

$$\Psi_{\mathbf{DP},\psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbb{Z}/2\mathbb{Z}, \mathbf{x}) = M_{\{2\},4} + \rho(M_{\{1,2\},4} + M_{\{1,3\},4} + M_{\{2,3\},4} + 2M_{[3],4}).$$

Example 10. Consider the linearized combinatorial Hopf monoid $(\mathbf{M}, \chi)$, and let $\mathbf{m}$ be a matroid on a finite set N. Then a function $f \colon N \to \mathbb{N}$ is χ-proper for a matroid if it is maximized by a unique basis. If we let $F_\chi(\mathbf{m})$ denote the set of all χ-proper functions of $\mathbf{m}$, then $\mathrm{Aut}(\mathbf{m})$ acts on $F_\chi(\mathbf{m})$. Let $\mathfrak{G} \subseteq \mathrm{Aut}(\mathbf{m})$. Given $\mathfrak{g} \in \mathfrak{G}$, we see that $\Psi_{\mathbf{M},\chi}(\mathbf{m}, \mathfrak{G}, \mathbf{x}; \mathfrak{g})$ counts the m-generic functions f of $\mathbf{g}$ that are fixed by $\mathfrak{g}$, weighted by $\prod_{v \in N} x_{f(v)}$. We observe that $\Psi_{\mathbf{G},\chi}(\mathbf{g}, \{e\}, \mathbf{x})$ is the Billera-Jia-Reiner quasisymmetric function associated to a matroid.

Let $\mathbf{m}$ be the uniform matroid on four elements of rank two. That is, $\mathbf{m}$ have vertices $\{0, 1, 2, 3\}$, and every subset of size two is a basis. Let $\mathbb{Z}/4\mathbb{Z}$ act on $\mathbf{m}$ by cyclic rotation of the vertices. We let ρ denote the regular representation, and sgn denote the sign representation. Then

$$\Psi_{\mathbf{M},\chi}(\mathbf{m}, \mathbb{Z}/4\mathbb{Z}, \mathbf{x}) = (\rho + 1 + \mathrm{sgn})M_{\{2\},4} + 3\rho(M_{\{1,2\},4} + M_{\{2,3\},4}) + 6\rho M_{[3],4}.$$

4 Balanced Relative Simplicial Complexes

We discuss coloring complexes associated to linearized combinatorial Hopf monoids that have a balanced convex character. These geometric objects are used to prove Theorem 1.

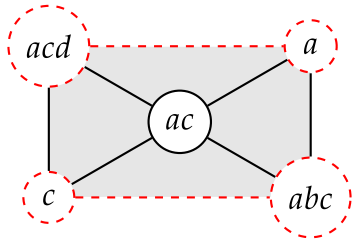

Figure 3: A coloring complex Σ. Dashed lines correspond to faces that are not in Σ.

Definition 11. A *balanced relative simplicial complex* of dimension $d-1$ on a vertex set V is a collection Φ of subsets of V, along with a function $\kappa\colon V \to [d]$ with the following properties:

1. For every $\rho \subseteq \sigma \subseteq \tau$, if $\rho, \tau \in \Phi$, then $\sigma \in \Phi$.
2. For every $\rho \in \Phi$, there exists $\sigma \in \Phi$ such that $\rho \subseteq \sigma$ and $|\sigma| = d$,
3. For every $\rho \in \Phi$, we have $\kappa(\rho) := \{\kappa(v) : v \in \rho\}$ has size $|\rho|$.

The name comes from the fact that there exists simplicial complexes (Γ, Σ) with $\Gamma \subseteq \Sigma$, and $\Phi = \Sigma \setminus \Gamma$. A bijection $\mathfrak{g}\colon V \to V$ is an *automorphism* of Φ if it satisfies the following two properties:

1. For every $v \in V$, we have $\kappa(\mathfrak{g}v) = \kappa(v)$.
2. For every $\{v_1, \ldots, v_k\} \in \Phi$, we have $\{\mathfrak{g}(v_1), \ldots, \mathfrak{g}(v_k)\} \in \Phi$.

Let $\mathrm{Aut}(\Phi)$ be the group of automorphisms of Φ, and fix a subgroup $\mathfrak{G} \subseteq \mathrm{Aut}(\Phi)$. For $\mathfrak{g} \in \mathfrak{G}$, define

$$\mathrm{F}(\Phi, \mathfrak{G}, \mathbf{x}) = \sum_{\sigma \in \Phi : \mathfrak{g}\sigma = \sigma} M_{\kappa(\sigma), d+1}.$$

As we vary $\mathfrak{g} \in \mathfrak{G}$, we obtain a quasisymmetric class function, which we call the *flag quasisymmetric class function* associated to $(\Phi, \mathfrak{G})$.

As an example, consider the balanced relative simplicial complex Φ on $2^{\{a,b,c,d\}} \setminus \{\emptyset, \{a,b,c,d\}\}$ appearing in Figure 3. We denote the vertex $\{a,b,c\}$ as *abc* for simplicity. We also let $\kappa(S) = |S|$. Then Φ is a balanced relative simplicial complex. We see that $\mathrm{Aut}(\Phi)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. There are four irreducible representations of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $\mathbb{Z}/2\mathbb{Z}$ act on Φ as the 180° rotation of Figure 3. Then

$$F(\Phi, \mathbb{Z}/2\mathbb{Z}, \mathbf{x}) = M_{\{2\},4} + \rho(M_{\{1,2\},4} + M_{\{1,3\},4} + M_{\{2,3\},4} + 2M_{[3],4}).$$

Theorem 12. *Let Φ be a balanced relative simplicial complex of dimension $d-1$, and let $\mathfrak{G} \subseteq \mathrm{Aut}(\Phi)$. Then $\mathrm{F}(\Phi, \mathfrak{G}, \mathbf{x})$ is M-increasing.*

Given $S \subseteq [d]$, we let V_S denote the linear span of $\{\sigma \in \Phi : \kappa(\sigma) = S\}$. We see that $\mathfrak{G}$ acts on V_S, and the resulting character is $F_{\mathfrak{G},S}$. Given $\sigma \in \Phi$, and $S \subset \kappa(\sigma)$, we let $\sigma|_S = \{v \in \sigma : \kappa(v) \in S\}$. Given $S \subseteq T \subseteq [d]$, we let $\theta\colon V_T(\Phi) \to V_S(\Phi)$ be given by

$$\theta(\sigma) = \begin{cases} \sigma|_S & \text{if } \sigma|_S \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$$

In the full version, we show that θ is surjective and $\mathfrak{G}$-invariant.

4.1 Convex characters

Definition 13. We say that φ is a *balanced convex character* if, for every $\mathbf{H}$-structure $\mathbf{h} \in \mathbf{H}_N$ the following conditions are satisfied:

1. If $|N| = 1$, then $\varphi(\mathbf{h}) = 1$.
2. If $|N| > 1$, then there exists proper $S \subset N$ such that $\Delta_{S,N\setminus S}(\mathbf{h}) \neq 0$.
3. If $\varphi(\mathbf{h}) = 1$ and we have $\subseteq N$ such that $\Delta_{S,N\setminus S}(\mathbf{h}) \neq 0$, then $\varphi(\mathbf{h}|_S) = \varphi(\mathbf{h}/S) = 1$.

Note that the second condition implies that $\mathbf{h}$ is not primitive for any $\mathbf{H}$-structure $\mathbf{h}$. However, since the $\mathbf{H}$-structures form bases for the linearized species $\mathbb{K}(\mathbf{H})$, we are still allowing for $\mathbb{K}(\mathbf{H})$ to have primitive elements.

Example 14. Let $\mathbf{H}$ be a linearized Hopf monoid. If, for every $\mathbf{H}$-structure $\mathbf{h}$, there exists a proper $S \subset N$ such that $\Delta_{S,N\setminus S}(\mathbf{h}) \neq 0$, then χ is a balanced convex character. Similarly, ψ is a balanced convex character for the Hopf monoid of double posets $\mathbf{DP}$.

Definition 15. Let $(\mathbf{H}, \varphi)$ be a linearized combinatorial Hopf monoid. Let N be a finite set, and let V be the collection of all proper subsets of N, and define $\kappa\colon V \to [|N|]$ by $\kappa(S) = |S|$. Given $\mathbf{h} \in \mathbf{H}_N$, define the *$\varphi$-coloring complex* of $\mathbf{h}$ to be the relative simplicial complex on V given by

$$\Sigma_\varphi(\mathbf{h}) = \{\{F_1, \ldots, F_k\} : \emptyset \subset F_1 \subset \cdots \subset F_k \subset N \text{ and } \prod_{i=0}^{k} \varphi(\mathbf{h}|_{F_{i+1}}/F_i) = 1\}.$$

where we set $F_0 = \emptyset$ and $F_{k+1} = N$.

Example 16. Let $\mathbf{G}$ be the linearized combinatorial Hopf monoid of graphs. Then the character χ is balanced convex. The φ-coloring complex $\Sigma_\chi(\mathbf{g})$ consists of chains $S_1 \subset S_2 \subset \cdots \subset S_k \subset N$ such that $S_i \setminus S_{i-1}$ is an independent set for all i. This relative simplicial complex can be represented as $\Sigma \setminus \Gamma(\mathbf{g})$, where Σ is the Coxeter complex of type A, and $\Gamma(\mathbf{g})$ is collection of flags $S_1 \subset S_2 \subset \cdots \subset S_k \subset N$ where $S_{i+1} \setminus S_i$ must contain an edge for some i. The subcomplex $\Gamma(\mathbf{g})$ is the coloring complex of a graph studied by Steingrímsson [10].

Example 17. Let **DP** be the linearized combinatorial Hopf monoid of double posets. Let $(\mathbf{p}_1, \mathbf{p}_2)$ be the double poset appearing in Figure 2. Then $\Sigma_\psi(\mathbf{p}_1, \mathbf{p}_2)$ is the balanced relative simplicial complex appearing in Figure 3.

Theorem 18. *Let $(\mathbf{H}, \varphi)$ be a linearized combinatorial Hopf monoid. Suppose that φ is a balanced convex character. Let N be a finite set, and let V be the collection of all proper subsets of N, and define $\kappa\colon V \to [|N| - 1]$ by $\kappa(S) = |S|$. Given $\mathbf{h} \in \mathbf{H}_N$, let $\Sigma_\varphi(\mathbf{h})$ be the φ-coloring complex. Then $\Sigma_\varphi(\mathbf{h})$ is a balanced relative simplicial complex of dimension $|N| - 2$.*

We have $\mathrm{Aut}(\mathbf{h}) \subseteq \mathrm{Aut}(\Sigma_\varphi(\mathbf{h}))$. *If* $\mathfrak{G} \subseteq \mathrm{Aut}(\mathbf{h})$, *then*

$$\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x}) = \mathrm{F}(\Sigma_\varphi(\mathbf{h}), \mathfrak{G}, \mathbf{x}).$$

The proof is related to prior work in [11]. Given a φ-proper coloring f, recall that $N_i(f) = \{v : f(v) \leq i\}$. We consider the set $F(f) = \{N_i(f) : i \in \mathbb{N}\} = \{F_1, \ldots, F_k\}$ where $F_1 \subset F_2 \subset \cdots \subset F_k = N$. This defines a function Σ between $F_\varphi(\mathbf{h})$ and collections of chains of subsets of N. The definition of balanced convex character ensures that $\Sigma(F_\varphi(\mathbf{h}))$ is a balanced relative simplicial complex. The function is $\mathfrak{G}$-invariant, and given $F \in \Sigma_\varphi(\mathbf{h})$, we have

$$\sum_{f \in F_\varphi(\mathbf{h}) : F(f) = F} \mathbf{x}^f = M_{\kappa(F),|N|}.$$

5 Future Work

We can also study the basis of fundamental quasisymmetric functions $F_{S,n}$. Given a group $\mathfrak{G}$, we let $\chi_1, \ldots, \chi_n$ be the irreducible characters for $\mathfrak{G}$. When we express $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x})$ in terms of χ_i and $F_{\alpha,n}$, are the resulting coefficients positive integers? We say that $(\mathbf{h}, \mathfrak{G})$ is F-effective in this case. This would imply that $\sum_{n \geq 0} \Psi^O_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, n) t^n = \frac{h(t)}{(1-t)^{d+1}}$ where the coefficients of $h(t)$ are positive. In a previous paper [12], we studied this question for $\Psi_{\mathbf{DP},\psi}(\mathbf{d}, \mathfrak{G}, \mathbf{x})$, finding examples where the answer was 'no', as well as conditions for which the answer is 'yes'.

Conjecture 19. *Let $\mathbf{H}$ be a linearized combinatorial Hopf monoid with balanced convex character. Suppose that $\Sigma_\varphi(\mathbf{h})$ is relatively shellable for all $\mathbf{h}$. Then $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x})$ is F-effective.*

Another interesting problem is to study combinatorial reciprocity theorems. We have a combinatorial reciprocity theorem for $\Psi_{\mathbf{DP},\psi}(\mathbf{d}, \mathfrak{G}, \mathbf{x})$ which we proved in [12]. Is there a general method for constructing such combinatorial reciprocity theorems?

References

[1] M. Aguiar and F. Ardila. "Hopf monoids and generalized permutahedra". 2017. arXiv: 1709.07504.

[2] M. Aguiar, N. Bergeron, and F. Sottile. "Combinatorial Hopf algebras and generalized Dehn-Sommerville relations". *Compos. Math.* **142**.1 (2006), pp. 1–30. DOI.

[3] M. Aguiar and S. Mahajan. *Monoidal Functors, Species and Hopf Algebras*. Vol. 29. CRM Monograph Series. With forewords by Kenneth Brown and Stephen Chase and André Joyal. Providence, RI: American Mathematical Society, 2010, pp. lii+784.

[4] M. Aguiar and S. Mahajan. "Hopf monoids in the category of species". *Contemp. Math.* **585** (2013), pp. 17–124. DOI.

[5] L. J. Billera, N. Jia, and V. Reiner. "A quasisymmetric function for matroids". *European J. Combin.* **30**.8 (2009), pp. 1727–1757. DOI.

[6] P. J. Cameron and K. K. Kayibi. "Orbital chromatic and flow roots". *Combin. Probab. Comput.* **16**.3 (2007), pp. 401–407. DOI.

[7] W. Fulton and J. Harris. *Representation theory*. Vol. 129. Graduate Texts in Mathematics. A first course, Readings in Mathematics. Springer-Verlag, New York, 1991, pp. xvi+551. DOI.

[8] D. Grinberg. "Double posets and the antipode of QSym". *Electron. J. Combin.* **24**.2 (2017), Paper 2.22, 47.

[9] K. Jochemko. "Order polynomials and Pólya's enumeration theorem". *Electron. J. Combin.* **21**.2 (2014), Paper 2.52, 11.

[10] E. Steingrímsson. "The coloring ideal and coloring complex of a graph". *J. Algebraic Combin.* **14**.1 (2001), pp. 73–84. DOI.

[11] J. White. "Coloring complexes and combinatorial Hopf monoids". 2016. arXiv:1611.04079.

[12] J. White. "The chromatic quasisymmetric class function of a digraph". *Ann. Comb.* (2021). Link.