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Row-strict dual immaculate functions and 0-Hecke modules

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Abstract. We introduce a new basis of quasisymmetric functions, the row-strict dual immaculate functions. We construct a cyclic, indecomposable 0-Hecke algebra module for these functions. Our row-strict dual immaculate functions are related to the dual immaculate functions of Berg–Bergeron–Saliola–Serrano–Zabrocki (2014–15) by the involution ψ on the ring QSym of quasisymmetric functions. We give an explicit description of the effect of ψ on the associated 0-Hecke modules, via the poset induced by the 0-Hecke action on standard immaculate tableaux. This remarkable poset reveals other 0-Hecke submodules and quotient modules, often cyclic and indecomposable, notably for a row-strict analogue of the extended Schur functions studied in Assaf–Searles (2019).

Keywords: 0-Hecke algebra, dual immaculate functions, indecomposable module, quasisymmetric functions

1 Introduction

In this paper we introduce a new family of quasisymmetric functions, *the row-strict dual immaculate functions*. Our focus here is the study of the associated 0-Hecke algebra modules, which we define and analyse.

The row-strict dual immaculate functions $\mathcal{R}\mathfrak{S}^*_\alpha$ are initially defined as generating functions for certain types of tableaux of composition shape α. By identifying the correct descent set, we show that the functions $\mathcal{R}\mathfrak{S}^*_\alpha$ expand positively in the basis of fundamental quasisymmetric functions, and also that they are the image of the well-studied dual immaculate functions of [3], under the involutive algebra automorphism ψ of the Hopf algebra QSym of quasisymmetric functions.

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The descent set determines a 0-Hecke algebra action on the set $\mathrm{SIT}(\alpha)$ of standard immaculate tableaux of composition shape α, yielding a cyclic indecomposable module whose quasisymmetric characteristic is $\mathcal{R}\mathfrak{S}^*_\alpha$. The action defines a partial order on $\mathrm{SIT}(\alpha)$ which turns out to be dual to the partial order in [3], see Lemma 14. The resulting poset $P\mathcal{R}\mathfrak{S}^*(\alpha)$ has remarkable properties, leading to the discovery of several other 0-Hecke modules with interesting quasisymmetric characteristics. Among these is an analogue of the extended Schur functions defined by Assaf and Searles [1]. An examination of the poset (see Figure 1) reveals various subposets that are closed under either our action or the dual immaculate action of [3]. We investigate the resulting submodules.

The duality in the poset reflects the action of the involution ψ on QSym: we show that the cyclic generators for the dual immaculate module and for the row-strict dual immaculate module are respectively the top and bottom elements of the poset; see Definition 15. Similarly, the cyclic generators for the extended Schur Hecke-module of [13] and our row-strict extended Schur Hecke-module (see Theorem 25) are the top and bottom elements of the interval $[S^{col}_\alpha, S^{row}_\alpha]$, cf. Definition 15. These ideas can be used to explain, for example, the passage between the modules in [14] and [2], and are developed further in the full paper [10].

Our proofs are technical, relying heavily on straightening algorithms which produce saturated chains in the poset $P\mathcal{R}\mathfrak{S}^*(\alpha)$. See Theorem 17. We prove indecomposability by following the pioneering work in [14] and [3], with considerable technical modifications.

Table 1 provides a summary of our results and a comparison with prior work.

2 Background

We refer the reader to [8] for basic definitions.

A composition of n is a positive sequence of integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ summing to n, which we depict as a collection of left-justified boxes with α_i boxes in row i, where row 1 is the bottom row, in the "French" convention.

It is well known that compositions of n are in bijection with subsets of $\{1, 2, \ldots, n-1\}$. Write $\alpha \vDash n$ for a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n; the corresponding set is $\mathrm{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$. Given a subset $S = \{s_1, s_2, \ldots, s_j\}$ of $\{1, \ldots, n-1\}$, the corresponding composition of n is $\mathrm{comp}(S) = (s_1, s_2 - s_1, \ldots, s_j - s_{j-1}, n - s_j)$.

A function $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is *quasisymmetric* if the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is the same as the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ for every $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $i_1 < i_2 < \cdots < i_k$. The set of all quasisymmetric functions forms a ring graded by degree, $\mathrm{QSym} = \bigoplus_n \mathrm{QSym}_n$, where each QSym_n is a vector space over $\mathbb{Q}$ with bases indexed by compositions of n.

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n, the *fundamental quasisymmetric function* indexed by α is

$$F_\alpha(x_1, x_2, \ldots) = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in \mathrm{set}(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Note that $\{F_\alpha : \alpha \vDash n\}$ is a basis for QSym_n, the *fundamental* basis.

The *complement* of a composition α, denoted α^c is the composition obtained from α by taking the complement of the set corresponding to α. That is, $\alpha^c = \mathrm{comp}(\mathrm{set}(\alpha)^c)$. In QSym we have the involutive automorphism ψ, defined on the fundamental basis by

$$\psi(F_\alpha) = F_{\alpha^c}. \tag{2.1}$$

The algebra $\mathrm{NSym} = \mathbb{Q}\langle \mathbf{e}_1, \mathbf{e}_2, \ldots \rangle$ of *noncommutative symmetric functions*, a Hopf algebra dual to QSym, see [6], is generated by noncommuting indeterminates $\mathbf{e}_n$ of degree n. We briefly review concepts we will need from the work of Berg–Bergeron–Saliola–Serrano–Zabrocki, who introduced the *immaculate functions* $\mathfrak{S}_\alpha$ as a basis of NSym. Their dual in QSym are the *dual immaculate functions*, $\mathfrak{S}^*_\alpha$. These functions can be defined as the generating function for *immaculate tableaux*.

Definition 1 ([3, Definition 2.1])**.** Given $\alpha \vDash n$, an immaculate tableau of shape α is a filling, D, of the cells of the diagram of α with positive integers such that

1. The leftmost column entries strictly increase from bottom to top;
2. The row entries weakly increase from left to right.

A *standard* immaculate tableau of shape $\alpha \vDash n$ is one that is filled with distinct entries taken from $\{1, 2, \ldots, n\}$.

Given an immaculate tableau D, define $x^D = x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}$, where d_i is the number of i's in the tableau D.

Definition 2. The *dual immaculate function* indexed by $\alpha \vDash n$ is $\mathfrak{S}^*_\alpha = \sum_D x^D$, summed over all immaculate tableaux of shape α.

Theorem 3 ([3, Definition 2.3, Proposition 3.1])**.** *The set* $\{\mathfrak{S}^*_\alpha\}_{\alpha \vDash n}$ *is a basis for* QSym_n. *Given a standard immaculate tableau S, its $\mathfrak{S}^*$-descent set $\mathrm{Des}_{\mathfrak{S}^*}(S)$ of S is*

$$\mathrm{Des}_{\mathfrak{S}^*}(S) = \{i : i+1 \textit{ appears strictly above } i \textit{ in } S\}.$$

Then $\mathfrak{S}^*_\alpha = \sum_S F_{\mathrm{comp}(\mathrm{Des}_{\mathfrak{S}^*}(S))}$, *summed over all standard immaculate tableaux of shape α.*

For $\alpha = (1, 2)$, the unique standard immaculate tableau $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ has $\mathfrak{S}^*$-descent set $\{1\}$; thus $\mathfrak{S}^*_{(1,2)} = F_{\mathrm{comp}\{1\}} = F_{(1,2)}$.

3 Row-strict dual immaculate functions

In this section we define a new quasisymmetric function, which we call the *row-strict dual immaculate function*. We will show that these functions form a basis of QSym.

Definition 4. Let $\alpha \vDash n$. A row-strict immaculate tableau of shape α is a filling U with positive integers such that

1. The leftmost column entries weakly increase from bottom to top;
2. The row entries strictly increase from left to right.

The *row-strict dual immaculate function* indexed by α is $\mathcal{R}\mathfrak{S}^*_\alpha = \sum_U x^U$ where the sum is over all row-strict immaculate tableaux of shape α, and $x^U = x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}$, where d_i is the number of i's in the tableau U.

Note that standard row-strict immaculate tableaux coincide with standard immaculate tableaux. We denote the set of standard immaculate tableaux of shape α by $\mathrm{SIT}(\alpha)$.

Theorem 5 ([11])**.** *Define the $\mathcal{R}\mathfrak{S}^*$-descent set of a standard immaculate tableau S by*

$$\mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}(S) = \{i : i+1 \textit{ is weakly below } i \textit{ in } S\}.$$

Then $\mathcal{R}\mathfrak{S}^*_\alpha = \sum_S F_{\mathrm{comp}(\mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}(S))}$, summed over all standard immaculate tableaux of shape α.

For $\alpha = (1,2)$, the unique standard immaculate tableau $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}$ has $\mathcal{R}\mathfrak{S}^*$-descent set $\{2\}$; thus $\mathcal{R}\mathfrak{S}^*_{(1,2)} = F_{\mathrm{comp}\{2\}} = F_{(2,1)}$.

Clearly for any standard immaculate tableau S, $\mathrm{Des}_{\mathfrak{S}^*}(S) = \mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}(S)^c$, and hence applying the involution ψ immediately gives $\psi(\mathfrak{S}^*_\alpha) = \mathcal{R}\mathfrak{S}^*_\alpha$. Consequently, we have:

Theorem 6. $\{\mathcal{R}\mathfrak{S}^*_\alpha \mid \alpha \vDash n\}$ *is a basis for* QSym_n.

4 A 0-Hecke algebra action for $\mathcal{R}\mathfrak{S}^*$

Recall that the symmetric group S_n can be defined via generators $s_i, 1 \le i \le n-1$, the adjacent transpositions, subject to the relations

$${s_i}^2 = 1; \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}; \quad s_i s_j = s_j s_i, \ |i-j| \ge 2.$$

Definition 7 ([9])**.** Let $\mathbb{K}$ be any field. The 0-Hecke algebra $H_n(0)$ is the $\mathbb{K}$-algebra with generators $\pi_i, 1 \le i \le n-1$ and relations

$${\pi_i}^2 = \pi_i; \quad \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}; \quad \pi_i \pi_j = \pi_j \pi_i, \ |i-j| \ge 2.$$

The algebra $H_n(0)$ has dimension $n!$ over $\mathbb{K}$, with basis elements $\{\pi_\sigma : \sigma \in S_n\}$, where $\pi_\sigma = \pi_{i_1} \cdots \pi_{i_m}$ if $\sigma = s_{i_1} \cdots s_{i_m}$ is a reduced word. This is well-defined by standard Coxeter group theory, see [9].

It is known [12] that the 0-Hecke algebra admits precisely 2^{n-1} simple modules L_α, one for each composition $\alpha \vDash n$, and all one-dimensional. The well-known Frobenius characteristic defined on the Grothendieck ring of the symmetric groups has the following analogue for finite-dimensional $H_n(0)$-modules. See also [5].

Definition 8 ([7, Section 5.4])**.** Let M be a finite-dimensional $H_n(0)$-module; let $M = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = \mathbb{K}$ be a composition series of submodules for M, where each successive quotient M_i/M_{i+1} is simple, and thus equal to L_{α^i} for some composition $\alpha^i \vDash n$. The *quasisymmetric characteristic* of the module M, $\mathrm{ch}(M)$, is then defined to be the quasisymmetric function equal to the sum of fundamentals $\sum_{i=1}^k F_{\alpha^i}$. In particular $\mathrm{ch}(L_\alpha) = F_\alpha$ for each $\alpha \vDash n$.

The following is our restatement of the main result of [3].

Theorem 9 ([3, Theorem 3.12])**.** *There is an indecomposable cyclic* 0*-Hecke algebra module* $\mathcal{W}_\alpha$ *whose quasisymmetric characteristic is the dual immaculate function* $\mathfrak{S}^*_\alpha$, $\mathrm{ch}(\mathcal{W}_\alpha) = \mathfrak{S}^*_\alpha$. *The module* $\mathcal{W}_\alpha$ *has dimension equal to the number of standard immaculate tableaux of shape* α. *The* $\mathfrak{S}^*$*-action of the* 0*-Hecke algebra generator* π_i *on the set of standard immaculate tableaux of shape* α, *for* $\alpha \vDash n$, *may be described as follows:*

$$\pi_i^{\mathfrak{S}^*}(T) = \begin{cases} T, & \textit{if } i+1 \textit{ is in a row weakly below } i, \\ 0 & \textit{if } i, i+1 \textit{ are in column 1 of } T, \\ s_i(T) & \textit{if } i+1 \textit{ is strictly above } i \textit{ in } T \\ & \textit{and } i, i+1 \textit{ are } \textbf{\textit{not}} \textit{ in column 1}, \end{cases} \tag{4.1}$$

where $s_i(T)$ *is the standard immaculate tableau obtained from* T *by swapping* i *and* $i+1$.

The goal of this section is to find an analogous module $\mathcal{V}_\alpha$ for our newly defined row-strict dual immaculate functions, that is, so that $\mathrm{ch}(\mathcal{V}_\alpha) = \mathcal{R}\mathfrak{S}^*_\alpha$.

Following [3], we consider the vector space $\mathcal{V}_\alpha$ whose basis vectors are the standard immaculate tableaux of shape α. Define, for each $1 \le i \le n-1$ and each standard immaculate tableau T of shape α, the $\mathcal{R}\mathfrak{S}^*$-action of the generator π_i on T to be

$$\pi_i(T) = \pi_i^{\mathcal{R}\mathfrak{S}^*}(T) = \begin{cases} T & \text{if } i+1 \text{ is strictly above } i\ , \\ 0 & \text{if } i \text{ and } i+1 \text{ are in the same row of } T, \\ s_i(T) & \text{if } i+1 \text{ is strictly below } i \text{ in } T, \end{cases} \tag{4.2}$$

where $s_i(T)$ is as above. To avoid cumbersome notation, we write simply $\pi_i(T)$ for the row-strict immaculate action, using $\pi_i^{\mathcal{R}\mathfrak{S}^*}$ and $\pi_i^{\mathfrak{S}^*}$ only when there is explicit need to distinguish between the actions of (4.2) and (4.1). We refer to these as the $\mathcal{R}\mathfrak{S}^*$-action and the $\mathfrak{S}^*$-action respectively. Likewise we may refer to the resulting $H_n(0)$-modules as the $\mathcal{R}\mathfrak{S}^*$-Hecke module and the $\mathfrak{S}^*$-Hecke module respectively.

Example 10. For the standard immaculate tableau $S = \begin{array}{|c|c|c|c|} \hline 6 \\ \hline 4 & 5 & 8 & 10 \\ \hline 3 & 7 \\ \hline 1 & 2 & 9 \\ \hline \end{array}$, from (4.2) we have $\pi_i^{\mathcal{R}\mathfrak{S}^*}(T) = T$ for $i \in \{2,3,5,7,9\}$, $\pi_1^{\mathcal{R}\mathfrak{S}^*}(T) = 0 = \pi_4^{\mathcal{R}\mathfrak{S}^*}(T)$, and $\pi_6^{\mathcal{R}\mathfrak{S}^*}(T) = s_6(T) = \begin{array}{|c|c|c|c|} \hline \mathbf{7} \\ \hline 4 & 5 & 8 & 10 \\ \hline 3 & \mathbf{6} \\ \hline 1 & 2 & 9 \\ \hline \end{array}$, $\quad \pi_8^{\mathcal{R}\mathfrak{S}^*}(T) = s_8(T) = \begin{array}{|c|c|c|c|} \hline 6 \\ \hline 4 & 5 & \mathbf{9} & 10 \\ \hline 3 & 7 \\ \hline 1 & 2 & \mathbf{8} \\ \hline \end{array}$.

Theorem 11. *The operators $\pi_i^{\mathcal{R}\mathfrak{S}^*}$ define an action of $H_n(0)$ on the vector space $\mathcal{V}_\alpha$.*

Figure 1 shows the row-strict dual immaculate action on the standard immaculate tableaux of shape 223, with labelled arrows indicating which generator maps one tableau to another. We note that this poset is the dual of the poset appearing in [3].

The cover relation for our poset $P\mathcal{R}\mathfrak{S}^*_\alpha$ for standard immaculate tableaux of shape $\alpha \vDash n$ is

$$S \prec_{\mathcal{R}\mathfrak{S}^*_\alpha} T \text{ if and only if there exists } i \text{ such that } T = \pi_i^{\mathcal{R}\mathfrak{S}^*}(S), \tag{4.3}$$

with respect to the **row-strict** 0**-Hecke** action defined by Theorem 11.

On the other hand, the cover relation for the poset $P\mathfrak{S}^*_\alpha$ for standard immaculate tableaux of shape $\alpha \vDash n$, as described in [3], is

$$S \prec_{\mathfrak{S}^*_\alpha} T \text{ if and only if there exists } i \text{ such that } S = \pi_i^{\mathfrak{S}^*}(T), \tag{4.4}$$

with respect to the **dual immaculate** 0**-Hecke** action defined by Theorem 9.

The following lemma shows that in each case, a cover relation is determined by a unique generator of the 0-Hecke algebra.

Lemma 12. *Let $\alpha \vDash n$ and $T \in \mathrm{SIT}(\alpha)$. If $\pi_i(T) = s_i(T), \pi_j(T) = s_j(T) \in \mathrm{SIT}(\alpha)$ and $\pi_i(T) = \pi_j(T)$, then necessarily $i = j$.*

By means of a filtration defined on the poset $P\mathcal{R}\mathfrak{S}^*_\alpha$, just as in [14] and [3], we show that

Theorem 13. *Let $\alpha \vDash n$ and $T_1 \in \mathrm{SIT}(\alpha)$ be the minimal element under any total order imposed on the poset $P\mathcal{R}\mathfrak{S}^*_\alpha$. Then T_1 determines a unique $H_n(0)$-module $\mathcal{V}_\alpha$ whose quasisymmetric characteristic is the row-strict dual immaculate function $\mathcal{R}\mathfrak{S}^*_\alpha$.*

However, we analyse the poset $P\mathcal{R}\mathfrak{S}^*_\alpha$ in more detail. We observe first that

Lemma 14. *Let $\alpha \vDash n$ and $S, T \in \mathrm{SIT}(\alpha)$. Then $S \prec_{\mathfrak{S}^*_\alpha} T$ if and onlf if $S \prec_{\mathcal{R}\mathfrak{S}^*_\alpha} T$. Hence the two posets $P\mathfrak{S}^*_\alpha$ and $P\mathcal{R}\mathfrak{S}^*_\alpha$ are isomorphic.*

Definition 15. Let $\alpha \vDash n$ be of length $\ell = \ell(\alpha)$. We single out three special standard immaculate tableaux:

Define S^0_α to be the standard tableau of shape α with entries $1, 2, \ldots, \ell$ in column 1, increasing bottom to top, and then fill the remaining rows, top to bottom, left to right with consecutive integers starting at $\ell + 1$ and ending at n.

Define S^{row}_α to be the *row superstandard* tableau of shape α, whose rows are filled left to right, bottom to top, beginning with the first row, with the numbers $1, 2, \ldots, n$ in consecutive order.

Define S^{col}_α to be the *column superstandard* tableau of shape α, whose columns are filled bottom to top and left to right, beginning with the first column, with the numbers $1, 2, \ldots, n$ in consecutive order.

Example 16. We have

$$S^0_{43423} = \begin{array}{llll} \mathbf{5} & 6 & 7 & \\ \mathbf{4} & 8 & & \\ \mathbf{3} & 9 & 10 & 11 \\ \mathbf{2} & 12 & 13 & \\ \mathbf{1} & 14 & 15 & 16 \end{array}, \qquad S^{row}_{43423} = \begin{array}{llll} \mathbf{14} & 15 & 16 & \\ \mathbf{12} & 13 & & \\ \mathbf{8} & 9 & 10 & 11 \\ \mathbf{5} & 6 & 7 & \\ \mathbf{1} & 2 & 3 & 4 \end{array}, \qquad S^{col}_{43423} = \begin{array}{llll} \mathbf{5} & 10 & 14 & \\ \mathbf{4} & 9 & & \\ \mathbf{3} & 8 & 13 & 16 \\ \mathbf{2} & 7 & 12 & \\ \mathbf{1} & 6 & 11 & 15 \end{array}.$$

By means of two important straightening algorithms, we show that

Theorem 17. *The poset $P\mathcal{R}\mathfrak{S}^*_\alpha$ has a unique bottom element S^0_α and a unique top element S^{row}_α. Also, for any $T \in \mathrm{SIT}(\alpha)$, there are saturated chains from S^0_α to T, and from T to S^{row}_α.*

Example 18. Let $\alpha = 223$, so that $S^0_\alpha = \begin{array}{lll} 3 & 4 & 5 \\ 2 & 6 & \\ 1 & 7 & \end{array}$, and let $T = \begin{array}{lll} 5 & 6 & 7 \\ 2 & 4 & \\ 1 & 3 & \end{array}$. First we straighten column 1 of T to match column 1 of S^0_α: Start with the lowest entry a_j in column 1 of T such that $a_j \neq j$, and swap it with $a_j - 1$; continue in this manner until column 1 has the entries $3, 2, 1$ from top to bottom.

$$T = T_0 \xleftarrow{\pi_4} T_1 = \begin{array}{lll} \mathbf{4} & 6 & 7 \\ 2 & 5 & \\ 1 & 3 & \end{array} \xleftarrow{\pi_3} T_2 = \begin{array}{lll} \mathbf{3} & 6 & 7 \\ 2 & 5 & \\ 1 & 4 & \end{array}.$$

Next we work on the top row of T_2, starting with the smallest entry which differs from the corresponding entry in the same cell of S^0_α, and continuing until the top rows match:

$$T_2 \xleftarrow{\pi_5} T_3 = \begin{array}{lll} 3 & \mathbf{5} & 7 \\ 2 & 6 & \\ 1 & 4 & \end{array} \xleftarrow{\pi_4} T_4 = \begin{array}{lll} 3 & \mathbf{4} & 7 \\ 2 & 6 & \\ 1 & 5 & \end{array} \xleftarrow{\pi_6} T_5 = \begin{array}{lll} 3 & 4 & \mathbf{6} \\ 2 & 7 & \\ 1 & 5 & \end{array} \xleftarrow{\pi_5} T_6 = \begin{array}{lll} 3 & 4 & \mathbf{5} \\ 2 & 7 & \\ 1 & 6 & \end{array}.$$

Now move down to the next row from the top, and proceed in the same manner, finding the smallest entry that differs from the corresponding entry in S^0_α:

$$T_6 \xleftarrow{\pi_6} T_7 = \begin{array}{lll} 3 & 4 & 5 \\ 2 & \mathbf{6} & \\ 1 & 7 & \end{array} = S^0_\alpha.$$

We end up with $T = \pi_4\pi_3\pi_5\pi_4\pi_6\pi_5\pi_6(S^0_\alpha)$.

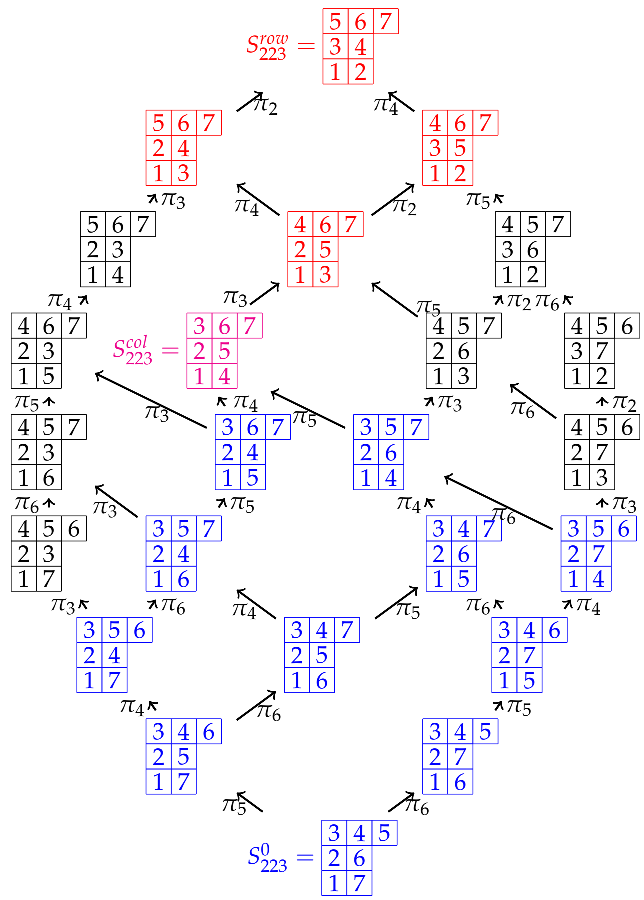

Figure 1: The row-strict dual immaculate poset $\mathcal{PRS}^*_{223}$; the red tableaux and S^{col}_{223} are increasing along rows and up columns, so are in $\mathrm{SET}(223) = [S^{col}_{223}, S^{row}_{223}]$; the blue tableaux and S^{col}_{223} are the elements in $\mathrm{SIT}^*(223) = [S^0_{223}, S^{col}_{223}]$, with the smallest elements in the first column.

From this we obtain:

Theorem 19. *The $\mathcal{RS}^*$-Hecke module $\mathcal{V}_\alpha$ of Theorem 13, with $\mathrm{ch}(\mathcal{V}_\alpha) = \mathcal{RS}^*_\alpha$, is cyclic, and generated by the unique minimal element S^0_α of the poset $\mathcal{PRS}^*_\alpha$.*

We also recover the result of [3]:

Theorem 20. *The $\mathfrak{S}^*$-Hecke module $\mathcal{W}_\alpha$ of Theorem 9, with $\mathrm{ch}(\mathcal{W}_\alpha) = \mathfrak{S}^*_\alpha$, is cyclic, and generated by the unique maximal element S^{row}_α of the poset $\mathcal{PRS}^*_\alpha$.*

A long series of technical lemmas culminates in our main result for this section, namely:

Theorem 21. *Let $\mathcal{V}_\alpha$ be the cyclic $\mathcal{R}\mathfrak{S}^*$-Hecke module generated by S^0_α. Then $\mathcal{V}_\alpha$ is indecomposable, and* $\mathrm{ch}(\mathcal{V}_\alpha) = \mathcal{R}\mathfrak{S}^*_\alpha$.

5 A row-strict analogue for extended Schur functions

This section is motivated by a closer examination of the poset $P\mathcal{R}\mathfrak{S}^*_\alpha$.

Definition 22 ([13])**.** Let $\mathrm{SET}(\alpha)$ be the set of all standard immaculate tableaux of shape α, called *s*tandard *e*xtended *t*ableaux, in which **all** columns increase from bottom to top.

Searles [13] constructs a quotient $H_n(0)$-module with basis $\mathrm{SET}(\alpha)$, from a larger parent module, whose quasisymmetric characteristic is the extended Schur function $\mathcal{E}_\alpha$ defined in [4, Section 7] and [1], and whose cyclic generator is S^{row}_α. First we restate Searles' formulation of the Assaf–Searles theorem as follows:

Theorem 23 ([1],[13, Theorem 2.7])**.** *Let $\alpha \vDash n$. Then the extended Schur function $\mathcal{E}_\alpha$ expands positively in the fundamental basis of* QSym *as follows: $\mathcal{E}_\alpha = \sum_{T \in \mathrm{SET}(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\mathfrak{S}^*}(T))}$.*

Theorem 24 ([4])**.** *The extended Schur functions $\{\mathcal{E}_\alpha\}_{\alpha \vDash n}$ form a basis of* QSym_n*, such that when the composition α is a partition λ of n, $\mathcal{E}_\lambda$ coincides with the Schur function s_λ.*

Let $\mathcal{Z}_\alpha = \mathrm{span}\langle \mathrm{SET}(\alpha) \rangle$. Figure 1 shows the elements of $\mathrm{SET}(223)$ in red; notice that they form a closed interval $[S^{col}_\alpha, S^{row}_\alpha]$ of $P\mathcal{R}\mathfrak{S}^*_{223}$ with respect to our $\mathcal{R}\mathfrak{S}^*$-Hecke action defined in (4.2). This is no accident.

Theorem 25. *The tableau S^{col}_α is the unique minimal element of* $\mathrm{SET}(\alpha)$*. The $H_n(0)$-submodule $\mathcal{Z}_\alpha$ of $\mathcal{V}_\alpha$ is cyclically generated by S^{col}_α and is indecomposable. It has characteristic*

$$\mathcal{R}\mathcal{E}_\alpha = \sum_{T \in \mathrm{SET}(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}(T))} = \psi(\mathcal{E}_\alpha).$$

Hence the functions $\{\mathcal{R}\mathcal{E}_\alpha\}_{\alpha \vDash n}$ also form a basis for QSym_n*. Furthermore, when the composition α is a partition λ of n, $\mathcal{R}\mathcal{E}_\lambda$ coincides with the Schur function s_{λ^t}, where λ^t is the transpose of λ.*

We call the $\mathcal{R}\mathcal{E}_\alpha$ *row-strict extended Schur* functions.

The extended Schur functions are dual to the *shin* basis of noncommutative symmetric functions of [4]. Hence our row-strict extended Schur functions $\mathcal{R}\mathcal{E}_\alpha$ also give rise to a dual basis of NSym, which we call the *Rshin* basis.

Next we examine the quotient module $\mathcal{V}_\alpha / \mathcal{Z}_\alpha$ arising from the submodule $\mathcal{Z}_\alpha$ of $\mathcal{V}_\alpha$.

Definition 26. Let $\mathrm{NSET}(\alpha)$ be the set of standard tableaux of shape α where all rows increase left to right, but at least one **column** does *not* increase from bottom to top.

Theorem 27. *The quotient module $\overline{\mathcal{V}}_\alpha = \mathcal{V}_\alpha / \mathcal{Z}_\alpha$ is also an $H_n(0)$-module for the $\mathcal{R}\mathfrak{S}^*$-action, with basis indexed by* $\mathrm{NSET}(\alpha) \cap \mathrm{SIT}(\alpha)$. *The module $\overline{\mathcal{V}}_\alpha$ is nonzero if and only if α has at least two parts of size greater than or equal to 2. In the latter case it is cyclically generated by $S^0_\alpha + \mathcal{Z}_\alpha$ and indecomposable, with characteristic equal to the quasisymmetric function*

$$\overline{\mathcal{RE}}_\alpha = \sum_{T \in \mathrm{NSET}(\alpha) \cap \mathrm{SIT}(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}(T))}.$$

Equivalently, $\mathcal{R}\mathfrak{S}^*_\alpha - \overline{\mathcal{RE}}_\alpha = \sum_{T \in \mathrm{SET}(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}(T))} = \mathcal{RE}_\alpha$.

Define S^{row*}_α to be the standard immaculate tableau in $\mathrm{SIT}(\alpha)$ whose first column consists of $1, 2, \ldots, \ell(\alpha)$, and whose remaining cells are filled with the entries $\ell + 1, \ldots, n$ in consecutive order along rows, bottom to top and left to right. Clearly S^{row*}_α has strictly increasing columns bottom to top, and hence $S^{row*}_\alpha \in \mathrm{SIT}^*(\alpha) \cap \mathrm{SET}(\alpha)$.

For example, we have

$$S^{row*}_{223} = S^{col}_{223} = \begin{array}{|c|c|c|}\hline 3 & 6 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 1 & 4 \\ \cline{1-2}\end{array}, \qquad S^{row*}_{332} = \begin{array}{|c|c|c|}\hline 3 & 8 \\ \hline 2 & 6 & 7 \\ \hline 1 & 4 & 5 \\ \hline\end{array} \neq S^{col}_{332}, \qquad S^{row*}_{1323} = \begin{array}{|c|c|c|}\hline 4 & 8 & 9 \\ \hline 3 & 7 \\ \hline 2 & 5 & 6 \\ \hline 1 \\ \cline{1-1}\end{array} \neq S^{col}_{1323}.$$

Let $\mathrm{SIT}^*(\alpha)$ denote the set of standard immaculate tableaux whose first column consists of the integers $\{1, 2, \ldots, \ell(\alpha)\}$. Examining the poset $P\mathcal{R}\mathfrak{S}^*(\alpha)$ again, we observe that the tableaux in blue in Figure 1 form a closed interval $[S^0_\alpha, S^{col}_\alpha = S^{row*}_\alpha]$ for the **dual immaculate action** of Theorem 9 (reverse the arrows, thanks to Lemma 14). We obtain

Theorem 28. *Let $\alpha \vDash n$, and let $\mathcal{W}_\alpha$ be the $H_n(0)$-module with basis* $\mathrm{SIT}(\alpha)$ *as in Theorem 9. Then*

1. $\mathrm{SIT}^*(\alpha) = [S^0_\alpha, S^{row*}_\alpha]$ *spans an $H_n(0)$-submodule $\mathcal{X}_\alpha$ of $\mathcal{W}_\alpha$, of dimension* $\binom{n-\ell(\alpha)}{\alpha_1 - 1, \ldots, \alpha_\ell - 1}$.
2. *$\mathcal{X}_\alpha$ is cyclically generated by S^{row*}_α; it has characteristic*
$$\mathrm{ch}(\mathcal{X}_\alpha) = \sum_{T \in \mathrm{SIT}^*(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\mathfrak{S}^*}(T))}.$$
3. *Assume α has at least two parts of size greater than 1. Then the quotient module $\mathcal{W}_\alpha / \mathcal{X}_\alpha$ is nonzero, indecomposable, and cyclically generated by $S^{row}_\alpha + \mathcal{X}_\alpha$, with characteristic*
$$\mathrm{ch}(\mathcal{W}_\alpha / \mathcal{X}_\alpha) = \sum_{T \in \mathrm{SIT}(\alpha) \setminus \mathrm{SIT}^*(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\mathfrak{S}^*}(T))}.$$

If α has at most one part greater than 1, then $\mathcal{W}_\alpha = \mathcal{X}_\alpha$ is the $H_n(0)$-irreducible L_α.

We have the following pleasing expression for the characteristic of the module $\mathcal{X}_\alpha$.

Proposition 29. *Let $\alpha \vDash n$ have length ℓ, and let $\overline{\alpha}$ be the composition $(\alpha_1 - 1, \alpha_2 - 1, \ldots, \alpha_\ell - 1)$ of $n - \ell$, where we omit any part that is zero. Then*

$$\mathrm{ch}(\mathcal{X}_\alpha) = \sum_{k \geq \ell} e_{\ell-1}(x_1, \ldots, x_{k-1}) x_k h_{\overline{\alpha}}(x_k, x_{k+1}, \ldots),$$

where e_r is the rth elementary symmetric function, $h_\beta = h_{\beta_1} h_{\beta_2} \cdots$ is the product of the homogeneous symmetric functions indexed by the parts of the composition β.

Finally, we observe (contrast with [13]) that the poset $P\mathfrak{S}^*(\alpha)$ itself gives a quotient module of $\mathcal{W}_\alpha$ whose characteristic is the extended Schur function $\mathcal{E}_\alpha$. Reversing arrows in Figure 1 shows that the tableaux **not** in $\mathrm{SET}(\alpha)$ are closed under the $\mathfrak{S}^*$-action.

Proposition 30. *The set* $\mathrm{NSET}(\alpha) \cap \mathrm{SIT}(\alpha)$ *indexes a basis for a submodule $\mathcal{Y}_\alpha$ of $\mathcal{W}_\alpha$ for the $\mathfrak{S}^*$-action. The resulting quotient module $\mathcal{W}_\alpha / \mathcal{Y}_\alpha$ has basis* $\mathrm{SET}(\alpha)$ *with characteristic $\mathcal{E}_\alpha$, and is cyclically generated by $S^{row}_\alpha + \mathcal{Y}_\alpha$. It is indecomposable.*

A summary of our results for row-strict dual immaculates appears in Table 1 below.

Immaculate Tableaux $\to$	$\mathfrak{S}^*_\alpha$ [3] Dual immaculate	$\mathcal{R}\mathfrak{S}^*_\alpha$ Row-strict dual imm.
1st Col bottom to top	strict $\nearrow$	weak $\nearrow$
Rows left to right	weak $\nearrow$	strict $\nearrow$
Descents (for fund. expansion)	$\{i : i+1 \text{ strictly above } i\}$	$\{i : i+1 \text{ weakly below } i\}$
Action of ψ in QSym	$\mathfrak{S}^*_\alpha(x_1, \ldots, x_n)$	$\mathcal{R}\mathfrak{S}^*_\alpha = \psi(\mathfrak{S}^*_\alpha)$
0-Hecke action on $\mathrm{SIT}(\alpha)$	*Note that standard*	*tableaux are the same*
$\pi_i(T) = T$	$i+1$ weakly below i	$i+1$ strictly above i
$\pi_i(T) = 0$	$i, i+1$ in 1st column	$i, i+1$ in same row
$T, s_i(T)$ standard, and $\pi_i(T) = s_i(T)$	$i+1$ strictly above i, $i, i+1$ NOT both in 1st column	$i+1$ strictly below i
Partial order on $\mathrm{SIT}(\alpha)$	Poset $P\mathfrak{S}^*(\alpha) = [S^0_\alpha, S^{row}_\alpha]$	Poset $P\mathcal{R}\mathfrak{S}^*(\alpha) \simeq P\mathfrak{S}^*(\alpha)$
Cover relation	$S \prec_{\mathfrak{S}^*_\alpha} T$ if and onlf if $S = \pi_i^{\mathfrak{S}^*}(T)$	$S \prec_{\mathcal{R}\mathfrak{S}^*_\alpha} T$ if and onlf if $T = \pi_i^{\mathcal{R}\mathfrak{S}^*}(S)$
Imm. module generated by	top element: $\mathcal{W}_\alpha = \langle S^{row}_\alpha \rangle$	bottom element: $\mathcal{V}_\alpha = \langle S^0_\alpha \rangle$
Indecomposable?	Yes [3]	Yes
Extended Schur fn basis	$\mathcal{E}_\alpha$, $\mathcal{E}_\lambda = s_\lambda$[1]	$\psi(\mathcal{E}_\alpha) = \mathcal{R}\mathcal{E}_\alpha$, $\mathcal{R}\mathcal{E}_\lambda = s_{\lambda^t}$
Module, Extended Schur fn Basis $\mathrm{SET}(\alpha)$	cyclic $\langle S^{row}_\alpha \rangle$, indecomp. [13] (quotient of larger module)	cyclic $\langle S^{col}_\alpha \rangle$, indecomp. submodule of $\mathcal{V}_\alpha$
Quotient module, Ext Schur fn Basis $\mathrm{NSET}(\alpha) \cap \mathrm{SIT}(\alpha)$	None	Yes, cyclic $\langle S^{row}_\alpha \rangle$, indecomp. quotient of $\mathcal{V}_\alpha$

Table 1: $\mathfrak{S}^*_\alpha$ versus the new basis $\mathcal{R}\mathfrak{S}^*_\alpha$.

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