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Soliton Cellular Automata for the Affine General Linear Lie Superalgebra

Mitchell Ryan*[1] and Benjamin Solomon†[1]

[1]*School of Mathematics and Physics, University of Queensland, St. Lucia, QLD 4072, Australia*

Abstract. The box-ball system (BBS) is a cellular automaton that is an ultradiscrete analogue of the Korteweg–de Vries equation, a non-linear PDE used to model water waves. In 2001, Hikami and Inoue generalised the BBS to the general linear Lie superalgebra $\mathfrak{gl}(m|n)$. We further generalise the Hikami–Inoue BBS using the Kirillov–Reshetikhin crystals for $\widehat{\mathfrak{gl}}(m|n)$ devised by Kwon and Okado in 2021, where we find similar solitonic behaviour under certain conditions.

Keywords: soliton, crystal bases, cellular automaton, integrable systems

1 Introduction

The Takahashi–Satsuma box-ball system (BBS) [10] is an ultradiscrete dynamical system that can be derived from a discretisation of the soliton solutions to the Korteweg–de Vries (KdV) equation using a limiting procedure [11]. This ultradiscrete system can be formulated using the crystal theory of quantum affine algebras [6].

The crystal theoretic formulation makes use of the 'classical' crystal B_ℓ, which is the crystal basis of an ℓ-fold symmetric tensor representation of $U_q(\mathfrak{sl}_n)$ promoted to the Kirillov–Reshetikhin (KR) crystal of $U_q'(\widehat{\mathfrak{sl}}_n)$ [8] by adding 0-arrows. States of the system are then defined as elements of $(B_\ell)^{\otimes\infty}$. The time evolution of the state is realised as the action of a row-to-row transfer matrix as $q \to 0$ that is constructed using the unique isomorphism between the tensor product of KR crystals called the combinatorial R-matrix, $R\colon B \otimes B' \to B' \otimes B$. The time evolution of a state can be described by repeated applications of the R-matrix. Like the KdV equation, there exist states with soliton solutions; that is, states containing objects called solitons that move with speed corresponding to their length and are stable under collisions (this stability is called *scattering*).

In 2001, Hikami and Inoue generalised the BBS using crystals for the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ and showed that similar behaviour held in this generalised system [7]. We further generalise the BBS using the KR crystals for $\widehat{\mathfrak{gl}}(m|n)$ developed by

*research@mitchell-ryan.com
†hart.loire@gmail.com. Partially supported by the AMSI Vacation Research Scholarships 2020–21.

Kwon and Okado [9]. Such a crystal can be parameterised by a young diagram Y, where the crystal is identified with the set of semistandard Young tableaux (SSYT) of shape Y, denoted $B(Y)$. In our generalised BBS, we define states as elements of $\left(B(Y^{r,1})\right)^{\otimes\infty}$, where $Y^{r,1}$ represents a Young diagram of height r and width 1. We similarly have an R-matrix giving a bijection of tensor products of crystals $B(Y^{r_1,s_1}) \otimes B(Y^{r_2,s_2}) \to B(Y^{r_2,s_2}) \otimes B(Y^{r_1,s_1})$. This allows us to define the time evolution of the system in parallel to the classical case. When $r = 1$, this is precisely the Hikami–Inoue BBS [7].

Example 1. Consider the $U_q(\widehat{\mathfrak{gl}}(3|1))$ crystal $B(Y^{2,1})$. We build the BBS in $(B(Y^{2,1}))^{\otimes\infty}$ starting at time $t = 0$ in the following diagram, where the maximal weight element ($\begin{smallmatrix}\overline{3}\\ \overline{2}\end{smallmatrix}$) is represented as a dot.

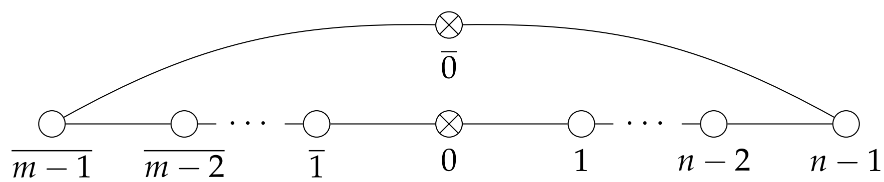
$$
\begin{array}{lccccccccccccccccccc}
t=0 & \cdot & \cdot & \cdot & \cdot & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \cdot & \cdot & \begin{smallmatrix}\overline{2}\\1\end{smallmatrix} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
t=1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \cdot & \begin{smallmatrix}\overline{2}\\1\end{smallmatrix} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
t=2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \begin{smallmatrix}\overline{2}\\1\end{smallmatrix} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
t=3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \begin{smallmatrix}\overline{2}\\1\end{smallmatrix} & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
t=4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \cdot & \begin{smallmatrix}\overline{2}\\1\end{smallmatrix} & \begin{smallmatrix}\overline{3}\\1\end{smallmatrix} & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

The above diagram shows the time evolution of the state after 4 time steps. At time $t = 1$, we can see that both $\begin{smallmatrix}\overline{3}&\overline{3}\\1&1\end{smallmatrix}$ and $\begin{smallmatrix}\overline{2}\\1\end{smallmatrix}$ have moved with speed proportional to their length. They collide at times $t = 2$ and $t = 3$, before separating into two solitons of the same lengths at $t = 4$ (stability under collisions). This demonstrates solitonic behaviour in our generalised system. Note that, at $t = 4$, $\begin{smallmatrix}\overline{2}&\overline{3}\\1&1\end{smallmatrix}$ is one step ahead (to the right) of where $\begin{smallmatrix}\overline{3}&\overline{3}\\1&1\end{smallmatrix}$ would be if there had been no collision. Similarly, $\begin{smallmatrix}\overline{3}\\1\end{smallmatrix}$ is one step behind where $\begin{smallmatrix}\overline{2}\\1\end{smallmatrix}$ would be. This phenomenon is called the *phase shift* and is a shadow of the nonlinearity.

The R-matrix can be explicitly calculated with the RSK algorithm, using the modified Schensted's Bumping Algorithm outlined in Section 2.1. For our generalised system, we present conditions that are sufficient for an 'object' to move with speed corresponding to its length; these results are presented in Theorem 14. However, they do not always have solitonic interactions as illustrated in Example 16. We also investigate the scattering of two-soliton states and define an explicit structure that is sufficient for stability under collisions, thus providing sufficient conditions for solitonic behaviour. Moreover, we describe the phase shift of these solitons in terms of the energy function, analogous to the classical case. These results are presented in Theorem 15.

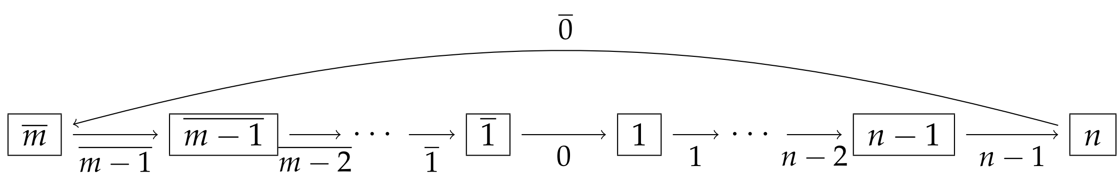
2 Background

The BBS described by Hatayama, Kuniba, Okado, Takagi and Yamada [5, 6] is derived from type A_n affine Lie algebra $\widehat{\mathfrak{sl}}_n$. In the super context, the structure is derived from the affine general linear superalgebra $\widehat{\mathfrak{gl}}(m|n)$ and its quantum group $U_q(\widehat{\mathfrak{gl}}(m|n))$ (in the sense of [9]). Let $I = I_{\text{even}} \sqcup I_{\text{odd}}$ be the indexing set of simple roots, where $I_{\text{even}} = \{\overline{m-1}, \ldots, \overline{1}, 1, \ldots, n-1\}$ and $I_{\text{odd}} = \{0, \overline{0}\}$. It is useful to set $I_- = \{\overline{m-1}, \ldots, \overline{1}\}$ and $I_+ = \{1, \ldots, n-1\}$, so that $I_{\text{even}} = I_- \sqcup I_+$. The Dynkin diagram for $\widehat{\mathfrak{gl}}(m|n)$ is:



where $\bigotimes$ denotes an isotropic simple root.

The fundamental representation of $U_q(\widehat{\mathfrak{gl}}(m|n))$ is an $(m+n)$-dimensional super vector space $\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_-$. Th representation admits a *crystal basis* $\{v_b \mid b \in B\}$ with $B = B_- \sqcup B_+$ where $B_- = \{\overline{m}, \overline{m-1}, \ldots, \overline{1}\}$ and $B_+ = \{1, \ldots, n-1, n\}$, which gives rise to the following *crystal graph*:



where $\boxed{b'} \xrightarrow{i} \boxed{b}$ if and only if $f_i v_{b'} = v_b$ (equivalently, $e_i v_b = v_{b'}$). For further explanation of crystals for $U_q(\widehat{\mathfrak{gl}}(m|n))$, see [9]. With the crystal in mind, we can define an ordering on B by $\overline{m} < \cdots < \overline{1} < 1 < \cdots < n$.

Let $\mathbf{V}^{\otimes N}$ be the N-th tensor power of the fundamental representation. It can be shown that all tensor powers with $N \geq 1$ are completely reducible. Moreover, the summands are in bijection with Young diagrams of $(m|n)$-hook shape [1, 9]. Given a summand W corresponding to the Young diagram Y, this bijection identifies the crystal basis elements of W with the semistandard Young tableaux (SSYT) of shape Y. In this context, a tableau is called *semistandard* if the rows are weakly (resp. strictly) increasing for indices in I_- (resp. I_+) and the columns are weakly (resp. strictly) increasing for indices in I_+ (resp. I_-). We refer the reader to Bump and Schilling for more information on SSYT [2]. For $i \in I_{\text{even}}$, the action of the crystal operators e_i and f_i can be computed by a *signature rule* similar to that for $U'_q(\widehat{\mathfrak{sl}}_n)$-crystals [12]. Let $Y^{r,s}$ be a rectangular Young diagram with height r and width s and let $B(Y^{r,s})$ be the set of SSYT of shape $Y^{r,s}$.

Take an arbitrary tableau

$$x = \begin{array}{|c|c|c|c|} \hline t_{11} & t_{12} & \cdots & t_{1s} \\ \hline t_{21} & t_{22} & \cdots & t_{2s} \\ \hline \cdots & \cdots & \ddots & \vdots \\ \hline t_{r1} & t_{r2} & \cdots & t_{rs} \\ \hline \end{array} \in B(Y^{r,s}).$$

We define a function, col by

$$\mathrm{col}(x) = \underbrace{t_{1s} \ldots t_{rs}}_{t_{*s}} \cdots \underbrace{t_{12} \ldots t_{r2}}_{t_{*2}} \underbrace{t_{11} \ldots t_{r1}}_{t_{*1}}.$$

Moreover, for $x, y \in B(Y^{r,s})$, we define $\mathrm{col}(x \otimes y) = \mathrm{col}(x)\,\mathrm{col}(y)$.

Definition 2. For some positive integer d, let $x \in B(Y^{r,s})^{\otimes d}$ and let $i \in I_{\text{even}}$ with $i = k \in I_+$, (resp. $i = \overline{k} \in I_-$). We define the *i-signature*, denoted $\mathrm{sg}_i(x)$, to be the sequence of $+$ and $-$ obtained by deleting all letters in $\mathrm{col}(x)$ which are not k or $k+1$ (resp. $\overline{k}$ or $\overline{k+1}$), and then replacing all k (resp. $\overline{k}$) with a $-$ symbol and replacing all $k+1$ (resp. $\overline{k+1}$) with a $+$ symbol.

We define the *reduced i-signature*, denoted $\mathrm{rsg}_i(x)$, to be equal to the i-signature, except with $+-$ pairs (in that order) successively deleted, so that $\mathrm{rsg}_i(x)$ is of the form

$$\underbrace{- \cdots -}_{a}\underbrace{+ \cdots +}_{b}$$

(where a or b can be zero).

For a tableau $x \in B(Y^{r,s})$ and for $i \in I_{\text{even}}$ where $i = k \in I_+$ (resp. $i = \overline{k} \in I_-$):

- To evaluate $f_k(x)$ (resp. $e_{\overline{k}}(x)$), find the rightmost $-$ symbol in $\mathrm{rsg}_i(x)$ and change the corresponding $\boxed{k}$ in x to $\boxed{k+1}$ (resp. $\boxed{\overline{k}}$ in x to $\boxed{\overline{k+1}}$). If there are no $-$ symbols, then $f_k(x) = 0$ (resp. $e_{\overline{k}}(x) = 0$).

- To evaluate $e_k(x)$ (resp. $f_{\overline{k}}(x)$), find the leftmost $+$ symbol in $\mathrm{rsg}_i(x)$ and change the corresponding $\boxed{k+1}$ in x to $\boxed{k}$ (resp. $\boxed{\overline{k+1}}$ in x to $\boxed{\overline{k}}$). If there are no $+$ symbols, then $e_k(x) = 0$ (resp. $f_{\overline{k}}(x) = 0$).

The operators e_0 and f_0 have a different algorithm:

- If the first occurrence of $\overline{1}$ in $\mathrm{col}(x)$ is before the first occurrence of 1, then $f_0(x)$ replaces the corresponding $\boxed{\overline{1}}$ in x with $\boxed{1}$, and $e_0(x) = 0$.

- If the first occurrence of 1 in $\mathrm{col}(x)$ is before the first occurrence of $\overline{1}$, then $e_0(x)$ replaces the corresponding $\boxed{1}$ in x with $\boxed{\overline{1}}$, and $f_0(x) = 0$

Example 3. We will compute $e_{\overline{3}}(x)$ for

$$x = \begin{array}{|c|c|c|}\hline \overline{4} & \overline{3} & \overline{3} \\ \hline \overline{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array}.$$

Then,

$$\begin{array}{rcccccccccc} \mathrm{col}(x) & = & \overline{3} & 3 & 3 & \overline{3} & 1 & 2 & \overline{4} & \overline{3} & 1 \\ \mathrm{sg}_{\overline{3}}(x) & = & - & & & - & & & + & - & \\ \mathrm{rsg}_{\overline{3}}(x) & = & - & & & - & & & & & \end{array}.$$

The rightmost $-$ corresponds to the bolded number below,

$$\begin{array}{rcccccccccc} \mathrm{col}(x) & = & \overline{3} & 3 & 3 & \boldsymbol{\overline{3}} & 1 & 2 & \overline{4} & \overline{3} & 1 \\ \mathrm{rsg}_{\overline{3}}(x) & = & - & & & - & & & & & \end{array} \quad \rightsquigarrow \quad \begin{array}{|c|c|c|}\hline \overline{4} & \boldsymbol{\overline{3}} & \overline{3} \\ \hline \overline{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array},$$

so we replace this $\boxed{\overline{3}}$ with $\boxed{\overline{4}}$ to get

$$e_{\overline{3}}(x) = \begin{array}{|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array}.$$

2.1 Combinatorial R-Matrix and energy function

Consider two $U_q(\widehat{\mathfrak{gl}}(m|n))$-crystals $B(Y^{r_1,s_1})$ and $B(Y^{r_2,s_2})$. Then there exists a unique isomorphism called the *combinatorial R-matrix*

$$R\colon B(Y^{r_1,s_1}) \otimes B(Y^{r_2,s_2}) \to B(Y^{r_2,s_2}) \otimes B(Y^{r_1,s_1})$$

that commutes with e_i and f_i (for all $i \in I$) [9]. To describe the action of the combinatorial R-matrix, we use a modified version of Schensted's Bumping Algorithm.

For inserting $i \in B$ into a tableau x, which we will denote $i \to x$, the bumping algorithm is as follows:

1. For $i \in B_+$, (resp. $i \in B_-$): if none of the boxes in the first column of x are strictly larger than i (resp. larger than or equal to i) then add a box with i in it at the bottom of the column.

2. Otherwise, for the topmost $\boxed{j}$ with $j > i$ (resp. $j \geq i$) in the first column, replace $\boxed{j}$ with $\boxed{i}$. Then, insert j into the second column following analogous steps 1 and 2.

3. Repeat until the bumped number can be put in a new box.

Example 4.

$$\overline{2} \to \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & 1 & 3 \\ \hline \overline{2} & 1 & 2 & 5 \\ \hline \end{array}$$

$$= \begin{array}{c} \overline{2} \\ \downarrow \\ \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & 1 & 3 \\ \hline \mathbf{\overline{2}} & 1 & 2 & 5 \\ \hline \end{array} \end{array} = \begin{array}{c} \overline{2} \\ \downarrow \\ \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & 1 & 3 \\ \hline \overline{2} & \mathbf{1} & 2 & 5 \\ \hline \end{array} \end{array} = \begin{array}{c} 1 \\ \downarrow \\ \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & 1 & 3 \\ \hline \overline{2} & \overline{2} & \mathbf{2} & 5 \\ \hline \end{array} \end{array} = \begin{array}{c} 2 \\ \downarrow \\ \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & 1 & \mathbf{3} \\ \hline \overline{2} & \overline{2} & 1 & 5 \\ \hline \end{array} \end{array} = \begin{array}{c} 3 \\ \downarrow \\ \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & 1 & 2 \\ \hline \overline{2} & \overline{2} & 1 & 5 \\ \hline \end{array} \end{array}$$

$$= \begin{array}{|c|c|c|c|c|}\hline \overline{3} & \overline{3} & 1 & 2 & 3 \\ \hline \overline{2} & \overline{2} & 1 & 5 \\ \cline{1-4} \end{array}$$

Proposition 5 ([9, Theorem 7.9])**.** *The combinatorial R-matrix maps $x \otimes y$ to $\widetilde{y} \otimes \widetilde{x}$ if and only if* $\mathrm{col}(y) \to x = \mathrm{col}(\widetilde{x}) \to \widetilde{y}$

Example 6. Set,

$$x = \begin{array}{|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \quad y = \begin{array}{|c|}\hline \overline{3} \\ \hline 1 \\ \hline 2 \\ \hline \end{array}, \quad \widetilde{y} = \begin{array}{|c|}\hline \overline{3} \\ \hline 1 \\ \hline 3 \\ \hline \end{array}, \quad \widetilde{x} = \begin{array}{|c|c|c|}\hline \overline{4} & \overline{4} & 1 \\ \hline \overline{3} & \overline{3} & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array}.$$

Then, $R(x \otimes y) = \widetilde{y} \otimes \widetilde{x}$. Indeed, let us first compute

$$\mathrm{col}(y) \to x = \overline{3}12 \to \begin{array}{|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array} = 12 \to \begin{array}{|c|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} & 3 \\ \hline \overline{3} & \overline{3} & 1 \\ \cline{1-3} 1 & 2 & 3 \\ \cline{1-3} \end{array} = 2 \to \begin{array}{|c|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} & 3 \\ \hline \overline{3} & \overline{3} & 1 \\ \cline{1-3} 1 & 2 & 3 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} & 3 \\ \hline \overline{3} & \overline{3} & 1 \\ \cline{1-3} 1 & 2 & 3 \\ \cline{1-3} 1 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}.$$

We similarly find that

$$\mathrm{col}(\widetilde{x}) \to \widetilde{y} = \begin{array}{|c|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} & 3 \\ \hline \overline{3} & \overline{3} & 1 \\ \cline{1-3} 1 & 2 & 3 \\ \cline{1-3} 1 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}.$$

Remark 7. The R-matrix can be explicitly computed using the RSK algorithm. For more information of the RSK algorithm, we refer the reader to [3].

Definition 8. We call a function $H \colon B(Y^{r_1,s_1}) \otimes B(Y^{r_2,s_2}) \to \mathbb{Z}$ an *energy function* if, for all $b = x \otimes y \in B(Y^{r_1,s_1}) \otimes B(Y^{r_2,s_2})$, we have $H(f_i b) = H(b)$ and $H(e_i b) = H(b)$ for $i \in I \setminus \{\overline{0}\}$, and

$$H(e_{\overline{0}} b) = H(b) + \begin{cases} 1 & \text{in case LL,} \\ 0 & \text{in case LR or RL,} \\ -1 & \text{in case RR,} \end{cases}$$

where in case LL, $e_{\overline{0}}$ applied to both $x \otimes y$ and $R(x \otimes y)$ acts on the left factor both times; in case LR $e_{\overline{0}}$ applies to the left factor of $x \otimes y$ and the right factor of $R(x \otimes y)$, etc.

The energy function exists and is unique up to additive constant [9]. Moreover, we can compute the energy function using the bumping algorithm:

Proposition 9 ([9, Theorem 7.9])**.** *Up to additive constant, $H(x \otimes y)$ is given by the number of nodes in* $\mathrm{col}(y) \to x$ *that are strictly to the right of the* $\max(s_1, s_2)$*-th column.*

By convention, we will choose the additive constant so that the maximum value of H is zero. Explicitly, if $\widetilde{H}(x \otimes y)$ is given by the number of nodes as in Proposition 9, with additive constant equal to 0, then we define $H(x \otimes y) = \widetilde{H}(x \otimes y) - \min(r_1, r_2)\min(s_1, s_2)$.

Example 10. Set x and y as in Example 6. We know that

$$\mathrm{col}(y) \to x = \begin{array}{|c|c|c|c|} \hline \overline{4} & \overline{4} & \overline{3} & 3 \\ \hline \overline{3} & \overline{3} & 1 \\ \cline{1-3} 1 & 2 & 3 \\ \cline{1-3} 1 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} .$$

We have that $\max(s_1, s_2) = \max(3, 1) = 3$, and the number of nodes to the right of the third column is 1. So, $H(x \otimes y) = 1 - \min(r_1, r_2)\min(s_1, s_2) = -2$.

3 Super Box-Ball System

A BBS possesses a *vacuum element* representing the absence of a ball. We require that the combinatorial R-matrix act as an identity on this element; that is, if u is the vacuum element then $R(u \otimes u) = u \otimes u$. We define the vacuum element to be the genuine highest weight element of $B(Y^{r,1})$, which will have the desired property. More generally, the genuine highest weight element for $B(Y^{r,s})$ has the form

$$u_s = \underbrace{\begin{array}{|c|c|c|} \hline \overline{m} & \cdots & \overline{m} \\ \hline \overline{m-1} & \cdots & \overline{m-1} \\ \hline \vdots & \ddots & \vdots \\ \hline \overline{m-r+1} & \cdots & \overline{m-r+1} \\ \hline \end{array}}_{s} .$$

The vacuum element is then denoted by u_1.

We can think of the elements of $B(Y^{r,1}) \setminus \{u_1\}$ as representing different balls in the system. Within the super BBS we have the notion of a *state*, which consists of $B(Y^{r,1})$ elements in a one dimensional lattice. More precisely, a *state* is of the form

$$b_0 \otimes b_1 \otimes \cdots \otimes b_K \otimes (u_1)^{\otimes\infty} \in (B(Y^{r,1}))^{\otimes\infty},$$

where $b_i \in B(Y^{r,1})$ can be any element (including u_1).

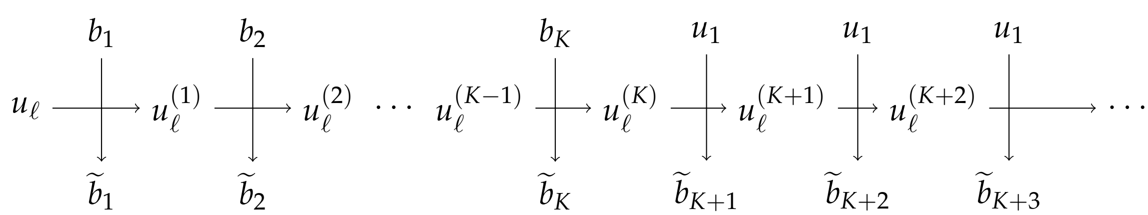
The state evolves with time by the effect of the *carrier* which 'picks up' and 'puts down' particles. The carrier is an element of $B(Y^{r,\ell})$, which changes based on its location in the state, and is initialised as the genuine highest weight element u_ℓ. The action of moving the carrier through the state is performed by the combinatorial R-matrix. In particular, this is performed by functions R_a where

$$R_a = \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{a} \otimes R \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots .$$

We can then define the *time evolution operator*, T_ℓ, by

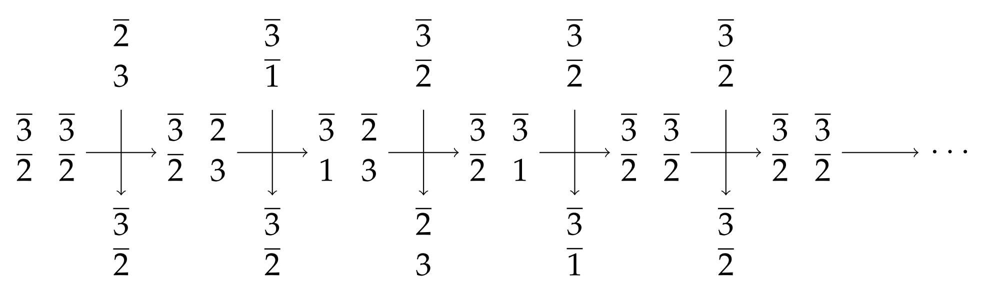
$$T_\ell(b) \otimes u_\ell = \cdots R_3 R_2 R_1 R_0 (u_\ell \otimes b)$$

for any state b. This is well-defined because there are finitely many non-vacuum elements in the state, so we eventually have $R(u_\ell \otimes u_1) = u_1 \otimes u_\ell$. The time evolution operator computes the state for the next time step. Pictorially, we can represent the computation of the time evolution $T_\ell(b_1 \otimes \cdots \otimes b_K \otimes (u_1)^{\otimes\infty}) = \bigotimes_{j=1}^{\infty} \widetilde{b}_j$ as follows:



where $R(u_\ell^{(j)} \otimes b_{j+1}) = \widetilde{b}_{j+1} \otimes u_\ell^{(j+1)}$.

Example 11. For $U_q(\widehat{\mathfrak{gl}}(3|3))$ crystals,



That is,

$$p = \boxed{\begin{matrix}\overline{2}\\ 3\end{matrix}} \otimes \boxed{\begin{matrix}\overline{3}\\ \overline{1}\end{matrix}} \otimes u_1 \otimes u_1 \otimes u_1 \otimes \cdots \quad\Longrightarrow\quad T_2(p) = u_1 \otimes u_1 \otimes \boxed{\begin{matrix}\overline{2}\\ 3\end{matrix}} \otimes \boxed{\begin{matrix}\overline{3}\\ \overline{1}\end{matrix}} \otimes u_1 \otimes \cdots .$$

Proposition 12. *Time evolution operators commute: $T_\ell T_{\ell'}(p) = T_{\ell'} T_\ell(p)$.*

The proof of this fact is identical to Theorem 3.1 of [4], and relies on the Yang–Baxter equation: $(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R)$. This is proved for $U_q(\widehat{\mathfrak{gl}}(m|n))$-crystals in [9, Theorem 7.11].

The time evolution operator also respects the crystal structure, *i.e.*, T_ℓ commutes with the crystal operators, with restrictions as outlined in Lemma 13.

Lemma 13. *For all $i \in I \setminus \{\overline{0}, \overline{m-r}\}$, and for a state p, we have that $T_\ell(e_i(p)) = e_i(T_\ell(p))$ and $T_\ell(f_i(p)) = f_i(T_\ell(p))$.*

The proof is similar to Lemma 2.8 in [12]. This lemma allows us to prove results by only considering the highest weight elements with respect to the $U_q(\widehat{\mathfrak{gl}}(m|n))$-crystal where the operators $f_{\overline{0}}$, $e_{\overline{0}}$, $f_{\overline{m-r}}$ and $e_{\overline{m-r}}$ have been removed. Note that such a crystal is isomorphic to a $U_q(\mathfrak{gl}(r)) \otimes U_q(\mathfrak{gl}(m-r,n))$-crystal.

4 Solitons

4.1 States with a single soliton

We first consider solitonic behaviour for single soliton states. The following theorem provides a large class of states which have one of the properties we desire of solitons. Namely, that speed corresponds to length.

Theorem 14. *Let*

$$x = \begin{array}{|c|}\hline x_{11} \\ \hline x_{21} \\ \hline \vdots \\ \hline x_{r1} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline x_{12} \\ \hline x_{22} \\ \hline \vdots \\ \hline x_{r2} \\ \hline \end{array} \otimes \cdots \otimes \begin{array}{|c|}\hline x_{1s} \\ \hline x_{2s} \\ \hline \vdots \\ \hline x_{rs} \\ \hline \end{array} \in (B(Y^{r,1}))^{\otimes s}.$$

Suppose the factors of the tensor product in reverse order

$$\begin{array}{|c|c|c|c|}\hline x_{1s} & \cdots & x_{12} & x_{11} \\ \hline x_{2s} & \cdots & x_{22} & x_{21} \\ \hline \vdots & \ddots & \vdots & \vdots \\ \hline x_{rs} & \cdots & x_{r2} & x_{r1} \\ \hline \end{array}$$

form a SSYT and that there exists a row number k ($1 \leq k \leq r$) such that

$$x_{ij} < \overline{m-r} \qquad \text{for all } j \text{ and for } i < k,$$
$$x_{ij} \geq \overline{m-r} \qquad \text{for all } j \text{ and for } i \geq k.$$

Then, $(T_\ell)^t(u_1^{\otimes c} \otimes x \otimes u_1^{\otimes \infty}) = u_1^{\otimes(c+t\min\{s,\ell\})} \otimes x \otimes u_1^{\otimes \infty}$ for all positive integers t.

We prove this theorem by direct computation using the RSK insertion algorithm for the R-matrix.

4.2 Scattering of two solitons

Consider a state containing two solitons of different lengths such that the longer soliton is positioned to the left of the shorter soliton. If these two solitons are sufficiently distanced, they behave separately and move as shown in the Subsection 4.1. They continue to propagate in this way until the longer soliton becomes 'too close' to the shorter soliton, where they collide.

In contexts analysed by other authors, such as $\widehat{\mathfrak{sl}}_n$ [6], colliding solitons interact to form two new solitons of the same length but with the longer soliton now on the right. This behaviour is called *scattering*. We have already seen scattering for $\widehat{\mathfrak{gl}}(m|n)$ solitons in Example 1. This subsection is devoted to describing sufficient conditions for when a state will exhibit solitonic behaviour.

Let u be a SSYT, and let $u_\downarrow$ denote the bottom row of u, and $u_\uparrow$ denote the other rows of u. We will only consider the case where $u_\downarrow$ only has entries greater than or equal to $\overline{m-r}$, and $u_\uparrow$ only has entries strictly less than $\overline{m-r}$ (where r is the height of u). In the notation from Theorem 14, we are only considering the case where $k = r$.

Theorem 15. *Let U and V be elements of $(B(Y^{r,1}))^{\otimes d_1}$ and $(B(Y^{r,1}))^{\otimes d_2}$ respectively, with $d_1 > d_2$. Assume U and V satisfy the assumptions of Theorem 14 with $k = r$. Let*

$$p = \cdots \otimes U \otimes \cdots \otimes V \otimes \cdots$$

where the ellipses $(\cdots)$ represent omitted vacuum states. If t is a sufficiently large integer and $\ell > d_2$, then

$$(T_\ell)^t(p) = \cdots \otimes \widetilde{V} \otimes \cdots \otimes \widetilde{U} \otimes \cdots$$

for some $\widetilde{V} \in (B(Y^{r,1}))^{\otimes d_2}$ and $\widetilde{U} \in (B(Y^{r,1}))^{\otimes d_1}$. The elements U and V are related to $\widetilde{U}$ and $\widetilde{V}$ via their SSYT. Let $u, v, \widetilde{v}, \widetilde{u}$ be the SSYT corresponding to $U, V, \widetilde{V}, \widetilde{U}$, respectively. Then,

$$\widetilde{v}_\uparrow \otimes \widetilde{u}_\uparrow = R(u_\uparrow \otimes v_\uparrow) \qquad \text{and} \qquad \widetilde{v}_\downarrow \otimes \widetilde{u}_\downarrow = R(u_\downarrow \otimes v_\downarrow).$$

The phase shift is given by $\delta = 2d_2 + H(u_\downarrow \otimes v_\downarrow) + H(u_\uparrow \otimes v_\uparrow)$.

By Lemma 13, it is sufficient to prove the theorem for highest weight states. By Proposition 12, we have that $(T_\ell)^t = (T_{d_2+1})^{-t'}(T_\ell)^t(T_{d_2+1})^{t'}$. Therefore, if we prove the theorem for T_{d_2+1} and choose t' sufficiently large (so that the solitons have already collided), we can prove the theorem in the general case. With these simplifications, we can then proceed by direct (and tedious) computation. The assumption that $k = r$ is essential:

Example 16. Consider a state composed of elements from the $U_q(\widehat{\mathfrak{gl}}(3|3))$ crystal $B(Y^{2,1})$.

$$\begin{array}{l}
t=0 \quad \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{1}}{2}\ \genfrac{}{}{0pt}{}{\overline{1}}{1}\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{1}}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
t=1 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{1}}{2}\ \genfrac{}{}{0pt}{}{\overline{1}}{1}\ \cdot\ \genfrac{}{}{0pt}{}{\overline{1}}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
t=2 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{1}}{2}\ \genfrac{}{}{0pt}{}{\overline{3}}{\overline{1}}\ \genfrac{}{}{0pt}{}{1}{1}\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
t=3 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{3}}{\overline{1}}\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \genfrac{}{}{0pt}{}{\overline{3}}{2}\ \genfrac{}{}{0pt}{}{1}{1}\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \cdot\ \cdot\ \cdot \\
t=4 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{3}}{\overline{1}}\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \genfrac{}{}{0pt}{}{\overline{3}}{2}\ \genfrac{}{}{0pt}{}{1}{1}\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \cdot
\end{array}$$

We observe that the two objects $\genfrac{}{}{0pt}{}{\overline{1}}{2}\genfrac{}{}{0pt}{}{\overline{1}}{1}$ and $\genfrac{}{}{0pt}{}{\overline{1}}{1}$ satisfy Theorem 14 with $k = 1$. But upon collision they are unstable.

However, the assumptions of Theorem 15 are not necessary, and there exist two-soliton states not satisfying these assumptions.

Example 17. Consider the following time evolution of a BBS composed of elements from the $U_q(\widehat{\mathfrak{gl}}(4|1))$-crystal with $r = 2$:

$$\begin{array}{l}
t=0 \quad \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
t=1 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
t=2 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
t=3 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}$$

We observe that the two objects $\genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}$ and $\genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}$ satisfy Theorem 14 with $k = 1$ and are stable upon collision. However, $\genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}\genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}$ and $\genfrac{}{}{0pt}{}{\overline{2}}{\overline{1}}$ do not satisfy the assumptions of Theorem 15.

Remark 18. States with an arbitrary number of solitons can be reduced to multiple collisions of two solitons. Moreover, it is a consequence of the Yang–Baxter equation that the states after all collisions have occurred are independent of the order of collisions.

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