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# The Monopole-Dimer Model for Cartesian Products of Graphs: Extended Abstract

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**Abstract.** The monopole-dimer model is a signed variant of the monomer-dimer model which has determinantal structure. We extend the monopole-dimer model for planar graphs introduced by the second author (Math. Phys. Anal. Geom., 2015) to Cartesian products thereof and show that the partition function of this model can be expressed as a determinant of a generalised signed adjacency matrix. We then give an explicit product formula for three-dimensional grid graphs a la Kasteleyn and Temperley–Fischer, in which case the partition function turns out to be fourth power of a polynomial when all grid lengths are even. Finally, we generalise this product formula to $k$ dimensions, again obtaining an explicit product formula.

**Saaransh** (सारांश)**.** मोनोपोल–डाइमर मॉडल मोनोमर–डाइमर मॉडल का एक चिन्हित प्रकार है, जिसमें निर्धारक संरचना होती है। हम दूसरे लेखक (Math. Phys. Anal. Geom., 2015) द्वारा पेश किए गए मोनोपोल–डाइमर मॉडल, जोकि तलीय ग्राफ के लिये परिभाषित है, का तलीय ग्राफों के कार्तीय गुणन के लिए विस्तार करते हैं। अतः दर्शाते हैं कि इस मॉडल का विभाजन फलन एक सामान्यीकृत एडजेसेंसी आव्यूह के निर्धारक के रूप में व्यक्त किया जा सकता है। इसके अलावा हम त्रि–आयामी ग्रिड ग्राफ के लिए एक स्पष्ट उत्पाद सूत्र प्रस्तुत करते हैं जैसा कि कास्टेलिन तथा टेम्परली–फिशर ने द्वि–आयामी ग्रिड ग्राफ के लिए किया था। त्रि–आयामी ग्रिड ग्राफ का विभाजन फलन सभी ग्रिड लंबाई सम होने पर एक बहुपद की चौथी घात है। अंत में, हम इस उत्पाद सूत्र को k आयामों के लिए सामान्यीकृत करते हैं, तथा फिर से एक स्पष्ट उत्पाद सूत्र प्राप्त करते हैं।

**Keywords:** dimer model, monopole-dimer model, Cartesian product, plane graph, Pfaffian orientation

## 1 Introduction

The *dimer model* originally arose as the study of the physical process of adsorption of diatomic molecules (like oxygen) on the surface of a solid. Abstractly it can be thought of as enumerating perfect matchings in an edge-weighted graph. For planar graphs, Kasteleyn [8] solved the problem completely by showing that the partition function can be written as a Pfaffian of a certain adjacency matrix built using a special class of orientations called Pfaffian orientations on the graph. An immediate corollary of Kasteleyn's result is that the

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Pfaffian is independent of the orientation. For the case of two-dimensional grid graphs $Q_{m,n}$, Kasteleyn [9] and Temperley–Fisher [5, 10] independently gave an explicit product formula. For example, when $m$ and $n$ are even, horizontal (resp. vertical) edges have weight $a$ (resp. $b$), the partition function can be written as

$$2^{mn/2} \prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left( a^2 \cos^2 \frac{i\pi}{m+1} + b^2 \cos^2 \frac{j\pi}{n+1} \right). \tag{1.1}$$

This formula is remarkable because although each factor is a degree-two polynomial in $a$ and $b$ with not-necessarily rational coefficients, the product turns out to be a polynomial with nonnegative integer coefficients. In particular, when $a = b = 1$, it is not obvious from this formula that the resulting product is an integer.

There have been attempts to generalise the dimer model while preserving this nice structure. The natural physical generalisation is the *monomer-dimer model*, which represents adsorption of a gas cloud consisting of both monoatomic and diatomic molecules. The abstract version here is the enumeration of all matchings of a graph. This is known to be a computationally difficult problem [7] and the partition function here does not have such a clean formula. However, when there is a single monomer on the boundary of a plane graph, the partition function can indeed be written as a Pfaffian [11]. A lower bound for the partition function of the monomer-dimer model for $d$-dimensional grid graphs has been obtained by Hammersley–Menon [6] by generalising the method of Kasteleyn and Temperley–Fisher.

In another direction, a signed version of monomer-dimer model called the *monopole-dimer model* has been introduced by the second author [2] for planar graphs. Configurations of the monopole-dimer model can be thought of as superpositions of two monomer-dimer configurations having monomers (called monopoles there) at the same locations. Thus, one ends up with even loops and isolated vertices. What makes the monopole-dimer model less physical is that configurations have a signed weight. On the other hand, the partition function here can be expressed as a determinant. Moreover, it is a perfect square for a $2m \times 2n$ grid graph. A combinatorial interpretation of the square root is given in [3].

In this work, we generalise the monopole-dimer model to certain non-planar graphs in a canonical way. We formulate this for Cartesian products of planar graphs in Section 3. We first show that the partition function is a determinant of a generalised adjacency matrix built using Pfaffian orientations. As in the dimer model, we see immediately that the determinant is independent of the orientation.

We then focus on the special family of grid graphs in higher dimensions. We give an explicit product formula for the partition function of the monopole-dimer model on three-dimensional grid graphs in Section 4 generalising (1.1). One peculiar feature of this partition function is that it is a fourth power of a polynomial when all side lengths are even. Just as for the partition function of the monopole-dimer model for two-dimensional grids, it would be interesting to obtain a combinatorial interpretation of the fourth root. We then briefly discuss the higher dimensional case in Section 5 and give a similar explicit product formula in the even case.

The proofs will appear in a longer version [1]. We begin with the background definitions and previous results in Section 2.

## 2 Background

We begin by recalling basic terminology from graph theory. A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is the set of *vertices* of $G$ and $E(G)$ is a collection of two-element subsets of $V(G)$, known as *edges*. We will work with undirected graphs and we will always assume that the graphs are finite and naturally labelled from $\{1, 2, \ldots, |V(G)|\}$. We allow multiple edges, but no loops. Recall that a *planar* graph is a graph which can be embedded in the plane, *i.e.* it can be drawn in such a way that no edges will cross each other. Such an embedding of a planar graph is referred as a *plane graph* and it divides the whole plane into regions, each of which is called a *face*. An *orientation* on a graph $G$ is the assignment of arrows to its edges. A graph with an orientation is called an *oriented graph*. An orientation on a labelled graph obtained by orienting its edges from lower to higher labelled vertex is called a *canonical orientation*.

**Definition 1.** An orientation on a plane graph $G$ is said to be *Pfaffian* if it satisfies the property that each simple loop enclosing a bounded face has an odd number of clockwise oriented edges. A Pfaffian orientation is said to possess the *clockwise-odd property*.

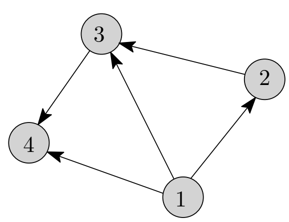



**Figure 1:** An oriented graph on 4 vertices.

For example, the orientation in Figure 1 is a Pfaffian orientation. Kasteleyn has shown that every plane graph has a Pfaffian orientation [8]. A *dimer covering* or *perfect matching* is a collection of edges in the graph $G$ such that each vertex is covered in exactly one edge. The set of all dimer coverings of $G$ will be denoted as $\mathcal{M}(G)$. Let $G$ be an edge-weighted graph on $2n$ vertices with edge-weight $w_e$ for $e \in E(G)$. Then the *dimer model* is the collection of all dimer covers together with the weight of each dimer covering $M \in \mathcal{M}(G)$ given by $w(M) = \prod_{e \in M} w_e$. The *partition function* of the dimer model on $G$ is then defined as

$$Z_G := \sum_{M \in \mathcal{M}(G)} w(M).$$

To state Kasteleyn's celebrated result, recall that the *Pfaffian* of $2n \times 2n$ skew-symmetric matrix $A$ is given by

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma)\, A_{\sigma_1,\sigma_2} A_{\sigma_3,\sigma_4} \dots A_{\sigma_{2n-1},\sigma_{2n}},$$

and Cayley's theorem says that for such a matrix, $\det(A) = \mathrm{Pf}(A)^2$.

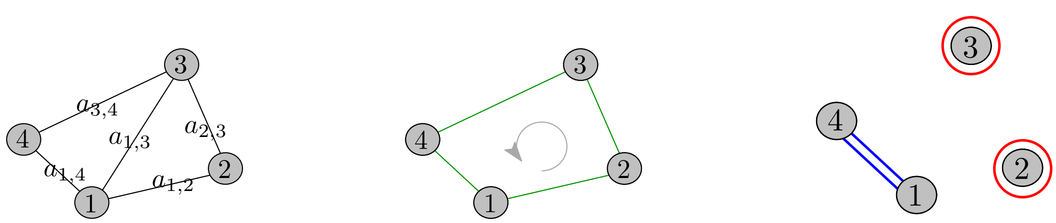
**Theorem 2** (Kasteleyn [8])**.** *If $G$ is a plane graph with Pfaffian orientation $\mathcal{O}$, then the partition function of the dimer model on $G$ is given by $Z_G = \mathrm{Pf}(K_G)$, where $K_G$ is a signed adjacency matrix defined by*

$$(K_G)_{u,v} = \begin{cases} w_e & \text{if } u \to v \text{ in } \mathcal{O}, \\ -w_e & \text{if } v \to u \text{ in } \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us now recall the loop-vertex model [2]. Let $G$ be a simple weighted graph on $n$ vertices with an orientation $\mathcal{O}$, vertex-weights $x(v)$ for $v \in V(G)$ and edge-weights $a_{v,v'} \equiv a_{v',v}$ for $(v,v') \in E(G)$. A *loop-vertex configuration* $C$ of $G$ is a subgraph of $G$ consisting of

- directed loops of even length (with length $> 2$),
- doubled edges (which can be thought of as loops of length 2),
- isolated vertices,

with the condition that each vertex of $G$ is either an isolated vertex or is covered in exactly one loop. The set of all loop vertex configurations of $G$ will be denoted as $\mathcal{L}(G)$. Figure 2 shows a graph and two loop-vertex configurations on it.




(a) A plane graph (b) The directed cycle (1234) (c) The doubled edge (14) and isolated vertices 2 and 3

**Figure 2:** A graph $G$ in (a) and two loop-vertex configurations on it in (b) and (c).

The *sign* of an edge $(v,v') \in E(G)$, is given by

$$\mathrm{sgn}(v,v') := \begin{cases} 1 & \text{if } v \to v' \text{ in } \mathcal{O}, \\ -1 & \text{if } v' \to v \text{ in } \mathcal{O}. \end{cases} \tag{2.1}$$

Let $\ell = (v_0, v_1, \ldots, v_{2k-1}, v_{2k} = v_0)$ be a directed even loop in $G$. The *weight* of the loop $\ell$ is given by

$$w(\ell) := -\prod_{i=0}^{2k-1} \operatorname{sgn}(v_i, v_{i+1})\, a_{v_i,v_{i+1}}. \tag{2.2}$$

Then the *loop-vertex model* on the pair $(G, \mathcal{O})$ is the collection $\mathcal{L}(G)$ with the weight of a configuration, $C = (\ell_1, \ldots, \ell_j; v_1, \ldots, v_k)$ consisting of loops $\ell_1, \ldots, \ell_j$ and isolated vertices $v_1, \ldots, v_k$, given by

$$w(C) = \prod_{i=1}^{j} w(\ell_i) \prod_{i=1}^{k} x(v_i). \tag{2.3}$$

The *(signed) partition function* of the loop-vertex model is defined as

$$\mathcal{Z}_{G,\mathcal{O}} := \sum_{C \in \mathcal{L}(G)} w(C).$$

*Example* 3. Let $G$ be a weighted graph on four vertices with vertex weights $x$ for all the vertices and edge weights as shown in Figure 2a. Then the weights of the configuration shown in Figures 2b and 2c are $a_{1,2}a_{2,3}a_{3,4}a_{1,4}$ and $x^2 a_{1,4}^2$. The partition function of the loop-vertex model on the graph in Figure 2a with canonical orientation is

$$\mathcal{Z}_{G,\mathcal{O}} = x^4 + a_{1,2}^2x^2 + a_{1,3}^2x^2 + a_{1,4}^2x^2 + a_{2,3}^2x^2 + a_{3,4}^2x^2 + a_{1,2}^2a_{3,4}^2 + a_{1,4}^2a_{2,3}^2 + 2a_{1,2}a_{2,3}a_{3,4}a_{1,4}.$$

**Theorem 4** ([2, Theorem 2.5])**.** *The partition function of the loop-vertex model on $(G, \mathcal{O})$ is*

$$\mathcal{Z}_{G,\mathcal{O}} = \det(\mathcal{K}_G),$$

*where $\mathcal{K}_G$ is a generalised adjacency matrix of $(G, \mathcal{O})$ defined as:*

$$\mathcal{K}_G(v, v') = \begin{cases} x(v) & \text{if } v = v', \\ a_{v,v'} & \text{if } v \to v' \text{ in } \mathcal{O}, \\ -a_{v,v'} & \text{if } v' \to v \text{ in } \mathcal{O}, \\ 0 & \text{if } (v, v') \notin E(G). \end{cases} \tag{2.4}$$

*Example* 5. The generalised adjacency matrix for the graph $G$ in Figure 2a with the canonical orientation is

$$\mathcal{K}_G = \begin{pmatrix} x & a_{1,2} & a_{1,3} & a_{1,4} \\ -a_{1,2} & x & a_{2,3} & 0 \\ -a_{1,3} & -a_{2,3} & x & a_{3,4} \\ -a_{1,4} & 0 & -a_{3,4} & x \end{pmatrix},$$

and

$$\det \mathcal{K}_G = x^4 + a_{1,2}^2x^2 + a_{1,3}^2x^2 + a_{1,4}^2x^2 + a_{2,3}^2x^2 + a_{3,4}^2x^2 + a_{1,2}^2a_{3,4}^2 + a_{1,4}^2a_{2,3}^2 + 2a_{1,2}a_{2,3}a_{3,4}a_{1,4},$$

which is that same as $\mathcal{Z}_{G,\mathcal{O}}$ from Example 3.

If $G$ is a simple vertex- and edge-weighted plane graph and $\mathcal{O}$ is a Pfaffian orientation on it, then the loop-vertex model is called the *monopole-dimer model*. In that case, it can be seen [2] that the weight of a loop $\ell = (v_0, v_1, \dots, v_{2k-1}, v_{2k} = v_0)$ can be written as

$$w(\ell) = (-1)^{\text{number of vertices enclosed by } \ell} \prod_{j=0}^{2k-1} a_{v_j, v_{j+1}}. \tag{2.5}$$

Then Theorem 4 shows that the partition function of the monopole-dimer model on a plane graph is given by a determinant which turns out to be independent of the Pfaffian orientation.

## 3 Monopole-dimer model on Cartesian products

We now extend the definition of the monopole-dimer model to Cartesian products of plane graphs. Let us first recall some more definitions.

The *degree* of a vertex is the number of edges incident to it and an *even* graph $G$ is one in which all the vertices have even degree. A *cycle decomposition* of an even graph $G$ is a family $\mathcal{D}$ consisting of edge-disjoint cycles of $G$ such that

$$\bigcup_{c \in \mathcal{D}} E(c) = E(G). \tag{3.1}$$

Veblen's theorem [4, Theorem 2.7] shows that a graph admits a cycle decomposition if and only if it is even.

**Definition 6.** We say that the *sign* of a cycle decomposition $\mathcal{D} = \{c_1, c_2, \dots, c_k\}$ of an even plane graph $G$ is given by

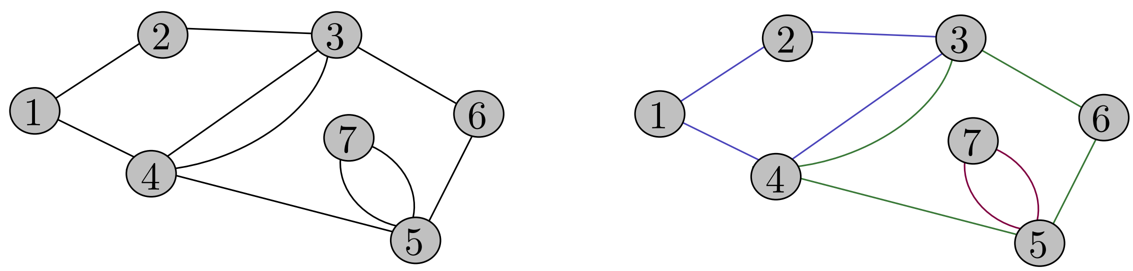
$$\operatorname{sgn}(\mathcal{D}) := \prod_{i=1}^{k} (-1)^{1+\text{number of vertices in } V(G) \text{ enclosed by } c_i}. \tag{3.2}$$

*Example* 7. For the even plane graph shown in Figure 3a, the sign of its cycle decomposition $\{(1,2,3,4), (3,4,5,6), (5,7)\}$ shown in Figure 3b is

$$(-1)^{1+0} \times (-1)^{1+1} \times (-1)^{1+0} = 1.$$

Recall that a *bipartite graph* is a graph $G$ whose vertex set can be partitioned into two subsets $X$ and $Y$ such that each edge of $G$ has one end in $X$ and other end in $Y$. Bipartite graphs only have cycles of even length.

**Lemma 8.** *Let $G$ be a connected, bipartite, even plane graph. Then all cycle decompositions of $G$ have the same sign.*




(a) A graph $H$ (b) $H$ split as $(1234)(3456)(57)$

**Figure 3:** (a) A plane graph on 7 vertices and (b) a cycle decomposition of it.

*Proof sketch.* Let $\mathcal{D}$ be a cycle decomposition of $G$. Since $G$ is even and connected, the boundary of the outer face of $G$ is a closed trail. For simplicity, we suppose the boundary is a single cycle $c$. Then by performing certain sign-preserving moves on $\mathcal{D}$, we obtain another cycle decomposition $\mathcal{D}_1$ containing $c$ which has the same sign as $\mathcal{D}$. Let $G_1$ be obtained from $G$ by removing all the edges of $c$ and the resulting isolated vertices. Note that although $G_1$ can be disconnected, the regions enclosed by its connected components $G_{1,1}, G_{1,2}, \dots, G_{1,t}$ of $G_1$ will not intersect. Now, $\mathcal{D}_1 \setminus \{c\}$ is a cycle decomposition of $G_1$. By again performing similar sign-preserving moves on $\mathcal{D}_1 \setminus \{c\}$, we obtain a cycle decomposition $\mathcal{D}_2$ of $G$ containing $c$ and $d_{1,1}, d_{1,2}, \dots, d_{1,t}$, the boundary cycles of $G_{1,1}, G_{1,2}, \dots, G_{1,t}$ respectively, such that $\operatorname{sgn} \mathcal{D}_1 = \operatorname{sgn} \mathcal{D}_2$. Now remove $d_{1,1}, d_{1,2}, \dots, d_{1,t}$ from $G_1$ to obtain $G_2$ and continue this process. Since $G$ is finite, this process must stop. In fact, it will stop at the cycle decomposition obtained by successively including outer boundaries of $G_1, G_2$ and so on. □

The *Cartesian product of two graphs* $G_1$ and $G_2$ is the graph $G_1 \square G_2$ with vertex set $V(G_1) \times V(G_2)$ and edge set

$$\left\{ ((u_1, u_2), (u_1', u_2')) \;\middle|\; \begin{array}{r} \text{either } u_1 = u_1',\ (u_2, u_2') \in E(G_2) \\ \text{or } u_2 = u_2',\ (u_1, u_1') \in E(G_1) \end{array} \right\}.$$

The above definition generalises to the *Cartesian product of $k$ graphs* denoted as $G_1 \square \cdots \square G_k$. We will call edges in $G_1 \square \cdots \square G_k$ of the form $((u_1, \dots, u_i, \dots, u_k), (u_1, \dots, u_i', \dots, u_k))$ as *$G_i$-edges*. Let $P_n$ denote the *path graph* on $n$ vertices. Figure 4 shows the Cartesian product $P_4 \square P_3$. We will use the notation $[n]$ for the set $\{1, \dots, n\}$.

**Definition 9.** We define the *oriented Cartesian product* of $(G_1, \mathcal{O}_1), \dots, (G_k, \mathcal{O}_k)$, denoted $L = (G_1 \square G_2 \square \cdots \square G_k, \mathcal{O})$, as the graph $G_1 \square G_2 \square \cdots \square G_k$ with orientation $\mathcal{O}$ given as follows. For each $i \in [k]$, if $u_i \to u_i'$ in $\mathcal{O}_i$, then $\mathcal{O}$ gives orientation $(u_1, \dots, u_i, \dots, u_k) \to (u_1, \dots, u_i', \dots, u_k)$ if $u_{i+1} + u_{i+2} + \cdots + u_k + (k - i) \equiv 0 \pmod 2$ and $(u_1, \dots, u_i', \dots, u_k) \to (u_1, \dots, u_i, \dots, u_k)$ otherwise.

**Definition 10.** The *$i$-projection* of a subgraph $S$ of $G_1 \square G_2 \square \cdots \square G_k$, denoted as $\widetilde{S}_i$, is the graph obtained by contracting all but $G_i$-edges of $S$.

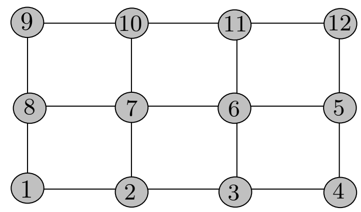



**Figure 4:** The Cartesian product, $P_4 \square P_3$, with a boustrophedon labelling; see Section 4.

Consider $k$ plane, bipartite, simple oriented graphs $G_1, G_2, \dots, G_k$ with Pfaffian orientations $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k$ respectively, and let $L$ be their oriented Cartesian product. Let $\ell = (w_0, w_1, \dots, w_{2k-1}, w_{2k} = w_0)$ be a directed even loop in $L$, and $\mathcal{L}_i$ be a cycle decomposition of the $i$-projection $\widetilde{\ell}_i$. For $i \in [k]$, let $G^{(i)}$ be the graph $G_1 \square \cdots \square G_{i-1} \square G_{i+1} \square \cdots \square G_k$. For $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k) \in V(G^{(i)})$, let $G^{(i)}_{v_1,\dots,v_{i-1},v_{i+1},\dots,v_k}$ be the induced subgraph of $L$ given by $\{(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k) \in V(L) \mid v \in V(G_i)\}$ and let $e_i$ be the number of edges lying both in $\ell$ and $G^{(i)}_{v_1,\dots,v_{i-1},v_{i+1},\dots,v_k}$ such that $v_{i+1} + \cdots + v_k + (k-i) \equiv 1 \pmod 2$. Then the *sign* of $\ell$ is defined by

$$\operatorname{sgn}(\ell) := -\prod_{i=1}^{k-1} (-1)^{e_i} \prod_{i=1}^{k} \operatorname{sgn}(\mathcal{L}_i). \tag{3.3}$$

Note that the sign of $\ell$ is well-defined by Lemma 8. Now suppose that $L$ has been given vertex weights $x(w)$ for $w \in V(L)$ and edge weights $a_e$ for $e \in E(L)$. Then the *weight of the loop* $\ell$ is defined as

$$w(\ell) := \operatorname{sgn}(\ell) \prod_{e \in E(\ell)} a_e. \tag{3.4}$$

**Definition 11.** The *(extended) monopole-dimer model* on the weighted oriented Cartesian product $L$ is the collection $\mathcal{L}$ of monopole-dimer configurations on $L$ where the weight of each configuration $C = (\ell_1, \dots, \ell_j; v_1, \dots, v_k)$ is

$$w(C) = \prod_{i=1}^{j} w(\ell_i) \prod_{i=1}^{k} x(v_i).$$

The *(signed) partition function* of the monopole-dimer model on the oriented Cartesian product $L$ is

$$\mathcal{Z}_L := \sum_{C \in \mathcal{L}} w(C).$$

Note that the definition of $\mathcal{Z}_L$ is independent of the Pfaffian orientations $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k$. The following theorem is a generalisation of Theorem 4 when $G$ is plane and $\mathcal{O}$ is Pfaffian.

**Theorem 12.** *Let* $G_1, G_2, \ldots, G_k$ *be* $k$ *simple, plane, and bipartite graphs with Pfaffian orientations* $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$ *respectively. The (signed) partition function of the monopole-dimer model for the weighted oriented Cartesian product* $L$ *of* $G_1, G_2, \ldots, G_k$ *is given by*

$$\mathcal{Z}_L = \det \mathcal{K}_G, \tag{3.5}$$

*where* $\mathcal{K}_G$ *is the generalised adjacency matrix defined in* (2.4) *for* $L$*.*

The proof strategy is similar to that of Theorem 4.

# 4 Three-dimensional grids

Recall that $P_n$ is the path graph on $n$ vertices. With the natural labelling on $P_n$, denote it with the canonical orientation as $(P_n, \mathcal{O}_n)$. Consider the two-dimensional grid graph $P_l \square P_m = \{(i,j) \mid i \in [l], j \in [m]\}$ whose vertex $(i,j)$ has label $2sl + i$ if $j = 2s+1$ and $2sl - i + 1$ if $j = 2s$. Such a 'snake-like' labelling is known as a *boustrophedon labelling* [6]. With the canonical orientation, denote this graph as $(P_l \square P_m, \mathcal{O}_{l,m})$. Figure 4 shows this labelling on $P_4 \square P_3$.

**Theorem 13.** *Let* $G$ *be the oriented Cartesian product of* $(P_l \square P_m, \mathcal{O}_{l,m})$ *with* $(P_n, \mathcal{O}_n)$*. Let vertex weights be* $x$ *for all vertices of* $G$*, edge weights be* $a$*,* $b$*,* $c$ *for the edges along the* $x$*-,* $y$*- and* $z$*-directions respectively. Then the partition function of the monopole-dimer model on* $G$ *is given by*

$$\mathcal{Z}_G = \prod_{j=1}^{\lfloor n/2 \rfloor} \prod_{s=1}^{\lfloor m/2 \rfloor} \prod_{k=1}^{\lfloor l/2 \rfloor} \left( x^2 + 4a^2 \cos^2 \frac{\pi k}{l+1} + 4b^2 \cos^2 \frac{\pi s}{m+1} + 4c^2 \cos^2 \frac{\pi j}{n+1} \right)^4$$
$$\times \begin{cases} 1 & l,n,m \in 2\mathbb{N}, \\ T^2_{n,m}(b,c;x) & l \notin 2\mathbb{N}, m,n \in 2\mathbb{N}, \\ T^2_{n,l}(a,c;x) & l,n \in 2\mathbb{N}, m \notin 2\mathbb{N}, \\ T^2_{n,m}(b,c;x)\, T^2_{n,l}(a,c;x)\, S_n(c;x) & l,m \notin 2\mathbb{N}, n \in 2\mathbb{N}, \\ T^2_{m,l}(a,b;x) & l,m \in 2\mathbb{N}, n \notin 2\mathbb{N}, \\ T^2_{n,m}(b,c;x)\, T^2_{m,l}(a,b;x)\, S_m(b;x) & l,n \notin 2\mathbb{N}, m \in 2\mathbb{N}, \\ T^2_{n,l}(a,c;x)\, T^2_{m,l}(a,b;x)\, S_l(a;x) & l \in 2\mathbb{N}, m,n \notin 2\mathbb{N}, \\ x\ T^2_{n,m}(b,c;x)\, T^2_{n,l}(a,c;x)\, T^2_{m,l}(a,b;x)\, S_n(c;x)\, S_m(b;x)\, S_l(a;x) & l,m,n \notin 2\mathbb{N}, \end{cases}$$

*where*

$$S_n(c;x) = \prod_{k=1}^{\lfloor n/2 \rfloor} \left( x^2 + 4c^2 \cos^2 \frac{\pi k}{n+1} \right),$$

*and*

$$T_{n,l}(a,b;x) = \prod_{j=1}^{\lfloor n/2 \rfloor} \prod_{k=1}^{\lfloor l/2 \rfloor} \left( x^2 + 4a^2 \cos^2 \frac{\pi k}{l+1} + 4b^2 \cos^2 \frac{\pi j}{(n+1)} \right).$$

The proof of Theorem 13 follows a similar strategy to that of [9, 5] for the two-dimensional dimer model. Here, one has to diagonalise a triple tensor product and various special cases have to be considered carefully.

We make a few remarks about this result. First, the orientation on $G$ is Pfaffian over all standard planes and $G$ is non-planar when at least two of $l, m, n$ are greater than 2. Second, although it is not obvious from Theorem 13, $\mathcal{Z}_G$ is always a polynomial in $x, a, b, c$ with nonnegative integer coefficients. Moreover, $\mathcal{Z}_G$ is the fourth power of a polynomial when $l, m$ and $n$ are all even and the square of a polynomial when exactly two of $l, m$ and $n$ are even. Third, the formula in Theorem 13 coincides with the already known partition function [2] of the two-dimensional grid graph for $l = 1$ or $m = 1$ or $n = 1$. Finally, although it is not obvious from the construction, the formula is symmetric in all three directions. That is to say, it is symmetric under any permutation interchanging $(a, l), (b, m)$ and $(c, n)$.

The boustrophedon labelling that induces the orientation over the three-dimensional grid graph $G$ of Theorem 13 is as follows. The vertex $(i, j, k)$ has label

$$\begin{cases} 2tlm + 2s + i & \text{if } j = 2s+1, k = 2t+1, \\ 2tlm + 2sl - i + 1 & \text{if } j = 2s, k = 2t+1, \\ 2tlm - 2s - i + 1 & \text{if } j = 2s+1, k = 2t, \\ 2tlm - 2sl + i & \text{if } j = 2s, k = 2t, \end{cases}$$

where $i \in [l], j \in [m]$ and $k \in [n]$. Figure 5 shows this labelling on the graph $P_3 \square P_2 \square P_3$.

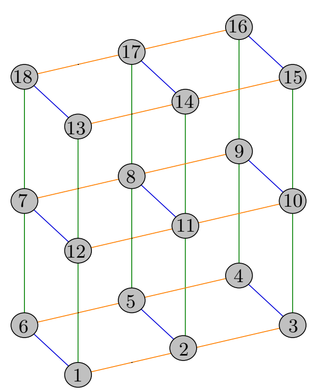



**Figure 5:** The boustrophedon labelling on $P_3 \square P_2 \square P_3$.

**Proposition 14.** *The partition function of the monopole-dimer model on the oriented Cartesian product of* $(P_l \square P_m, \mathcal{O}_{l,m})$ *with* $(P_n, \mathcal{O}_n)$ *is same as the partition function of the monopole-dimer model on the oriented Cartesian product of* $(P_l, \mathcal{O}_l)$ *with* $(P_m \square P_n, \mathcal{O}_{m,n})$.

*Proof sketch.* The orientation in both the cases is that induced from the same boustrophedon labelling. Moreover, the partition function of the monopole-dimer model is same as the partition function of the loop-vertex model on $P_l \square P_m \square P_n$ with the canonical orientation induced from boustrophedon labelling. $\square$

# 5 Higher-dimensional grid graphs

We now generalise our results to higher dimensional grid graphs. For the sake of brevity, we focus only on the case where all the side lengths are even. The general result will appear in [1].

**Theorem 15.** *Let $G$ be the oriented Cartesian product of $(P_{2m_1} \square P_{2m_2}, \mathcal{O}_{2m_1,2m_2}), (P_{2m_3}, \mathcal{O}_{2m_3}), \ldots, (P_{2m_k}, \mathcal{O}_{2m_k})$. Let vertex weights be $x$ for all vertices of $G$ and edge weights be $a_i$ for the $P_{2m_i}$-edges. Then the partition function of the monopole-dimer model on $G$ is given by*

$$\mathcal{Z}_G = \prod_{i_1=1}^{m_1} \cdots \prod_{i_k=1}^{m_k} \left( x^2 + \sum_{s=1}^{k} 4a_s^2 \cos^2 \frac{i_s \pi}{2m_s + 1} \right)^{2^{k-1}}. \tag{5.1}$$

The proof strategy is similar to that of [6, Section 4]. Using ideas similar to the proof of Proposition 14, it can be shown that for $s \in [k-1]$, the formula above coincides with the partition function of the monopole-dimer model on the oriented Cartesian product of $P_{2m_1}, P_{2m_2}, \ldots, P_{2m_{s-1}}, (P_{2m_s} \square P_{2m_{s+1}}), P_{2m_{s+2}}, \ldots, P_{2m_k}$.

As for the three-dimensional case, it is not obvious from the formula for $\mathcal{Z}_G$ that it is a polynomial with nonnegative integer coefficients. The formula is also symmetric under any permutation of $(a_1, m_1), \ldots, (a_k, m_k)$. Finally, (5.1) tells that the partition function of monopole-dimer model is the $2^{(k-1)}$'th power of a polynomial. Again a combinatorial interpretation of the underlying polynomial would be interesting.

We end with an example for a well-studied family of graphs.

*Example* 16. Consider the $d$-dimensional oriented hypercube, $Q_d$, built as an oriented Cartesian product of $d$ copies of $(P_2, \mathcal{O}_2)$ as in Theorem 15. Then the partition function of the monopole-dimer model on $Q_d$ is given by

$$\mathcal{Z}_{Q_d} = (x^2 + a_1^2 + \cdots + a_d^2)^{2^{d-1}}.$$

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