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Acyclic Reorientation Lattices and Their Lattice Quotients

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Abstract. We consider acyclic reorientation posets of directed acyclic graphs. We characterize those which are lattices and provide formulas to compute meets and joins in these lattices. We next characterize those which are distributive, semidistributive, congruence normal, or congruence uniform lattices. In the latter case, we introduce a combinatorial gadget to encode the join irreducibles acyclic reorientations and exploit it to describe the canonical representations, the congruence lattices, and the polytopal realizations of the quotients of these acyclic reorientation lattices.

Résumé. Nous considérons les ordres partiels sur les réorientations acycliques des graphes orientés acycliques. Nous caractérisons ceux qui sont des treillis et donnons des formules pour calculer les bornes inférieures et supérieures dans ces treillis. Nous caractérisons ensuite ceux qui sont des treillis distributifs, semi-distributifs, congruence normaux, ou congruence uniformes. Dans le dernier cas, nous introduisons des gadgets combinatoires qui encodent les réorientations acycliques sup-irréductibles et les exploitons pour décrire les représentations canoniques, les treillis de congruences, et les réalisations polytopales des quotients de ces treillis de réorientations acycliques.

Keywords: directed graphs, acyclic orientations, graphical zonotopes, lattices

Fix a (finite and simple) directed acyclic graph D. Consider the poset $\mathcal{AR}_D$ of all acyclic reorientations of D, ordered by inclusion of their sets of reversed arcs (with respect to D). Its minimal element is D, its maximal element is the reverse of D, its cover relations are given by flipping a single arc, and it is clearly self-dual under reversing all arcs. For instance, $\mathcal{AR}_D$ is (isomorphic to) the boolean lattice when D is a forest, and the weak order on permutations when D is a tournament. See Figure 1.

These acyclic reorientations posets and the underlying flip graphs have been extensively studied, in particular for counting [18], traversing [15], and generating [1, 17] all acyclic orientations of a graph. Here, we consider acyclic reorientation posets from a lattice theoretic perspective: after characterizing the directed acyclic graphs D for which $\mathcal{AR}_D$ is a lattice, we explore lattice properties of $\mathcal{AR}_D$, in particular the combinatorics and geometry of the lattice quotients of $\mathcal{AR}_D$ when it is semidistributive. The prototypical example is the Tamari lattice [19] seen as quotient of the weak order, its connection with non-crossing partitions, and its realization by the associahedron of [6, 16].

Many details and all proofs omitted in this extended abstract can be found in [8].

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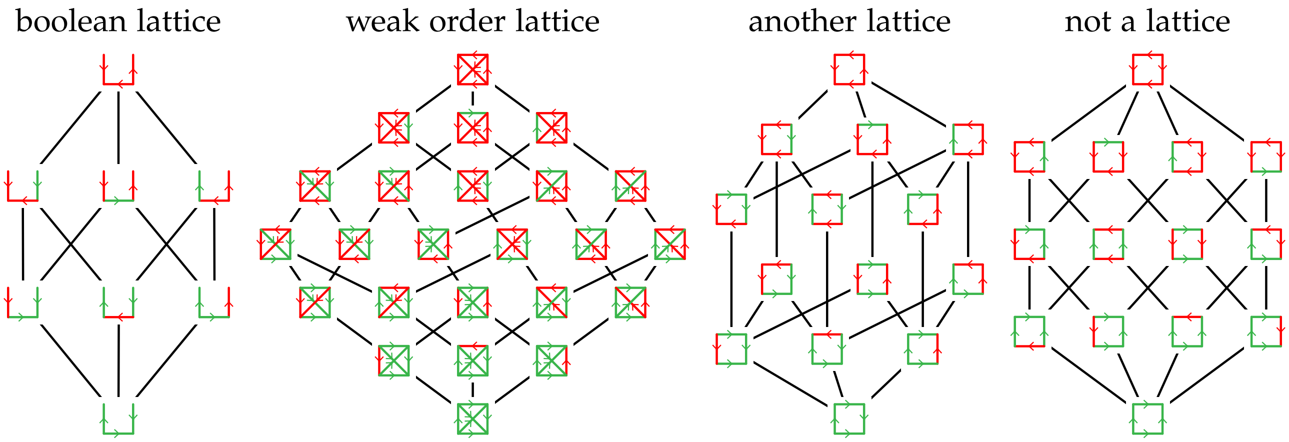


Figure 1: Some acyclic reorientation posets. The first three are lattices while the fourth is not. The first is boolean as D is a forest, the second is the weak order as D is a tournament. The red arcs are reversed while the green arcs are not.

1 Acyclic reorientation lattices

Recall that the *transitive reduction* (resp. *transitive closure*) of D is the directed graph obtained by deleting from (resp. adding to) D all arcs whose endpoints are connected by a directed path in D of length at least 2. These operations clearly play an important role for acyclic reorientations: for instance, note that an arc in an acyclic reorientation E of D is flippable if and only it belongs to the transitive reduction of E.

In this paper, we say that D is *vertebrate* when the transitive reduction of any induced subgraph of D is a forest. For instance, any forest and any tournament is vertebrate. Our starting observation is the following result illustrated in Figure 1.

Theorem 1. *The acyclic reorientation poset $\mathcal{AR}_D$ is a lattice if and only if D is vertebrate.*

There are at least two possible proofs of Theorem 1. The first is to observe that the acyclic reorientation poset of a vertebrate directed acyclic graph can be obtained from the acyclic reorientation lattice of its transitive reduction by a sequence of convex doublings [2]. The second is to characterize the sets of arcs of D whose reorientation is acyclic, and to use it to describe the join and meet operations in the acyclic reorientation lattice of a vertebrate directed acyclic graph. We now sketch the second approach.

It is classical that a subset B of $\binom{[n]}{2}$ is the inversion set of a permutation of $[n]$ if and only if both B and $\binom{[n]}{2} \smallsetminus B$ are transitive. We say that a subset B of arcs of D is (i) *closed* if all arcs of D in the transitive closure of B belong to B, (ii) *coclosed* if its complement is closed, and (iii) *biclosed* if it is both closed and coclosed.

Proposition 2. *If D is vertebrate, a set of edges is biclosed if and only if its reorientation is acyclic.*

With Proposition 2 at hand, we can refine Theorem 1 as follows. For the weak order on permutations, it is well-known that, for any permutations $\pi_1, \dots, \pi_k$ of $[n]$, the inversion set of $\pi_1 \vee \dots \vee \pi_k$ (resp. of $\pi_1 \wedge \dots \wedge \pi_k$) is the transitive closure (resp. the complement of the transitive closure) of the inversion sets (resp. of the complements of the inversion sets) of $\pi_1, \dots, \pi_k$. This generalizes for vertebrate directed acyclic graphs.

Theorem 3. *If D is vertebrate, then the join (resp. meet) of some acyclic reorientations $E_1, \dots, E_k$ of D is obtained by reversing all arcs of D that belong (resp. do not belong) to the transitive closure of the arcs reversed (resp. not reversed) in at least one of the reorientations $E_1, \dots, E_k$.*

2 Lattice properties

In this section, we assume that D is vertebrate and we study classical lattice properties of the acyclic reorientation lattice $\mathcal{AR}_D$, illustrated in Figure 2. We refer to [3] for a detailed reference on these lattice properties and just briefly recall the needed definitions.

Before starting, recall first that an element x is *join* (resp. *meet*) *irreducible* if it covers (resp. is covered by) a unique element denoted $x_\star$ (resp. $x^\star$). For instance, the join (resp. meet) irreducibles of the boolean lattice are the singletons (resp. complements of singletons), and the join (resp. meet) irreducibles in the weak order on permutations are the permutations with a single descent (resp. ascent). More generally, it is immediate that an acyclic reorientation E of D is join (resp. meet) irreducible in $\mathcal{AR}_D$ if and only if the transitive reduction of E contains a single reversed (resp. not reversed) arc.

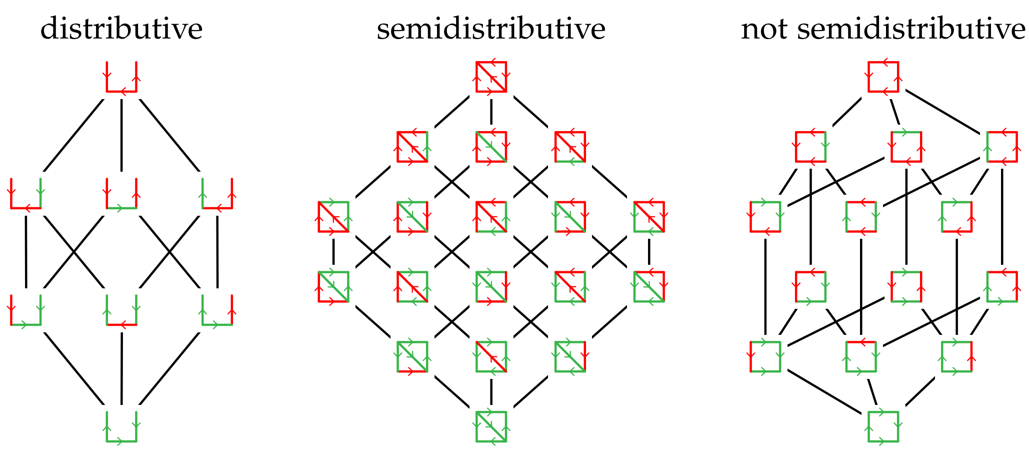


Figure 2: Some acyclic reorientation lattices. The first is distributive, the second is not distributive but semidistributive, the third is not semidistributive. They are all congruence normal, hence the first two are also congruence uniform.

2.1 Distributivity and semidistributivity

A finite lattice $(L, \leq, \wedge, \vee)$ is *distributive* if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (or equivalently $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$) for any $x, y, z \in L$. The following statement says that an acyclic reorientation lattice is distributive if and only if it is a boolean lattice.

Proposition 4. *The poset $\mathcal{AR}_D$ is a distributive lattice if and only if D is a forest.*

A finite lattice $(L, \leq, \wedge, \vee)$ is *join semidistributive* if the equality $x \vee y = x \vee z$ implies the distributivity law $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$. Equivalently, L is join semidistributive if for any cover relation $x \lessdot y$ in L, the set $K_\vee(x, y) := \{z \in L \mid x \vee z = y\}$ has a unique minimal element $k_\vee(x, y)$. Note that $k_\vee(x, y)$ is join irreducible. The *meet semidistributivity* and the maps $K_\wedge$ and $k_\wedge$ are defined dually. A lattice L is *semidistributive* if it is both join and meet semidistributive.

Our next statement characterizes semidistributivity for acyclic reorientation lattices. We say that D is *filled* when for any directed path π in D, if the arc joining the endpoints of π belongs to D, then all arcs joining any two vertices of π also belong to D. From now on, we abbreviate vertebrate and filled by *skeletal*. For instance, any forest and any tournament is skeletal. In fact, it is not difficult to check that the skeletal directed acyclic graphs are precisely the directed forests on which some directed paths are replaced by tournaments. For example, the first two graphs of Figure 2 are skeletal.

Proposition 5. *The poset $\mathcal{AR}_D$ is a semidistributive lattice if and only if D is skeletal.*

Semidistributivity enables to consider canonical representations. A *join representation* of $x \in L$ is a subset $J \subseteq L$ such that $x = \bigvee J$. Such a representation is *irredundant* if $x \neq \bigvee J'$ for any strict subset $J' \subsetneq J$. The irredundant join representations of an element $x \in L$ are ordered by containement of the lower ideals of their elements, *i.e.* $J \leq J'$ if and only if for any $y \in J$ there exists $y' \in J'$ such that $y \leq y'$ in L. The *canonical join representation* of x is the minimal irredundant join representation of x for this order when it exists. Its elements are called *canonical joinands* of x. The *canonical meet representations* and the *canonical meetands* are defined dually.

A classical result affirms that a finite lattice L is join (resp. meet) semidistributive if and only if any element of L admits a canonical join (resp. meet) representation. Moreover, in a join (resp. meet) semidistributive lattice, the canonical join (resp. meet) representation of $y \in L$ is $y = \bigvee_{x \lessdot y} k_\vee(x, y)$ (resp. $y = \bigwedge_{y \lessdot z} k_\wedge(y, z)$). Combining this description with Proposition 5, we obtain the following description generalizing [14].

Proposition 6. *Assume that D is skeletal. The canonical join (resp. meet) representation of an acyclic reorientation E of D is given by $E = \bigvee_a E_a$ (resp. $E = \bigwedge_a E_a$) where:*

- *a runs over all arcs of D reversed (resp. not reversed) in the transitive reduction of E,*
- *E_a is the acyclic reorientation of D where an arc is reversed (resp. not reversed) if and only if it is the only arc reversed (resp. not reversed) in E along a directed path in D joining the endpoints of a.*

2.2 Congruence normality and uniformity

A *congruence* of a finite lattice $(L, \leq, \wedge, \vee)$ is an equivalence relation on L such that $x \equiv x'$ and $y \equiv y'$ implies $x \vee y \equiv x' \vee y'$ and $x \wedge y \equiv x' \wedge y'$. The *lattice quotient* $L/{\equiv}$ is the lattice on the classes of $\equiv$, where for any two classes X and Y, the order is given by $X \leq Y$ if and only if $x \leq y$ for some $x \in X$ and $y \in Y$, and the join $X \vee Y$ (resp. meet $X \wedge Y$) is the class of $x \vee y$ (resp. $x \wedge y$) for any $x \in X$ and $y \in Y$. The lattice quotient $L/{\equiv}$ is isomorphic to the subposet of L induced by the minimal elements in their classes.

The set $\mathrm{con}(L)$ of all congruences of L, ordered by refinement, forms itself a distributive lattice where the meet is the intersection of relations and the join is the transitive closure of union of relations. For any $x, y \in L$, there is a unique minimal congruence $\mathrm{con}(x, y)$ in which $x \equiv y$. For a join irreducible element j of L, the congruence $\mathrm{con}(j_\star, j)$ is join irreducible in the congruence lattice $\mathrm{con}(L)$. The lattice L is called:

- *congruence normal* if $\mathrm{con}(j_\star, j) \neq \mathrm{con}(m, m^\star)$ for any join irreducible j and meet irreducible m such that $j \leq m$,
- *congruence uniform* if the map $j \mapsto \mathrm{con}(j_\star, j)$ (resp. $m \mapsto \mathrm{con}(m, m^\star)$) is a bijection between the join (resp. meet) irreducible elements of L and that of $\mathrm{con}(L)$.

A lattice is congruence uniform if and only if it is congruence normal and semidistributive. Using a standard characterization of congruence normal (resp. uniform) lattices in terms of convex (resp. interval) doublings [2], we obtain the following statements.

Proposition 7. *The poset $\mathcal{AR}_D$ is a congruence normal lattice for any vertebrate acyclic D.*

Proposition 8. *The poset $\mathcal{AR}_D$ is a congruence uniform lattice if and only if D is skeletal.*

3 Ropes

Assume that D is skeletal, so that $\mathcal{AR}_D$ is a congruence uniform lattice. We introduce ropes, non-crossing rope diagrams, and the subrope order, generalizing [14]. They provide models for the irreducibles, canonical representations, and congruences of $\mathcal{AR}_D$.

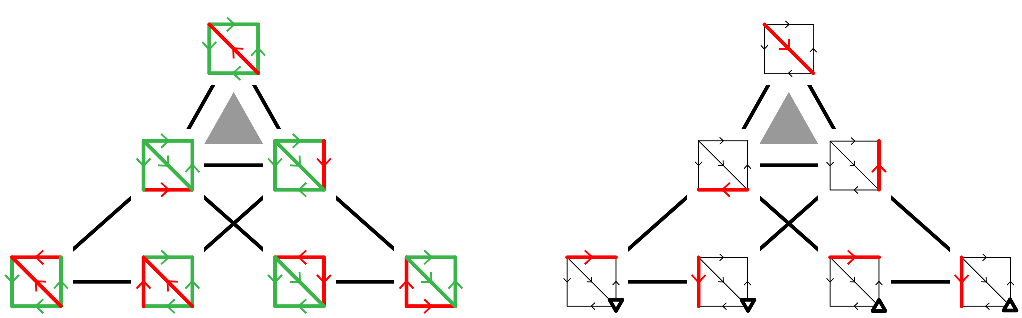


Figure 3: Correspondence between join irreducibles of $\mathcal{AR}_D$ and ropes of D.

3.1 Ropes and irreducibles

A *rope* of D is a quadruple $(u, v, \triangledown, \triangle)$ where (u, v) is an arc of D and $\triangledown \sqcup \triangle$ is a partition of the vertices that appear in the interior of a path from u to v in D (or equivalently since D is filled, the vertices w so that both (u, w) and (w, v) belong to D).

For an acyclic reorientation E of D and an arc $(u, v) \in D$, set $\rho^E_{u,v} := (u, v, \triangledown^E_{u,v}, \triangle^E_{u,v})$ where $\triangledown^E_{u,v}$ and $\triangle^E_{u,v}$ are defined by $\triangledown^E_{u,v} := \{w \in V \mid (u, w) \in D \smallsetminus E \text{ and } (w, v) \in D \cap E\}$ and $\triangle^E_{u,v} := \{w \in V \mid (u, w) \in D \cap E \text{ and } (w, v) \in D \smallsetminus E\}$. This map enables us to connect the ropes of D with the join and meet irreducibles of $\mathcal{AR}_D$:

- for a join (resp. meet) irreducible I of $\mathcal{AR}_D$, let $\rho_\vee(I)$ (resp. $\rho_\wedge(I)$) be the rope $\rho^I_{u,v}$ where (u, v) is the only arc reversed (resp. not reversed) in the transitive restriction of I,
- for a rope $\rho := (u, v, \triangledown, \triangle)$ on D, let $I_\vee(\rho)$ (resp. $I_\wedge(\rho)$) be the reorientation of D where an arc (w, w') of D is reversed (resp. not reversed) if and only if $w \in \triangle \cup \{u\}$ and $w' \in \triangledown \cup \{v\}$ (resp. $w \in \triangledown \cup \{u\}$ and $w' \in \triangle \cup \{v\}$).

These maps are illustrated in Figure 3. To represent a rope $(u, v, \triangledown, \triangle)$, we highlight in red the arc (u, v) and we mark with down and up triangles the vertices of $\triangledown$ and $\triangle$.

Proposition 9. *If D is skeletal, the two maps $\rho_\vee$ and $I_\vee$ (resp. $\rho_\wedge$ and $I_\wedge$) are inverse bijections between the join (resp. meet) irreducibles of $\mathcal{AR}_D$ and the ropes of D.*

3.2 Non-crossing rope diagrams and canonical representations

Two ropes $(u, v, \triangledown, \triangle)$ and $(u', v', \triangledown', \triangle')$ are *crossing* if there are distinct vertices $w \neq w'$ such that $w \in (\triangledown \cup \{u, v\}) \cap (\triangle' \cup \{u', v'\})$ and $w' \in (\triangle \cup \{u, v\}) \cap (\triangledown' \cup \{u', v'\})$. A *non-crossing rope diagram* is a collection of pairwise non-crossing ropes of D.

We now connect the non-crossing rope diagrams of D with the elements of $\mathcal{AR}_D$:

- for an acyclic reorientation E of D, let $\delta_\vee(E)$ (resp. $\delta_\wedge(E)$) be the set of ropes $\rho^E_{u,v}$ for all arcs (u, v) reversed (resp. not reversed) in the transitive reduction of E,
- for a non-crossing rope diagram δ of D, define the reorientation $E_\vee(\delta) := \bigvee_{\rho \in \delta} I_\vee(\rho)$ (resp. $E_\wedge(\delta) := \bigwedge_{\rho \in \delta} I_\wedge(\rho)$) of D.

Proposition 10. *If D is skeletal, the two maps $\delta_\vee$ and $E_\vee$ (resp. $\delta_\wedge$ and $E_\wedge$) are inverse bijections between acyclic reorientations of D and non-crossing rope diagrams of D.*

Proposition 10 enables us to rewrite Proposition 6 in terms of ropes, hence to connect the canonical join complex of $\mathcal{AR}_D$ (*i.e.* the simplicial complex of canonical join representations of $\mathcal{AR}_D$ [14]) to the noncrossing rope complex of D (*i.e.* the simplicial complex of non-crossing rope diagrams of D). These complexes are illustrated in Figure 3.

Corollary 11. *Assume that D is skeletal. The canonical join (resp. meet) representation of any acyclic reorientation E of D is $E = \bigvee_{\rho \in \delta_\vee(E)} I_\vee(\rho)$ (resp. $E = \bigwedge_{\rho \in \delta_\wedge(E)} I_\wedge(\rho)$). Hence, the canonical join (resp. meet) complex of $\mathcal{AR}_D$ is isomorphic to the non-crossing rope complex of D.*

3.3 Subrope order and congruences

Recall from Section 2.2 that the set $\mathrm{con}(L)$ of congruences of a lattice L, ordered by refinement, is a distributive lattice. When L is congruence uniform, $\mathrm{con}(L)$ is isomorphic to the set of lower ideals of the *forcing order* on join irreducibles of L, defined by $j \prec j'$ if $\mathrm{con}(j'_\star, j')$ refines $\mathrm{con}(j_\star, j)$. Moreover, for a congruence $\equiv$ of L corresponding to a lower ideal $\mathbb{I}$ of the forcing order,

- an element of L is minimal in its $\equiv$-class if and only if all the join irreducibles in its canonical join representation belong to $\mathbb{I}$,
- the canonical joinands of a congruence class X in $L/\!\equiv$ are the classes of the canonical joinands of the minimal element in X.

Dual statements hold using meets instead of joins. In view of these statements, understanding the congruences and quotients of a congruence uniform lattice amounts to understanding the forcing order on the join irreducibles of L and its lower ideals.

For acyclic reorientation lattices, the forcing order is not difficult to describe in terms of ropes. A rope $\rho := (u, v, \triangledown, \triangle)$ is a *subrope* of a rope $\rho' := (u', v', \triangledown', \triangle')$ if and only if $\{u, v\} \subseteq \{u', v'\} \cup \triangledown' \cup \triangle'$ and $\triangledown \subseteq \triangledown'$ while $\triangle \subseteq \triangle'$. The *subrope order* is the order on ropes of D defined by $\rho \prec \rho'$ if ρ is a subrope of ρ'.

Proposition 12. *Assume that D is skeletal. For any two join irreducibles J and J' of the acyclic reorientation lattice $\mathcal{AR}_D$, J forces J' if and only if $\rho_\vee(J)$ is a subrope of $\rho_\vee(J')$.*

Corollary 13. *If D is skeletal, the congruence lattice of $\mathcal{AR}_D$ is isomorphic to the lattice of lower ideals of the subrope order for D.*

Throughout the end of this paper, we denote by $\mathbb{I}_\equiv$ the lower ideal of the subrope order corresponding to a congruence $\equiv$ of $\mathcal{AR}_D$.

Corollary 14. *Assume that D is skeletal. For any congruence $\equiv$ of $\mathcal{AR}_D$,*

- *an acyclic reorientation E of D is minimal in its $\equiv$-class if and only if $\delta_\vee(E) \subseteq \mathbb{I}_\equiv$,*
- *$\mathcal{AR}_D/\!\equiv$ is isomorphic to the subposet of $\mathcal{AR}_D$ induced by $\{E \in \mathcal{AR}_D \mid \delta_\vee(E) \subseteq \mathbb{I}_\equiv\}$.*

A symmetric statement holds for maximal elements and $\delta_\wedge$.

Example 15. *Fix a pair $(\mho, \Omega)$ of subsets of vertices of D. The coherent congruence $\equiv_{(\mho,\Omega)}$ is the congruence of $\mathcal{AR}_D$ whose rope ideal is the set $\mathbb{I}_{(\mho,\Omega)}$ of ropes $(u, v, \triangledown, \triangle)$ of D with $\triangledown \subseteq \mho$ and $\triangle \subseteq \Omega$. In particular, we call*

- *sylvester congruence of $\mathcal{AR}_D$ the congruence $\equiv_{(V,\varnothing)}$, and Tamari lattice of D the quotient $\mathcal{AR}_D/\!\equiv_{(V,\varnothing)}$, generalizing [19]. See Figure 4.*
- *Cambrian congruences of $\mathcal{AR}_D$ the coherent congruences $\equiv_{(\mho,\Omega)}$ such that $\mho \sqcup \Omega = V$, and Cambrian lattices of D the corresponding quotients of $\mathcal{AR}_D$, generalizing [13].*

The coherent congruences of $\mathcal{AR}_D$ are the analogues of the permutree congruences of the weak order [9]. They share many interesting properties developed in [8, Sect. 5.4]. We conjecture that:

- *the graph of the Tamari lattice is regular if and only if D has no induced or or ,*
- *the Cambrian lattices of D are equinumerous if and only if D has no induced .*

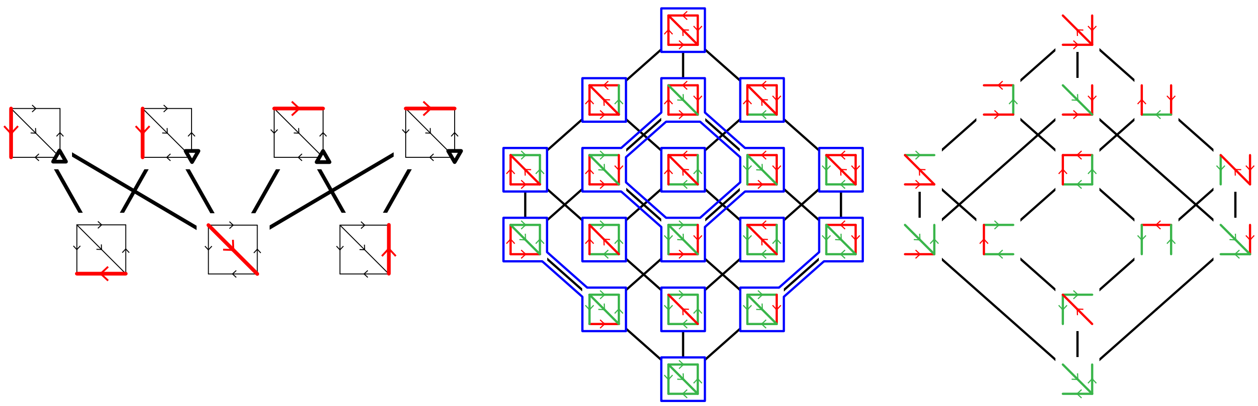


Figure 4: The subrope order, the sylvester congruence, and the Tamari lattice on D.

4 Quotient fans and quotientopes

We now switch to the geometric side of this paper. As originally observed in [4], the acyclic reorientation poset $\mathcal{AR}_D$ can be interpreted geometrically on the graphical fan of D or on the graphical zonotope of D. When D is skeletal, we consider the quotient fans of the congruences of $\mathcal{AR}_D$ and show that they are normal fans of quotientopes.

4.1 Graphical fan, shards, and quotient fans

Graphical fan. We work in the vector space $\mathbb{R}^V$ indexed by the vertex set V of D. We denote the standard basis by $(\boldsymbol{e}_v)_{v \in V}$, and let $\mathbf{1}_U := \sum_{u \in U} \boldsymbol{e}_u$ for $U \subseteq V$. The *graphical arrangement* $\mathcal{H}_D$ of D is the arrangement of the hyperplanes $\mathbb{H}_{uv} := \{\boldsymbol{x} \in \mathbb{R}^V \mid x_u = x_v\}$ for all arcs $(u, v) \in D$. It defines the *graphical fan* $\mathcal{F}_D$ of D, whose chambers are the closures of the connected components of $\mathbb{R}^V \smallsetminus \bigcup_{(u,v) \in D} \mathbb{H}_{uv}$. See Figure 5. The cones of $\mathcal{F}_D$ are in bijection with *ordered partitions* of D, *i.e.* pairs (μ, ω) where

- μ is a partition of V where each part induces a connected subgraph of D,
- ω is an acyclic reorientation on the quotient graph D/μ.

In particular, the chambers of $\mathcal{F}_D$ correspond to the acyclic reorientations of D, and the rays of $\mathcal{F}_D$ correspond to the *biconnected subsets* of D (*i.e.* a non-empty connected subset of V whose complement in its connected component of D is also non-empty and connected). The Hasse diagram of $\mathcal{AR}_D$ is isomorphic to the dual graph of the graphical fan $\mathcal{F}_D$, oriented by $\boldsymbol{\omega}_D := \sum_{(u,v) \in A} \boldsymbol{e}_v - \boldsymbol{e}_u$.

For instance, when D is a tournament on $[n]$, the graphical fan $\mathcal{F}_D$ is the braid fan, defined by the hyperplanes $\mathbb{H}_{ij}$ for $1 \le i < j \le n$. Its cones correspond to ordered partitions of $[n]$, its regions to permutations of $[n]$, its rays to proper subsets of $[n]$, and its dual graph is isomorphic to the Hasse diagram of the weak order on $\mathfrak{S}_n$.

Shards and quotient fan. Assume that D is skeletal, so that $\mathcal{AR}_D$ is a congruence uniform lattice. Generalizing [11, 12], define the *shard* of a rope $\rho := (u, v, \triangledown, \triangle)$ of D as

$$\Sigma_\rho := \{ \boldsymbol{x} \in \mathbb{R}^V \mid x_w \le x_u = x_v \le x_{w'} \text{ for any } w \in \triangledown \text{ and } w' \in \triangle \}.$$

For a congruence $\equiv$ of $\mathcal{AR}_D$, the *quotient fan* $\mathcal{F}_\equiv$ is then defined equivalently as follows:

- the chambers of $\mathcal{F}_\equiv$ are obtained by glueing the chambers of the graphical arrangement of D corresponding to acyclic reorientations in the same class of $\equiv$,
- the union of the walls of $\mathcal{F}_\equiv$ is the union of the shards Σ_ρ for ρ in the rope ideal $\mathbb{I}_\equiv$.

The Hasse diagram of the quotient $\mathcal{AR}_D/\equiv$ is isomorphic to the dual graph of the quotient fan $\mathcal{F}_\equiv$, oriented in the direction $\boldsymbol{\omega}_D := \sum_{(u,v)\in A} \boldsymbol{e}_v - \boldsymbol{e}_u$.

We call *sylvester fan* of D the quotient fan for the sylvester congruence. See Figure 5.

4.2 Graphical zonotope, shard polytopes, and quotientopes

Graphical zonotope. The *graphical zonotope* $\mathbb{Z}_D$ is the Minkowski sum of the segments $[\boldsymbol{e}_u, \boldsymbol{e}_v]$ for all $(u, v) \in A$. The graphical fan $\mathcal{F}_D$ is the normal fan of the graphical zonotope $\mathbb{Z}_D$. Hence, the faces of $\mathbb{Z}_D$ are in bijection with ordered partitions of D. In particular, the vertices of $\mathbb{Z}_D$ correspond to the acyclic reorientations of D, and the facets of $\mathbb{Z}_D$ correspond to the biconnected subsets of D. The Hasse diagram of $\mathcal{AR}_D$ isomorphic to the graph of $\mathbb{Z}_D$, oriented in the direction $\boldsymbol{\omega}_D := \sum_{(u,v)\in A} \boldsymbol{e}_v - \boldsymbol{e}_u$.

For instance, when D is a tournament on $[n]$, the graphical zonotope $\mathbb{Z}_D$ coincides up to a translation of vector $\mathbf{1}$ with the classical *permutahedron*, defined equivalently as:

- the convex hull of the points $\sum_{i\in[n]} \sigma_i \boldsymbol{e}_i$ for all permutations σ of $[n]$,
- the intersection of the hyperplane $\{ \boldsymbol{x} \in \mathbb{R}^n \mid \langle \mathbf{1} \mid \boldsymbol{x} \rangle = \binom{n+1}{2} \}$ with the halfspaces $\{ \boldsymbol{x} \in \mathbb{R}^n \mid \langle \mathbf{1}_U \mid \boldsymbol{x} \rangle \ge \binom{|U|+1}{2} \}$ for all proper subsets $\varnothing \ne U \subsetneq [n]$,
- the Minkowski sum of the vector $\mathbf{1}$ and the segments $[\boldsymbol{e}_i, \boldsymbol{e}_j]$ for all $1 \le i < j \le n$.

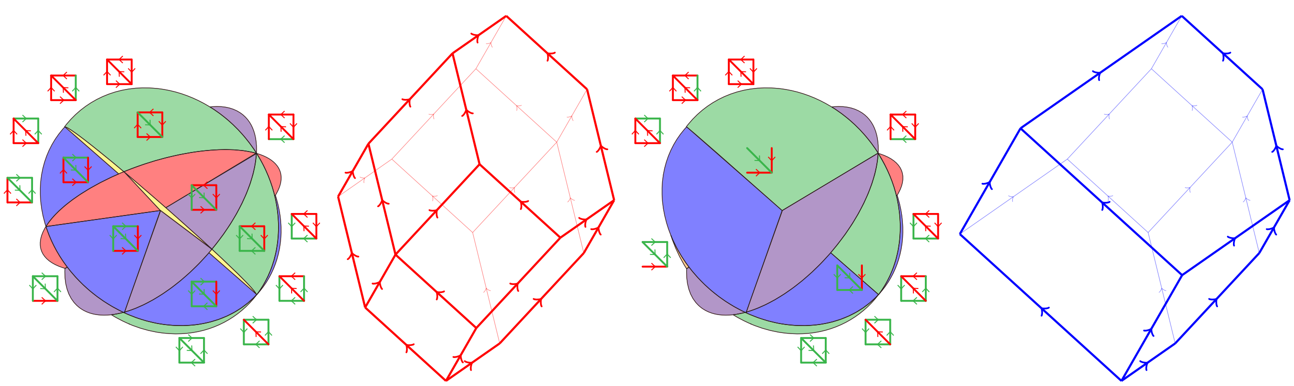
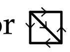
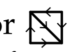


Figure 5: The graphical arrangement, the graphical zonotope, the sylvester fan, and the associahedron of a skeletal directed acyclic graph D.

Quotientopes and associahedra. Assume that D is skeletal, so that $\mathcal{AR}_D$ is a congruence uniform lattice. The main result of this section is the following statement.

Theorem 16. *Assume that D is skeletal. For any congruence $\equiv$ of $\mathcal{AR}_D$, the quotient fan $\mathcal{F}_\equiv$ is the normal fan of a polytope.*

We call *quotientope* any polytopal realization of the quotient fan $\mathcal{F}_\equiv$. We provide two general approaches to construct quotientopes in Theorems 17 and 19, and we discuss a third approach specific to the sylvester, Cambrian, and coherent congruences in Proposition 20.

We call *associahedron* for D any quotientope for the sylvester congruence $\equiv_{(V,\varnothing)}$. See Figure 5. Following Example 15, we conjecture that:
- the associahedron for D is simple if and only if D has no induced  or  or ,
- the quotientopes of all Cambrian congruences have isomorphic face lattices if and only if D has no induced .

Quotientopes from classical associahedra. Our first approach to realize the quotient fan $\mathcal{F}_\equiv$ as a polytope is based on the associahedra [16, 6, 5]. Recall first that when D is the increasing tournament on $[n]$, the sylvester fan is the normal fan of the classical associahedron, defined equivalently as:
- the convex hull of the points $\sum_{j\in[n]} \ell(T,j)\, r(T,j)\, \boldsymbol{e}_j$ for all binary trees T on n nodes, where $\ell(T,j)$ and $r(T,j)$ respectively denote the numbers of leaves in the left and right subtrees of the node j of T (labeled in inorder), see [6],
- the intersection of the hyperplane $\left\{\boldsymbol{x}\in\mathbb{R}^n \;\middle|\; \langle\, \mathbf{1} \mid \boldsymbol{x}\,\rangle = \binom{n+1}{2}\right\}$ with the halfspaces $\left\{\boldsymbol{x}\in\mathbb{R}^n \;\middle|\; \langle\, \mathbf{1}_{[a,b]} \mid \boldsymbol{x}\,\rangle \ge \binom{b-a+2}{2}\right\}$ for all intervals $1\le a\le b\le n$, see [16],
- the Minkowski sum of $\triangle_{[a,b]}$ for all intervals $1\le a\le b\le n$, where for $I\subseteq[n]$, $\triangle_I := \operatorname{conv}\{\boldsymbol{e}_i \mid i\in I\}$ is the face of the standard simplex $\triangle_{[n]}$ labeled by I, see [10].

Similar polytopal realizations were constructed for the quotient fans of the Cambrian congruences of the weak order in [5]. Here, we skip the precise vertex, facet, and Minkowski descriptions of these associahedra and refer to [5] for details. We just need to observe that, for the principal congruence whose rope ideal is the lower ideal generated by a rope ρ of D, the quotient fan $\mathcal{F}_\rho$ is the normal fan of an associahedron $\mathbb{A}_\rho$ obtained by embedding a Cambrian associahedron of [5] in $\mathbb{R}^V$. Mimicking [7, Theorem 1], we now observe that any quotient fan can be realized as a Minkowski sum of (low dimensional) Cambrian associahedra of [5].

Theorem 17. *Assume that D is skeletal. Consider any congruence $\equiv$ of $\mathcal{AR}_D$, and let $\rho_1,\dots,\rho_p$ denote the ropes generating the lower ideal $\mathbb{I}_\equiv$ of the subrope order. Then the quotient fan $\mathcal{F}_\equiv$ is*
- *the common refinement of the Cambrian fans $\mathcal{F}_{\rho_1},\dots,\mathcal{F}_{\rho_p}$,*
- *the normal fan of the Minkowski sum of the Cambrian associahedra $\mathbb{A}_{\rho_1},\dots,\mathbb{A}_{\rho_p}$.*

Quotientopes from shard polytopes. Our second approach to realize the quotient fan $\mathcal{F}_{\equiv}$ as a polytope is based on the shard polytopes introduced in [7]. Consider a rope $\rho := (u, v, \triangledown, \triangle)$ and let π denote the directed path from u to v in the transitive reduction of D. Define a *ρ-alternating matching* as a pair $(M_\triangledown, M_\triangle)$ with $M_\triangledown \subseteq \{u\} \cup \triangledown$ and $M_\triangle \subseteq \triangle \cup \{v\}$ such that $M_\triangledown$ and $M_\triangle$ are alternating along π. The *shard polytope* $\mathbb{SP}_\rho$ is the polytope of $\mathbb{R}^V$ defined as the convex hull of the vectors $\mathbf{1}_{M_\triangledown} - \mathbf{1}_{M_\triangle}$ for all ρ-alternating matchings $(M_\triangledown, M_\triangle)$. For instance, the shard polytope $\mathbb{SP}_\rho$ of a rope of the form $\rho := (u, v, \triangledown, \varnothing)$ is the face $\triangle_{\{u,v\} \cup \triangledown}$ of the standard simplex, translated by the vector $-\boldsymbol{e}_v$. The following statement is the fundamental property of shard polytopes.

Proposition 18. *Assume that D is skeletal. For any rope ρ of D, the union of the walls of the normal fan of the shard polytope $\mathbb{SP}_\rho$ contains the shard Σ_ρ and is contained in the union of the shards $\Sigma_{\rho'}$ for all subropes ρ' of ρ.*

Based on Proposition 18, we obtain polytopal realizations of all lattice quotients of $\mathcal{AR}_D$ as Minkowski sums of shard polytopes.

Theorem 19. *Assume that D is skeletal. For any congruence $\equiv$ of $\mathcal{AR}_D$ and any positive coefficients $\boldsymbol{s} \in (\mathbb{R}_{>0})^{\mathbb{I}_\equiv}$, the quotient fan $\mathcal{F}_\equiv$ is the normal fan of the Minkowski sum $\sum_{\rho \in \mathbb{I}_\equiv} \boldsymbol{s}_\rho \, \mathbb{SP}_\rho$.*

Associahedra as removahedra. Finally, we focus on quotientopes for coherent congruences and more specifically on associahedra. The following statement, visible in Figure 5, generalizes [16, 10].

Proposition 20. *Assume that D is skeletal. The sylvester fan $\mathcal{F}_{(V,\varnothing)}$ is the normal fan of the associahedron $\mathbb{A}_D$ defined equivalently as*

- *the polytope obtained from the graphical zonotope $\mathbb{Z}_D$ by deleting all facet inequalities corresponding to biconnected subsets U of D not connected in the transitive reduction of D,*
- *the Minkowski sum of the faces $\triangle_\pi$ of the standard simplex $\triangle_V$, for all directed paths π in the transitive reduction of D whose endpoints are connected by an arc of D.*

In contrast, we are still missing a simple vertex description of the associahedron $\mathbb{A}_D$ similar to that of [6] for the classical associahedron.

Note that the construction of Proposition 20 cannot provide a quotientope for any congruence of $\mathcal{AR}_D$. However, based on computer experiments, we conjecture that:

- For any coherent congruence $\equiv_{(\mho,\Omega)}$, the quotient fan $\mathcal{F}_{(\mho,\Omega)}$ is the normal fan of the polytope obtained by deleting from the facet description of the graphical zonotope $\mathbb{Z}_D$ the inequalities normal to the rays of the graphical fan $\mathcal{F}_D$ that are not rays of the quotient fan $\mathcal{F}_{(\mho,\Omega)}$. This would extend the permutreehedra of [9].
- For a Cambrian congruence (*i.e.* with $\mho \sqcup \Omega = V$), the resulting polytope coincides with the Minkowski sum of the shard polytopes of the ropes of $\mathbb{I}_{(\mho,\Omega)}$. This would extend the case of Cambrian congruences of the weak order treated in [7].

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