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On the Geometry of Flag Hilbert–Poincaré Series for Matroids

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Abstract. We extend the definition of coarse flag Hilbert–Poincaré series to matroids; these series arise in the context of local Igusa zeta functions associated to hyperplane arrangements. We study these series in the case of oriented matroids by applying geometric and combinatorial tools related to their topes. In this case, we prove that the numerators of these series are coefficient-wise bounded below by the Eulerian polynomial and equality holds if and only if all topes are simplicial. Moreover this yields a sufficient criterion for non-orientability of matroids of arbitrary rank.

Keywords: Coarse flag polynomial, Eulerian polynomials, Igusa zeta functions, oriented matroids

1 Introduction

We present the main results from [12], where details and omitted proofs can be found. The flag Hilbert–Poincaré series associated to a hyperplane arrangement, defined in [13], is a rational function in several variables connected to local Igusa zeta functions [5]. In fact, polynomial substitutions of the variables of the flag Hilbert–Poincaré series also yield motivic zeta functions associated to matroids [10]; see [20] for the topological analog. There are also substitutions yielding so-called ask zeta functions associated to certain modules of matrices [17]; see [13, Proposition 4.8].

Here we consider a specialization in variables Y and T, called the coarse flag Hilbert–Poincaré series, which seems to have remarkable combinatorial properties. In [13], it was shown that for most Coxeter hyperplane arrangements, the numerator of this specialization at $Y = 1$ is equal to an Eulerian polynomial. We generalize this to the setting of oriented matroids, a combinatorial abstraction of the face structure determined by real hyperplane arrangements. We show that the numerator can be better understood from the geometry of the topes, which are analogs of the chambers for real hyperplane arrangements. This settles a question by Voll and the second author [13, Question 1.7] for the case of real arrangements, asking about which properties of a hyperplane arrangement guarantee the equality to Eulerian polynomials mentioned above.

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1.1 Flag Hilbert–Poincaré series for matroids

Let M be a matroid, with ground set E, and $\mathcal{L}(M)$ its lattice of flats, with bottom and top elements denoted by $\widehat{0}$ and $\widehat{1}$, respectively. Relevant definitions concerning matroids and oriented matroids are given in [12, Section 2]. We call a matroid M *orientable* if there exists an oriented matroid whose underlying matroid is M. An orientable matroid M is *simplicial* if M has an oriented matroid structure such that the face lattice of every tope is a Boolean lattice — equivalently, for real hyperplane arrangements every chamber is a simplicial cone. For example, all Coxeter arrangements are simplicial.

Let $\mu_M\colon \mathcal{L}(M) \to \mathbb{Z}$ be the Möbius function on $\mathcal{L}(M)$, where $\mu_M(\widehat{0}) = 1$ and $\mu_M(X) = -\sum_{X'<X}\mu_M(X')$. A well-studied invariant of a matroid M is the *Poincaré polynomial*

$$\pi_M(Y) = \sum_{X\in\mathcal{L}(M)} \mu_M(X)(-Y)^{r(X)},$$

where $r(X)$ is the *rank* of X in $\mathcal{L}(M)$, viz. one less than the maximum over the number of elements of all flags from $\widehat{0}$ to X. If M is realized by a hyperplane arrangement $\mathcal{A}$, then its Poincaré polynomial captures topological and algebraic properties of $\mathcal{A}$ [15].

For a poset P let $\Delta(P)$ be the set of flags of P, and let $\Delta_k(P) \subseteq \Delta(P)$ be the set of flags of size k. If P has a bottom element $\widehat{0}$ and a top element $\widehat{1}$ set $\overline{P} = P \setminus \{\widehat{0}, \widehat{1}\}$. The *flag Poincaré polynomial* associated to $F = (X_1 < \cdots < X_\ell) \in \Delta(\overline{\mathcal{L}}(M))$, with $\ell \geqslant 0$, is the product of Poincaré polynomials on the minors determined by F,

$$\pi_F(Y) = \prod_{k=0}^{\ell} \pi_{M/X_k|X_{k+1}}(Y),$$

where $X_0 = \widehat{0}$ and $X_{\ell+1} = \widehat{1}$. Here, M/X_k is the contraction of $X_k \subseteq E$ from M, and $M|X_{k+1}$ is the restriction of M to $X_{k+1} \subseteq E$. The lattice $\mathcal{L}(M/X_k|X_{k+1})$ is isomorphic to the interval $[X_k, X_{k+1}]$ in $\mathcal{L}(M)$.

Definition 1.1. The *coarse flag Hilbert–Poincaré series* of a matroid M is

$$\mathsf{cfHP}_M(Y,T) = \frac{1}{1-T}\sum_{F\in\Delta(\overline{\mathcal{L}}(M))} \pi_F(Y)\left(\frac{T}{1-T}\right)^{|F|} = \frac{\mathcal{N}_M(Y,T)}{(1-T)^{r(M)}}.$$

We call $\mathcal{N}_M(Y,T)$ the *coarse flag polynomial*:

$$\mathcal{N}_M(Y,T) = \sum_{F\in\Delta(\overline{\mathcal{L}}(M))} \pi_F(Y)T^{|F|}(1-T)^{r(M)-1-|F|}.$$

1.2 Main results

For rational polynomials $f(T) = \sum_{k\geqslant 0} a_k T^k$ and $g(T) = \sum_{k\geqslant 0} b_k T^k$, we write $f(T) \leqslant g(T)$ if $a_k \leqslant b_k$ for all $k \geqslant 0$. We write $f(T) < g(T)$ to mean $f(T) \leqslant g(T)$ and $f(T) \neq g(T)$.

The *Eulerian polynomials* $E^{\mathsf{A}}_{r+1}(T)$ and $E^{\mathsf{B}}_{r+1}(T)$ are equal to the h-polynomials of the barycentric subdivisions of the boundaries of the r-dimensional simplex and the cross-polytope, respectively [16, Theorem 11.3]. The Eulerian polynomials are also defined by Coxeter-theoretic descent statistics [16, Section 11.4]. In [13, Theorem D], it was shown that for all Coxeter arrangements $\mathcal{A}$ of rank r, without an E_8-factor, $\mathcal{N}_{\mathcal{A}}(1,T)/\pi_{\mathcal{A}}(1) = E^{\mathsf{A}}_r(T)$. The next theorem generalizes this result.

Theorem 1.2. *Let M be an orientable matroid of rank r. Then*

$$E^{\mathsf{A}}_r(T) \leqslant \frac{\mathcal{N}_M(1,T)}{\pi_M(1)}, \tag{1.1}$$

and equality holds if and only if M is simplicial. Moreover,

$$\mathcal{N}_M(1,T^{-1}) = T^{r-1}\mathcal{N}_M(1,T).$$

The key insight in the proof for Theorem 1.2 is that in the orientable case $\mathcal{N}_M(1,T)$ is a sum of h-polynomials. Each of the summands is determined by the topes of M; see Proposition 3.2. A byproduct of Theorem 1.2 is a sufficient condition for non-orientability of matroids. The rank 3 case yields an inequality concerning the the number of rank 2 flats above every element in M.

Corollary 1.3. *Assume M is a simple matroid with rank 3, and suppose c is the number of rank 2 flats of M and s the sum of their sizes. If $3(c-1) < s$, then M is non-orientable.*

It is known that the Fano matroid is non-orientable, which is also shown by Corollary 1.3 since it has 7 lines, each with 3 points. There are a number of sufficient conditions for the non-orientability of matroids. Based on experiments using the database of non-orientable matroids [14], we report that the condition in Corollary 1.3 is independent from the sufficient condition in [6] for rank 3 matroids; see also [2, Proposition 6.6.1(i)]. Moreover, Corollary 1.3 is related to Corollary 2.6 in [7] where Cuntz and Geis proved that a rank 3 arrangement is simplicial if and only if its underlying matroid satisfies $3(c-1) = s$ in the notation above.

The lower bound in (1.1) raises the following question. How large or how small can the coefficients of the numerator of $\mathsf{cfHP}_M(1,T)/\pi_M(1)$ be? All of our results and computations suggest the following.

Conjecture 1.4. *For all matroids M of rank $r \geqslant 3$,*

$$(1+T)^{r-1} < \frac{\mathcal{N}_M(1,T)}{\pi_M(1)} < E^{\mathsf{B}}_r(T).$$

We note that $E_1^{\mathsf{A}}(T) = E_1^{\mathsf{B}}(T) = 1$ and $E_2^{\mathsf{A}}(T) = E_2^{\mathsf{B}}(T) = 1 + T$, and all matroids of rank 1 or 2 are both orientable and simplicial. For orientable matroids, the lower bound of Conjecture 1.4 holds by Theorem 1.2. Moreover, the upper bound in Conjecture 1.4 is reminiscent of similar "f-vector" bounds proved in [8, 19].

Theorem 1.5.

(1) If Conjecture 1.4 holds, then the bounds are sharp.

(2) Conjecture 1.4 holds for all matroids of rank 3*. Moreover for all orientable matroids, the upper bound holds for the linear term of the polynomials, so Conjecture 1.4 holds for all orientable matroids of rank* 4*.*

Remark 1.6. To prove Theorem 1.5 (1) we consider two extremal families of matroids for the lower and upper bounds: these are the projective geometries and uniform matroids, respectively. See [12, Section 5] for details.

In fact, more is known to hold for $\mathcal{N}_M(Y, T)$ in the case where $r(M) \leqslant 3$. We prove, in Proposition 2.2, that the numerator is nonnegative, palindromic, and when $Y = 1$ real-rooted. In particular, Conjecture E from [13] holds for all central hyperplane arrangements with rank at most 3. We are also interested in whether or not these three properties hold for the numerator of $\mathcal{N}_M(Y, T)$ for all matroids of rank larger than 3. For oriented matroids of rank 4, the polynomial $\mathcal{N}_M(1, T)$ is real-rooted, which follows from Theorem 1.2. This raises the following general question.

Question 1.7. *Is the polynomial $\mathcal{N}_M(1, T)$ real-rooted for all matroids M?*

Brenti and Welker asked whether the h-polynomial of the barycentric subdivision of a general polytope is real-rooted [3]. In the case of real hyperplane arrangements and their associated zonotopes, this question is related to Question 1.7 via our geometric interpretation of $\mathcal{N}_M(1, T)$ although the precise connection is not yet well understood.

We give parts of the proof for Theorem 1.2 in Section 3. Section 2 is devoted to general matroids of rank 3. There we also describe a pair of real hyperplane arrangements with the same coarse flag polynomial and different underlying matroids (Remark 2.3), answering a question of Voll and the second author [13]. We conclude with some examples in Section 4.

2 Matroids of rank 3

We explicitly determine the coarse flag Hilbert–Poincaré series for matroids of rank 3. First we require the next lemma, which follows from the definition of the Möbius function.

Lemma 2.1. *Let M be a simple rank 3 matroid on $E = [n]$. Let c be the number of rank 2 flats of M and s the sum of their sizes. Then*

$$\begin{aligned}\pi_M(Y) &= 1 + nY + (s-c)Y^2 + (1+s-n-c)Y^3 \\ &= \left(1 + (n-1)Y + (1+s-n-c)Y^2\right)(1+Y).\end{aligned}$$

Proposition 2.2. *For a simple rank 3 matroid M with ground set of size n, let c be the number of rank 2 flats of M and s the sum of their sizes. Then*

$$\mathcal{N}_M(Y,T) = \pi_M(Y) + \varphi_M(Y)T + Y^3\pi_M(Y^{-1})T^2,$$

where

$$\varphi_M(Y) = n + c - 2 + (2s - n + c)Y + (2s - n + c)Y^2 + (n + c - 2)Y^3.$$

Remark 2.3. With Proposition 2.2, we answer a question of Voll and the second author [13, Question 6.2], about whether there exists a distinct pair of arrangements with the same coarse flag polynomial. We describe a pair $\mathcal{A}$ and $\mathcal{B}$ of real arrangements in Figure 2.1 which we found in the database of [1] and are given by:

$$\begin{aligned}&\mathcal{A} : xyz(x+y)(x-y)(x+2y)(x+z)(y+z)(x+y+z) = 0, \\ &\mathcal{B} : xyz(x+y)(x+2y)(x-2y)(x+z)(2y+z)(2x+2y+z) = 0.\end{aligned}$$

They both contain nine hyperplanes with $c = 15$ and $s = 39$ using the above notation. The arrangement $\mathcal{A}$ has exactly two planes with three lines of intersection, whereas $\mathcal{B}$ has exactly one such plane, so they are inequivalent.

Corollary 2.4. *If M is a matroid of rank not larger than 3, then $\mathcal{N}_M(Y,T)$ has nonnegative coefficients and satisfies*

$$\mathcal{N}_M(Y^{-1},T^{-1}) = Y^{r(M)}T^{r(M)-1}\mathcal{N}_M(Y,T).$$

Moreover, the polynomial $\mathcal{N}_M(1,T)$ is real-rooted.

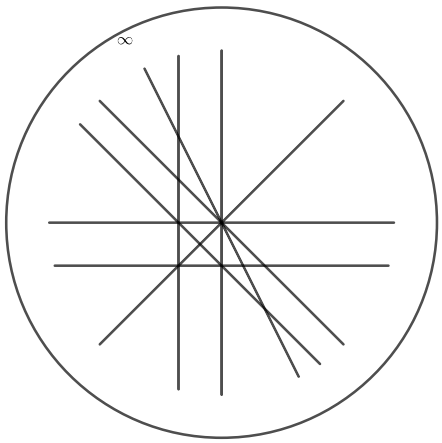
Proof. This is clear if $r(M) = 1$, and we assume that M is a simple matroid. If M has rank 2, then $M \cong U_{2,n}$, where n is the size of the ground set of M. Then,

$$\mathcal{N}_{U_{2,n}}(Y,T) = (1+Y)(1+(n-1)Y) + (1+Y)(n-1+Y)T,$$

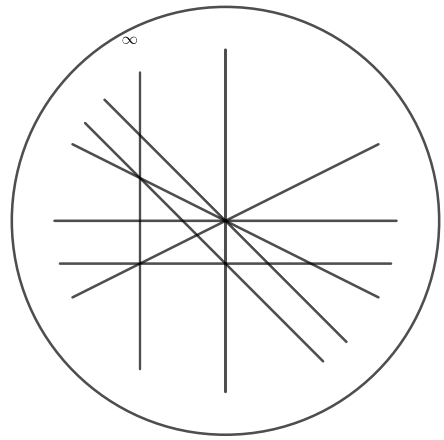
which satisfies the three properties.

If $r(M) = 3$, then from Proposition 2.2, $\mathcal{N}_M(Y,T)$ satisfies

$$\mathcal{N}_M(Y^{-1},T^{-1}) = Y^{-3}T^{-2}\mathcal{N}_M(Y,T).$$



(A) The arrangement $\mathcal{A}$.



(B) The arrangement $\mathcal{B}$.

Figure 2.1: Two projectivized pictures of the arrangements $\mathcal{A}$ and $\mathcal{B}$.

By a classical result of de Bruijn and Erdős [4], the number of elements is not larger than the number of rank 2 flats, so $c \geqslant n$. Thus, $\mathcal{N}_M(Y, T)$ has nonnegative coefficients. The discriminant of $\mathcal{N}_M(Y, T)$ as a polynomial in T is

$$\left((c+n)^2(1-Y)^2 - 4s(1-(c+1)Y+Y^2)\right)(1+Y)^4,$$

which is positive at $Y = 1$. □

Lemma 2.5. *For all matroids M with rank 3,*

$$(1+T)^2 < \frac{\mathcal{N}_M(1,T)}{\pi_M(1)} < E_3^{\mathsf{B}}(T) = 1 + 6T + T^2.$$

Proof. Without loss of generality, M is a simple matroid. By Proposition 2.2,

$$\frac{\mathcal{N}_M(1,T)}{\pi_M(1)} = 1 + \left(2 + \frac{4(c-1)}{s-(c-1)}\right)T + T^2, \tag{2.1}$$

where $c = |\mathcal{L}_2(M)|$ and $s = \sum_{X\in\mathcal{L}_2(M)} |X|$. Since $s \geqslant 2c$,

$$0 < \frac{4(c-1)}{s-(c-1)} < 4.$$ □

We note that equation (2.1) together with Theorem 1.2 proves Corollary 1.3.

3 Oriented matroids

Central to the proof of Theorem 1.2 is the face lattice of an oriented matroid $M = (E, \mathcal{C})$. For an oriented matroid M, we let $\mathcal{C}$ be the set of covectors of M, $\mathcal{F}(\mathcal{C})$ the face lattice, $\mathcal{T}(\mathcal{C})$ the set of topes, and $\mathrm{z}\colon \mathcal{C} \to 2^E$ the zero map sending C to $\{e \in E \mid C_e = 0\}$.

For a poset P, let $\widehat{\Delta}(P)$, resp. $\widehat{\Delta}_k(P)$, be the set of nonempty flags, resp. flags of length k, ending at a maximal element of P.

Lemma 3.1. *Let $M = (E, \mathcal{C})$ be an oriented matroid of rank r. Then for all $k \in [r]$,*

$$\left|\widehat{\Delta}_k(\overline{\mathcal{F}}(\mathcal{C}))\right| = \sum_{F \in \Delta_{k-1}(\overline{\mathcal{L}}(M))} \pi_F(1).$$

For a finite simplicial complex Σ, we write $f(\Sigma) := (f_0, \dots, f_d) \in \mathbb{N}_0^{d+1}$ for the *f-vector* of Σ, where f_k is the number of k-subsets in Σ—equivalently, the number of $(k-1)$-dimensional faces. Let $f(\Sigma; T) = \sum_{k=0}^{d} f_k T^k$ be the f-polynomial of Σ, and let $h(\Sigma; T) := (1-T)^d f(\Sigma; T/(1-T))$, which is the *$h$-polynomial* associated to Σ. The coefficients of $h(\Sigma; T)$ yield the *h-vector* $h(\Sigma)$ of Σ.

For a tope $\tau \in \mathcal{T}(\mathcal{C})$, we define a simplicial complex $\Sigma(\tau) := \Delta((\widehat{0}_{\mathcal{C}}, \tau))$, which is the set of flags in the open interval $(\widehat{0}_{\mathcal{C}}, \tau)$ in $\mathcal{F}(\mathcal{C})$ ordered by refinement. We write $\Sigma_k(\tau)$ for the flags of $\Sigma(\tau)$ with length k. If M is realizable over $\mathbb{R}$, then $\Sigma(\tau)$ is the barycentric subdivision of the boundary of the chamber determined by τ.

Proposition 3.2. *Let $M = (E, \mathcal{C})$ be an oriented matroid of rank r. Then*

$$\mathcal{N}_M(1, T) = \sum_{\tau \in \mathcal{T}(\mathcal{C})} h(\Sigma(\tau); T).$$

Proof. The flags in $\widehat{\Delta}(\overline{\mathcal{F}}(\mathcal{C}))$ are partitioned into subsets $\widehat{\Delta}((\widehat{0}_{\mathcal{C}}, \tau])$ for $\tau \in \mathcal{T}(\mathcal{C})$, and the latter are in bijection with the flags in $\Sigma(\tau)$. Thus,

$$\left|\widehat{\Delta}(\overline{\mathcal{F}}(\mathcal{C}))\right| = \sum_{\tau \in \mathcal{T}(\mathcal{C})} |\Sigma(\tau)|\,.$$

Applying Lemma 3.1, we have

$$\begin{aligned}\sum_{\tau \in \mathcal{T}(\mathcal{C})} h(\Sigma(\tau); T) &= \sum_{k=0}^{r-1} \sum_{\tau \in \mathcal{T}(\mathcal{C})} |\Sigma_k(\tau)|\, T^k (1-T)^{r-k-1} \\ &= \sum_{k=0}^{r-1} \sum_{\substack{F \in \Delta(\overline{\mathcal{L}}(M)) \\ |F| = k-1}} \pi_F(1) T^k (1-T)^{r-k-1} = \mathcal{N}_M(1, T). \qquad \square\end{aligned}$$

In order to prove the lower bound in Theorem 1.2, we work with the cd-index of an (Eulerian) poset. Details on the cd-index can be found in [18, Ch. 3.17].

Proposition 3.3. *Let $M = (E, \mathcal{C})$ be an oriented matroid of rank r. Then for all $\tau \in \mathcal{T}(\mathcal{C})$,*

$$E_r^{\mathsf{A}}(T) \leqslant h(\Sigma(\tau); T),$$

and equality holds if and only if τ is a simplicial tope.

Proof of Theorem 1.2. The first statement is proved using Propositions 3.2 and 3.3. Since $\beta_{[\widehat{0}_{\mathcal{C}}, \tau]}(S) = \beta_{[\widehat{0}_{\mathcal{C}}, \tau]}([r-1] \setminus S)$ by [18, Corollary 3.16.6], it follows that $h_k(\Sigma(\tau)) = h_{r-k-1}(\Sigma(\tau))$. Hence, the second statement follows by Proposition 3.2. $\square$

3.1 The upper bound for the linear term

In this section, we prove that $\mathcal{N}_M(1,T)$, for an oriented matroid $M=(E,\mathcal{C})$ of rank 4, is bounded above coefficient-wise by $\pi_M(1)E_r^{\mathsf{B}}(T)$. The next lemma determines the coefficients of $\mathcal{N}_M(1,T)$ in terms of the face lattice of M. To simplify notation, we define

$$\mathfrak{f}_k(\mathcal{C}) := \left|\widehat{\Delta}_{k+1}\left(\overline{\mathcal{F}}(\mathcal{C})\right)\right| = \left|\left\{F \in \Delta_{k+1}\left(\overline{\mathcal{F}}(\mathcal{C})\right) : F \text{ ends at a tope}\right\}\right|.$$

For $f(T)=\sum_{k\geqslant 0} a_k T^k$, let $f(T)[T^k]=a_k$.

Lemma 3.4. *Let $M=(E,\mathcal{C})$ be an oriented matroid of rank r. For $\ell \in [r-1]_0$,*

$$\mathcal{N}_M(1,T)[T^\ell] = \sum_{k=0}^{\ell}(-1)^{\ell-k}\mathfrak{f}_k(\mathcal{C})\binom{r-k-1}{\ell-k}.$$

Proposition 3.5. *If M is an orientable matroid of rank $r \geqslant 3$, then*

$$\mathcal{N}_M(1,T)[T] < \pi_M(1)E_r^{\mathsf{B}}(T)[T].$$

If M is rank 4, then $\mathcal{N}_M(1,T) < \pi_M(1)E_4^{\mathsf{B}}(T)$.

Proof. From Theorem 1.2, $\mathcal{N}_M(1,T)$ has degree $r-1$ and is palindromic. Therefore it suffices to just prove the inequality between the linear coefficients.

Suppose $\mathcal{C}$ is a set of covectors such that $M=(E,\mathcal{C})$ is an oriented matroid. The number $\mathfrak{f}_1(\mathcal{C})$ counts the flags of length two in $\overline{\mathcal{F}}(\mathcal{C})$ which end at a tope, and $\mathfrak{f}_0(\mathcal{C}) = |\mathcal{T}(\mathcal{C})|$. Using Proposition 4.6.9 of [2], we have the following inequality

$$\mathfrak{f}_1(\mathcal{C}) < \sum_{j=0}^{r-2} 2^{r-1-j}\binom{r-1}{j}|\mathcal{T}(\mathcal{C})|. \tag{3.1}$$

Using [16, Section 13.1], one can express the terms of $E_r^{\mathsf{B}}(T)$ in terms of alternating sums. The linear term is, thus, $3^{r-1}-r$.

$$\begin{aligned}
\mathcal{N}_M(1,T)[T] &= \mathfrak{f}_1(\mathcal{C}) - (r-1)\mathfrak{f}_0(\mathcal{C}) && \text{(Lemma 3.4)}\\
&< \left(\sum_{j=0}^{r-2} 2^{r-1-j}\binom{r-1}{j}|\mathcal{T}(\mathcal{C})|\right) - (r-1)|\mathcal{T}(\mathcal{C})| && \text{(Equation 3.1)}\\
&= (3^{r-1}-1)|\mathcal{T}(\mathcal{C})| - (r-1)|\mathcal{T}(\mathcal{C})|\\
&= \pi_M(1)E_r^{\mathsf{B}}[T].
\end{aligned}$$

The penultimate equality is seen by counting, in two different ways, the number of ways to color $r-1$ balls with three colors. $\square$

4 Examples

4.1 A uniform matroid with rank 3

Consider the matroid $M = U_{3,4}$. One set of covectors is defined by the real arrangement given by $xyz(x+y+z) = 0$. There are $14 = 2^4 - 2$ topes since $(+++-)$ and $(---+)$ are not topes. For instance, the inequality system given by $x > 0, y > 0, z > 0$ and $x + y + z < 0$ is infeasible. The topes with an even number of $+$ symbols are triangles, and the topes with an odd number of $+$ symbols are squares. Therefore, there are 8 triangles and 6 squares, so by Proposition 3.2,

$$\mathcal{N}_M(1,T) = 8(1+4T+T^2) + 6(1+6T+T^2) = 14 + 68T + 14T^2.$$

By Proposition 2.2, the coarse numerator for M is given by

$$\begin{aligned}\mathcal{N}_M(Y,T) = 1 + 4Y + 6Y^2 + 3Y^3 &+ (8 + 26Y + 26Y^2 + 8Y^3)T \\ &+ (3 + 6Y + 4Y^2 + Y^3)T^2.\end{aligned}$$

4.2 A uniform matroid with rank 4

Three non-Coxeter, uniform matroids in [13] had the seemingly rare property that

$$\mathcal{N}_M(1,T)/\pi_M(1) \in \mathbb{Z}[T].$$

These are the uniform matroids $U_{r,n}$ for $(r,n) \in \{(4,5),(4,7),(4,8)\}$, and we consider $(r,n) = (4,7)$. From Proposition 3.2, this integrality condition is equivalent to the integrality of the average of the h-vectors. To do this computation, we used the hyperplane arrangement package [11] of POLYMAKE [9].

The matroid $U_{4,7}$ can be realized as a hyperplane arrangement in $\mathbb{R}^4$, whose hyperplanes are given by

$$x_1x_2x_3x_4(x_1+x_2+x_3+x_4)(x_1+2x_2+3x_3+4x_4)(x_1+3x_2+2x_2+5x_4) = 0.$$

There are five different polytopes corresponding to chambers of this arrangement, and they can be seen in Figure 4.1. The chambers are 4-dimensional cones over these polytopes.

There are a total of 84 chambers; 22 are simplices, 22 are triangular prisms, 30 are the polytopes seen in Figure 4.1(C), six are the polytopes seen in Figure 4.1(D), and four are truncated simplices as seen in Figure 4.1(E). The h-vectors of the barycentric subdivisions are palindromic, and the first values different from 1 are 11, 17, 23, 29, and 29 respectively.

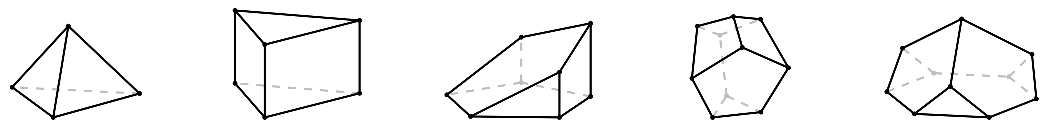


Figure 4.1: The five different polytopes arising as chambers in the $U_{4,7}$ arrangement.

Thus,

$$\begin{aligned}\mathcal{N}_{U_{4,7}}(1,T) &= 22(1+11T+11T^2+T^3)+22(1+17T+17T^2+T^3)\\ &\quad +30(1+23T+23T^2+T^3)+(4+6)(1+29T+29T^2+T^3).\end{aligned}$$

This has the nice coincidence that

$$\mathcal{N}_{U_{4,7}}(1,T) = 84(1+19T+19T^2+T^3).$$

Curiously, $(1,19,19,1)$ is the h-vector of the barycentric subdivision of the pyramid over a pentagon.

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