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Highest Weight Crystals for Schur Q-Functions

Eric Marberg*[1] and Kam Hung Tong†[1]

[1]*Department of Mathematics, Hong Kong University of Science and Technology*

Abstract. Work of Grantcharov *et al.* develops a theory of abstract crystals for the queer Lie superalgebra $\mathfrak{q}_n$. Such $\mathfrak{q}_n$-crystals form a monoidal category in which the connected normal objects have unique highest weight elements and characters that are Schur P-polynomials. We introduce a modified form of this category, with an extra crystal operator and a different tensor product, whose connected normal objects again have unique highest weight elements but now possess characters that are Schur Q-polynomials. The crystals in this category have some interesting features not present for ordinary $\mathfrak{q}_n$-crystals. For example, there is an action of the hyperoctahedral group exchanging highest and lowest weight elements. There are natural examples of normal $\mathfrak{q}_n$-crystal structures on shifted tableaux and factorized reduced words. We describe extended forms of these structures that give examples in our new category.

Keywords: crystals, Schur Q-functions, queer Lie superalgebras, shifted tableaux, involution words

1 Introduction

Crystals are an abstraction for the *crystal bases* of quantum group representations. Invented by Kashiwara [9, 10] and Lusztig [11, 12] in the 1990s, crystals may be viewed concretely as directed acyclic graphs with labeled edges, along with a map assigning weight vectors to each vertex, satisfying certain axioms. Isomorphisms of crystals correspond to weight-preserving graph isomorphisms, while subcrystals correspond to unions of weakly connected graph components.

For each finite-dimensional Lie superalgebra $\mathfrak{g}$ there is a category of (abstract) $\mathfrak{g}$-crystals. The structure of $\mathfrak{g}$ imposes different requirements for the weight map and edge labels. These categories have some common features. There is always a direct sum $\oplus$ operation for crystals corresponding to the disjoint union of directed graphs. There is also a more subtle notion of a crystal tensor product $\otimes$. There is also a character map ch assigning to each finite crystal its weight-generating function. Finally, there is a *standard crystal* $\mathbb{B}$ corresponding to the vector representation of the quantum group $U_q(\mathfrak{g})$.

*emarberg@ust.hk. Supported by RGC grants ECS 26305218 and GRF 16306120.
†khtongad@connect.ust.hk

These ingredients are enough to define a full subcategory of *normal $\mathfrak{g}$-crystals*: this consists of the $\mathfrak{g}$-crystals whose connected components are each isomorphic to a subcrystal of $\mathbb{B}^{\otimes m}$ for some $m \geq 0$. Such crystals form the smallest monoidal subcategory containing the standard crystal that is closed under isomorphisms, direct sums, and passage to subcrystals.

Defined in this way, the normal $\mathfrak{g}$-crystals are typically the abstract $\mathfrak{g}$-crystals that correspond directly to crystal bases of finite-dimensional integrable $U_q(\mathfrak{g})$-modules. This connection implies some desirable properties: for example, that each connected normal crystal has a unique *highest weight element* whose weight determines the crystal's isomorphism class.

Sections 2.1 and 2.2 review how this works in two cases that have been well-studied, when $\mathfrak{g} = \mathfrak{gl}_n$ is the complex general linear Lie algebra and when $\mathfrak{g} = \mathfrak{q}_n$ is the *queer Lie superalgebra*. Section 2.3 contains our main results, which establish similar formal properties of a new category of what we call *$\mathfrak{q}_n^+$-crystals*. In the categories of normal $\mathfrak{gl}_n$- and $\mathfrak{q}_n$-crystals, the connected objects have characters that are *Schur polynomials* and *Schur P-polynomials*, respectively. In our new category of normal $\mathfrak{q}_n^+$-crystals, the connected objects have characters *Schur Q-polynomials*.

Schur P- and Q-polynomials were first defined in work of Schur on the projective representations of the symmetric group but have applications in many other areas; see Definition 3.4. One application of $\mathfrak{gl}_n$- and $\mathfrak{q}_n$-crystals in combinatorics is to show that certain power series are *Schur positive* and *Schur P-positive*. A similar application of $\mathfrak{q}_n^+$ crystals is to demonstrate the stronger property of *Schur Q-positivity*.

2 Results

Throughout, $n \geq 2$ is a fixed positive integer and $[n] := \{1, 2, \dots, n\}$.

2.1 Crystals for Schur functions

Let $\mathcal{B}$ be a nonempty set with a *weight function* $\mathrm{wt}\colon \mathcal{B} \to \mathbb{Z}^n$ and an auxiliary element $0 \notin \mathcal{B}$. For each $i \in [n-1]$, assume that maps $e_i, f_i \colon \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ are given. We define $\varepsilon_i, \varphi_i \colon \mathcal{B} \to \mathbb{N} \sqcup \{\infty\}$ by $\varepsilon_i(b) := \max\left\{k \in \mathbb{N} : e_i^k(b) \neq 0\right\}$ and $\varphi_i(b) := \max\left\{k \in \mathbb{N} : f_i^k(b) \neq 0\right\}$. Write $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ for the standard basis of $\mathbb{Z}^n$.

Definition 2.1. The set $\mathcal{B}$ is an *(abstract) $\mathfrak{gl}_n$-crystal* if for all $b, c \in \mathcal{B}$ and $i \in [n-1]$:

(S1) One has $e_i(b) = c$ if and only if $f_i(c) = b$, in which case $\mathrm{wt}(c) - \mathrm{wt}(b) = \mathbf{e}_i - \mathbf{e}_{i+1}$.

(S2) Both $\varepsilon_i(b)$ and $\varphi_i(b)$ are finite and $\varphi_i(b) - \varepsilon_i(b) = \mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1}$.

The maps e_i and f_i are called *raising* and *lowering (crystal) operators*. An *isomorphism* of $\mathfrak{gl}_n$-crystals (or any of the other types of crystal appearing later) is a weight-preserving bijection that commutes with all crystal operators.

The *crystal graph* of $\mathcal{B}$ is the directed graph with vertex set $\mathcal{B}$ and edges $b \xrightarrow{i} c$ whenever $c = f_i(b)$. A weakly connected component of this graph is called a *full subcrystal*.

Example 2.2. The *standard $\mathfrak{gl}_n$-crystal* $\mathbb{B}_n$ has crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

and weight map $\mathrm{wt}(\boxed{i}) := \mathbf{e}_i$ for $i \in [n]$.

If $\mathcal{B}$ and $\mathcal{C}$ are nonempty sets then define $\mathcal{B} \otimes \mathcal{C} := \{b \otimes c : b \in \mathcal{B},\ c \in \mathcal{C}\}$. An essential feature of all categories of crystals is the existence of a nontrivial tensor product.

Theorem 2.3 (See [2, §2.3])**.** *Let $\mathcal{B}$ and $\mathcal{C}$ be $\mathfrak{gl}_n$-crystals. Then $\mathcal{B} \otimes \mathcal{C}$ has a unique $\mathfrak{gl}_n$-crystal structure with weight map $\mathrm{wt}(b \otimes c) := \mathrm{wt}(b) + \mathrm{wt}(c)$ and lowering operators*

$$f_i(b \otimes c) := \begin{cases} b \otimes f_i(c) & \text{if } \varepsilon_i(b) < \varphi_i(c) \\ f_i(b) \otimes c & \text{if } \varepsilon_i(b) \geq \varphi_i(c) \end{cases}$$

where it is understood that $b \otimes 0 = 0 \otimes c = 0$. If $\mathcal{D}$ is another $\mathfrak{gl}_n$-crystal then the map $(b \otimes c) \otimes d \mapsto b \otimes (c \otimes d)$ is a $\mathfrak{gl}_n$-crystal isomorphism $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} \cong \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$.

Let $\mathbb{1}$ be a $\mathfrak{gl}_n$-crystal with a single element, whose weight is $0 \in \mathbb{Z}^n$. A $\mathfrak{gl}_n$-crystal is *normal* if each of its full subcrystals is isomorphic to a full subcrystal of $\mathbb{B}_n^{\otimes m}$ for some $m \in \mathbb{N} := \{0, 1, 2, \ldots\}$, where we identify $\mathbb{B}_n^0$ with $\mathbb{1}$.

The *character* of a finite $\mathfrak{gl}_n$-crystal $\mathcal{B}$ is the Laurent polynomial $\mathsf{ch}(\mathcal{B}) := \sum_{b \in \mathcal{B}} x^{\mathrm{wt}(b)}$ where $x^{\mathrm{wt}(b)} := \prod_{i \in [n]} x_i^{\mathrm{wt}(b)_i}$. If $\mathcal{B}$ is a $\mathfrak{gl}_n$-crystal then a *$\mathfrak{gl}_n$-highest* (respectively, *$\mathfrak{gl}_n$-lowest*) *weight element* $b \in \mathcal{B}$ is an element with $e_i(b) = 0$ (respectively, $f_i(b) = 0$) for all i. The following theorem is well-known and the prototype for subsequent results.

Theorem 2.4 (See [2, Theorems 3.2 and 8.6])**.** *If $\mathcal{B}$ is a connected normal $\mathfrak{gl}_n$-crystal, then $\mathcal{B}$ has a unique $\mathfrak{gl}_n$-highest weight element, whose weight $\lambda \in \mathbb{Z}^n$ is a partition such that $\mathsf{ch}(\mathcal{B}) = s_\lambda(x_1, x_2, \ldots, x_n)$ is a Schur polynomial. For each partition $\lambda \in \mathbb{Z}^n$, there is a connected normal $\mathfrak{gl}_n$-crystal with highest weight λ, and finite normal $\mathfrak{gl}_n$-crystals with the same character are isomorphic.*

2.2 Crystals for Schur P-functions

Suppose $\mathcal{B}$ is a $\mathfrak{gl}_n$-crystal with maps $e_{\overline{1}}, f_{\overline{1}} \colon \mathcal{B} \to \mathcal{B} \sqcup \{0\}$, to be called the *queer raising and lowering operators*. The definitions of $\varepsilon_i, \varphi_i \colon \mathcal{B} \to \mathbb{N} \sqcup \{\infty\}$ extend to when $i = \overline{1}$. The following definition is a slight variation of [4, Def. 1.9]:

Definition 2.5. The $\mathfrak{gl}_n$-crystal $\mathcal{B}$ is an *(abstract) $\mathfrak{q}_n$-crystal* if for all $b, c \in \mathcal{B}$:

(P1) One has $e_{\overline{1}}(b) = c$ if and only if $f_{\overline{1}}(c) = b$, in which case $\mathrm{wt}(c) - \mathrm{wt}(b) = \mathbf{e}_1 - \mathbf{e}_2$ as well as $\varepsilon_i(b) = \varepsilon_i(c)$ and $\varphi_i(b) = \varphi_i(c)$ for all $i \in \{3, 4, \dots, n-1\}$.

(P2) If $i \in \{3, 4, \dots, n-1\}$ then e_i and f_i commute with $e_{\overline{1}}$ and $f_{\overline{1}}$.

(P3) If $\mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0$ then $(\varepsilon_{\overline{1}} + \varphi_{\overline{1}})(b) = 0$, and otherwise $(\varepsilon_{\overline{1}} + \varphi_{\overline{1}})(b) = 1$.

The *crystal graph* of a $\mathfrak{q}_n$-crystal has additional edges $b \xrightarrow{\overline{1}} c$ whenever $c = f_{\overline{1}}(b)$. A weakly connected component of this graph is called a *full $\mathfrak{q}_n$-subcrystal*.

Example 2.6. The *standard $\mathfrak{q}_n$-crystal* $\mathbb{B}_n$ has crystal graph

$$\boxed{1} \underset{1}{\overset{\overline{1}}{\rightrightarrows}} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

and weight map $\mathrm{wt}(\boxed{i}) := \mathbf{e}_i$ for $i \in [n]$.

Grantcharov *et al.* prove the following in [4, 5]. Our description of the tensor product here follows the "anti-Kashiwara" convention, which is opposite that of [4, 5].

Theorem 2.7 (See [4, 5])**.** *Suppose $\mathcal{B}$ and $\mathcal{C}$ are $\mathfrak{q}_n$-crystals. Then the $\mathfrak{gl}_n$-crystal $\mathcal{B} \otimes \mathcal{C}$ has a unique $\mathfrak{q}_n$-crystal structure with queer lowering operator*

$$f_{\overline{1}}(b \otimes c) := \begin{cases} b \otimes f_{\overline{1}}(c) & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0, \\ f_{\overline{1}}(b) \otimes c & \text{otherwise.} \end{cases}$$

If $\mathcal{D}$ is another $\mathfrak{q}_n$-crystal then the bijection $(b \otimes c) \otimes d \mapsto b \otimes (c \otimes d)$ is a $\mathfrak{q}_n$-crystal isomorphism $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} \cong \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$.

A $\mathfrak{q}_n$-crystal (exactly like $\mathfrak{gl}_n$-crystals) is *normal* if each of its full $\mathfrak{q}_n$-subcrystals is isomorphic to a full $\mathfrak{q}_n$-subcrystal of $\mathbb{B}_n^{\otimes m}$ for some $m \in \mathbb{N}$. The notion of highest and lowest weights for $\mathfrak{q}_n$-crystals is more involved than for $\mathfrak{gl}_n$-crystals, however.

Let $\mathcal{B}$ be a $\mathfrak{q}_n$-crystal. Fix $i \in [n-1]$ and $b \in \mathcal{B}$ and let $k := \varphi_i(b) - \varepsilon_i(b)$. Define a map $\sigma_i \colon \mathcal{B} \to \mathcal{B}$ by setting $\sigma_i(b) := e_i^{-k}(b)$ if $k \leq 0$ and $\sigma_i(b) := f_i^k(b)$ if $k \geq 0$. Inductively define $e_{\overline{i}} \colon \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for $i \in \{2, 3, \dots, n-1\}$ by $e_{\overline{i}} := \sigma_{i-1}\sigma_i e_{\overline{i-1}} \sigma_i \sigma_{i-1}$ and set

$$\sigma_{w_0} := (\sigma_1)(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1) \cdots (\sigma_{n-1} \cdots \sigma_2\sigma_1).$$

A *$\mathfrak{q}_n$-highest weight element* $b \in \mathcal{B}$ is an element with $e_i(b) = 0$ for $i = \overline{1}, 1, \dots, \overline{n-1}, n-1$. A *$\mathfrak{q}_n$-lowest weight element* $b \in \mathcal{B}$ is an element such that $\sigma_{w_0}(b)$ is $\mathfrak{q}_n$-highest weight.

The following contains several results in [4]; see, *e.g.*, [4, Theorem 2.5 and Corollary 4.6].

Theorem 2.8 ([4])**.** *If $\mathcal{B}$ is a connected normal $\mathfrak{q}_n$-crystal, then $\mathcal{B}$ has a unique $\mathfrak{q}_n$-highest weight element, whose weight $\lambda \in \mathbb{N}^n$ is a strict partition such that $\mathrm{ch}(\mathcal{B}) = P_\lambda(x_1, x_2, \dots, x_n)$ is a Schur P-polynomial. For each strict partition $\lambda \in \mathbb{N}^n$, there is a connected normal $\mathfrak{q}_n$-crystal with highest weight λ, and finite normal $\mathfrak{q}_n$-crystals with the same character are isomorphic.*

2.3 Crystals for Schur Q-functions

This section contains our main new results. Suppose $\mathcal{B}$ is a $\mathfrak{q}_n$-crystal with additional maps $e_0, f_0 \colon \mathcal{B} \to \mathcal{B} \sqcup \{0\}$. The definitions of $\varepsilon_i, \varphi_i \colon \mathcal{B} \to \mathbb{N} \sqcup \{\infty\}$ extend to when $i = 0$.

Definition 2.9. The $\mathfrak{q}_n$-crystal $\mathcal{B}$ is an *(abstract) $\mathfrak{q}_n^+$-crystal* if for all $b, c \in \mathcal{B}$:

(Q1) One has $e_0(b) = c$ if and only if $f_0(c) = b$, in which case $\mathrm{wt}(b) = \mathrm{wt}(c)$ as well as $\varepsilon_i(b) = \varepsilon_i(c)$ and $\varphi_i(b) = \varphi_i(c)$ for all $i \in [n-1]$ and also for $i = \overline{1}$ if $n \geq 2$.

(Q2) If $i \in \{2, 3, \dots, n-1\}$ then e_i and f_i commute with e_0 and f_0.

(Q3) If $\mathrm{wt}(b)_1 = 0$ then $(\varepsilon_0 + \varphi_0)(b) = 0$, and otherwise $(\varepsilon_0 + \varphi_0)(b) = 1$.

We draw the *crystal graph* of a $\mathfrak{q}_n^+$-crystal by adding arrows $b \xrightarrow{0} f_0(b) \neq 0$ to the $\mathfrak{q}_n$-crystal graph. This graph's weakly connected components are called *full $\mathfrak{q}_n^+$-subcrystals*.

Example 2.10. The *standard $\mathfrak{q}_n^+$-crystal* $\mathbb{B}_n^+$ has crystal graph

$$
\begin{tikzcd}
\boxed{1} \arrow[r, dashed, shift left, "\overline{1}"] \arrow[r, shift right, "1"'] \arrow[d, dotted, "0"'] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[r, "3"] & \cdots \arrow[r, "n-1"] & \boxed{n} \\
\boxed{1'} \arrow[r, dashed, shift left, "\overline{1}"] \arrow[r, shift right, "1"'] & \boxed{2'} \arrow[r, "2"] & \boxed{3'} \arrow[r, "3"] & \cdots \arrow[r, "n-1"] & \boxed{n'}
\end{tikzcd}
$$

and weight map $\mathrm{wt}(\boxed{i}) = \mathrm{wt}(\boxed{i'}) = \mathbf{e}_i$ for $i \in [n]$.

The tensor product for $\mathfrak{q}_n^+$-crystals is more complicated than for $\mathfrak{q}_n$-crystals.

Theorem 2.11. *Let $\mathcal{B}$ and $\mathcal{C}$ be $\mathfrak{q}_n^+$-crystals. Then the $\mathfrak{gl}_n$-crystal $\mathcal{B} \otimes \mathcal{C}$ has a unique $\mathfrak{q}_n^+$-crystal structure with lowering operators*

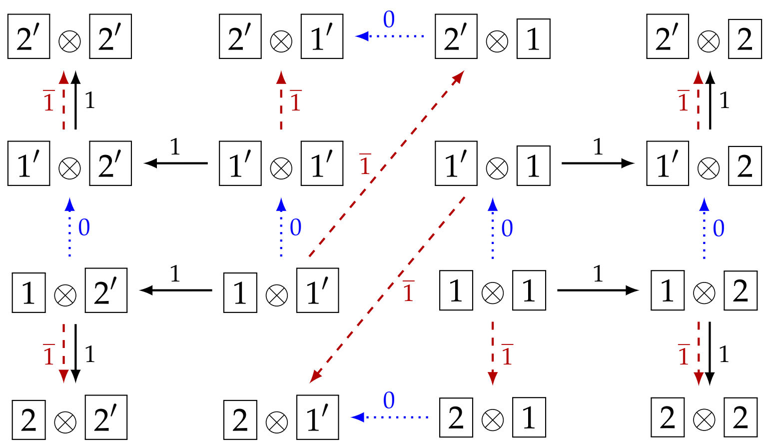
$$f_0(b \otimes c) := \begin{cases} f_0(b) \otimes c & \text{if } \mathrm{wt}(b)_1 \neq 0, \\ b \otimes f_0(c) & \text{if } \mathrm{wt}(b)_1 = 0, \end{cases}$$

and

$$f_{\overline{1}}(b \otimes c) := \begin{cases} b \otimes f_{\overline{1}}(c) & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0, \\ f_{\overline{1}} f_0(b) \otimes e_0(c) & \text{if } \mathrm{wt}(b)_1 = 1,\ f_{\overline{1}} f_0(b) \neq 0, \text{ and } e_0(c) \neq 0, \\ f_{\overline{1}} e_0(b) \otimes f_0(c) & \text{if } \mathrm{wt}(b)_1 = 1,\ f_{\overline{1}} e_0(b) \neq 0, \text{ and } f_0(c) \neq 0, \\ f_{\overline{1}}(b) \otimes c & \text{otherwise.} \end{cases}$$

If $\mathcal{D}$ is another $\mathfrak{q}_n^+$-crystal then the bijection $(b \otimes c) \otimes d \mapsto b \otimes (c \otimes d)$ is a $\mathfrak{q}_n^+$-crystal isomorphism $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} \cong \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$.

Example 2.12. The crystal graph of $\mathbb{B}_2^+ \otimes \mathbb{B}_2^+$ is



There are two full $\mathfrak{q}_2^+$-subcrystals, which are isomorphic.

A $\mathfrak{q}_n^+$-crystal is *normal* if each of its full $\mathfrak{q}_n^+$-subcrystals is isomorphic to a full $\mathfrak{q}_n^+$-subcrystal of $(\mathbb{B}_n^+)^{\otimes m}$ for some $m \in \mathbb{N}$, where $(\mathbb{B}_n^+)^0 := \mathbb{1}$.

Example 2.13. For $i \in \mathbb{Z}$ set $i' := i - \frac{1}{2}$. Define $\mathcal{W}_n^+(m)$ to be the set of words of length m with all letters in $\{1' < 1 < 2' < 2 < \cdots < n' < n\}$. We identify $\mathcal{W}_n^+(m) \cong (\mathbb{B}_n^+)^{\otimes m}$ by viewing $w = w_1 w_2 \cdots w_m$ as $\boxed{w_1} \otimes \boxed{w_2} \otimes \cdots \otimes \boxed{w_m}$. This gives $\mathcal{W}_n^+(m)$ a normal $\mathfrak{q}_n^+$-crystal structure, which we describe below. It suffices to explain the f_i operators.

Let $w = w_1 w_2 \cdots w_m \in \mathcal{W}_n^+(m)$ and $i \in [n-1]$. Consider the word formed by replacing each letter $w_k \in \{i', i\}$ by a right parenthesis ")" and each letter $w_j \in \{i+1, i+1'\}$ by a left parenthesis "(". The word $f_i(w)$ is constructed from w by adding 1 to the letter in w in the position of the last unpaired ")"; if no such position exists then $f_i(w) = 0$. For example, $f_1(131'22'131'2) = 132'22'131'2$.

To describe $f_{\overline{1}}(w)$, let $j \in [m]$ be minimal with $w_j \in \{1', 1\}$. If no such j exists or $w_i \in \{2, 2'\}$ for some $i \in [j-1]$, then $f_{\overline{1}}(w) = 0$. Otherwise let $k \in \{j+1, j+2, \dots, m\}$ be minimal with $w_k \in \{1', 1\}$. If no such k exists or if $w_j = w_k$ then $f_{\overline{1}}(w)$ is formed from w by adding 1 to w_j. If $w_j = 1$ and $w_k = 1'$ then $f_{\overline{1}}(w)$ is formed from w by changing w_j to $2'$ and w_k to 1. If $w_j = 1'$ and $w_k = 1$ then $f_{\overline{1}}(w)$ is formed from w by changing w_j to 2 and w_k to $1'$. For example, $f_{\overline{1}}(31'21'1) = 32'21'1$ and $f_{\overline{1}}(3121'1) = 32'211$.

Finally, to describe $f_0(w)$, let $j \in [m]$ be minimal with $w_j \in \{1', 1\}$. If no such j exists or $w_j = 1'$ then $f_0(w) = 0$. Otherwise $f_0(w)$ is formed from w by changing w_j to $1'$. For example, $f_0(3121'1) = 31'21'1$ and $f_0(31'21'1) = 0$.

Assume $\mathcal{B}$ is a $\mathfrak{q}_n^+$-crystal. For each $i \in [n]$ let $e_0^{[i]} := \sigma_{i-1} \cdots \sigma_2 \sigma_1 e_0 \sigma_1 \sigma_2 \cdots \sigma_{i-1}$ and $f_0^{[i]} := \sigma_{i-1} \cdots \sigma_2 \sigma_1 f_0 \sigma_1 \sigma_2 \cdots \sigma_{i-1}$. A *$\mathfrak{q}_n^+$-highest weight element* $b \in \mathcal{B}$ is a $\mathfrak{q}_n$-highest weight element with $e_0^{[i]}(b) = 0$ for all $i \in [n]$. A *$\mathfrak{q}_n^+$-lowest weight element* $b \in \mathcal{B}$ is a $\mathfrak{q}_n$-lowest weight element with $f_0^{[i]}(b) = 0$ for all $i \in [n]$.

Since $\mathbb{B}_n^+ \cong \mathbb{B}_n \sqcup \mathbb{B}_n$ as $\mathfrak{gl}_n$-crystals, a normal $\mathfrak{q}_n^+$-crystal is normal as a $\mathfrak{gl}_n$-crystal. However, normal $\mathfrak{q}_n^+$-crystals are not always normal as $\mathfrak{q}_n$-crystals. This means that results like Theorem 2.8 do not directly imply similar properties of normal $\mathfrak{q}_n^+$-crystals. In the particular, the proof of the following extension of Theorem 2.8 is nontrivial:

Theorem 2.14. *If $\mathcal{B}$ is a connected normal $\mathfrak{q}_n^+$-crystal, then $\mathcal{B}$ has a unique $\mathfrak{q}_n^+$-highest weight element, whose weight $\lambda \in \mathbb{N}^n$ is a strict partition with $\mathsf{ch}(\mathcal{B}) = Q_\lambda(x_1, x_2, \dots, x_n)$ is a Schur Q-polynomial. For each strict partition $\lambda \in \mathbb{N}^n$, there is a connected normal $\mathfrak{q}_n^+$-crystal with highest weight λ, and finite normal $\mathfrak{q}_n^+$-crystals with the same character are isomorphic.*

If $\mathcal{B}$ is a normal $\mathfrak{gl}_n$-crystal then there is a unique action of the symmetric group S_n on $\mathcal{B}$ in which the simple transposition $s_i = (i, i+1)$ acts as σ_i. The reverse permutation $w_0 = n \cdots 321 \in S_n$ acts as σ_{w_0} and interchanges highest and lowest weight elements.

On normal $\mathfrak{q}_n^+$-crystals, there is a natural action of hyperoctahedral group extending this S_n-action. This is an interesting feature of $\mathfrak{q}_n^+$-crystals that is not present for $\mathfrak{q}_n$-crystals. Suppose $\mathcal{B}$ is a $\mathfrak{q}_n^+$-crystal. The formula for $\sigma_i \colon \mathcal{B} \to \mathcal{B}$ makes sense when $i = 0$. Let W_n^{BC} denote the group whose elements are the permutations w of $\mathbb{Z}$ satisfying $w(-i) = -w(i)$ for all i and $w(i) = i$ for all $i > n$. This is the finite Coxeter group of type BC_n, with simple generators $t_0 := (-1, 1)$ and $t_i := (i, i+1)(-i, -i-1)$ for $i \in [n-1]$.

Theorem 2.15. *Let $\mathcal{B}$ be a normal $\mathfrak{q}_n^+$-crystal. There exists a unique action of W_n^{BC} on $\mathcal{B}$ in which t_0 and t_i for $i \in [n-1]$ act as σ_0 and σ_i, respectively. Moreover, the operator $\sigma_{w_0^+} := (\sigma_0)(\sigma_1\sigma_0)(\sigma_2\sigma_1\sigma_0)\cdots(\sigma_{n-1}\cdots\sigma_1\sigma_0)$ swaps $\mathfrak{q}_n^+$-highest and $\mathfrak{q}_n^+$-lowest weight elements.*

3 Constructions

Here, we describe explicit constructions realizing all connected normal $\mathfrak{q}_n^+$-crystals.

3.1 Crystal operators on increasing factorizations

Let $S_\mathbb{Z}$ be the group of permutations of $\mathbb{Z}$ that fix all but finitely many points. This is a Coxeter group generated by the simple transpositions $s_i = (i, i+1)$ for $i \in \mathbb{Z}$.

Fix $\pi \in S_\mathbb{Z}$. A *reduced word* for π is an integer sequence $a_1 a_2 \cdots a_l$ of shortest possible length such that $\pi = s_{a_1} s_{a_2} \cdots s_{a_l}$. Let $\mathcal{R}(\pi)$ denote the set of such words. For any set of words $\mathcal{W}$ let $\mathsf{Incr}_n(\mathcal{W})$ be the set of n-tuples $a = (a^1, a^2, \cdots, a^n)$ where each a^i is a strictly increasing (possibly empty) primed word such that $\mathsf{concat}(a) := a^1 a^2 \cdots a^n$ is in $\mathcal{W}$.

Morse and Schilling identify a $\mathfrak{gl}_n$-crystal structure on $\mathsf{Incr}_n(\mathcal{R}(\pi))$ in [16], which we recall below. The weight of $a = (a^1, a^2, \cdots, a^n)$ is $\mathrm{wt}(a) := (\ell(a^1), \ell(a^2), \dots, \ell(a^n))$.

Definition 3.1. Let v and w be strictly increasing words with letters in $\frac{1}{2}\mathbb{Z}$. Form a set of paired letters $\mathsf{pair}(v, w)$ by iterating over the letters in w from largest to smallest; at each iteration, the current letter w_j is paired with the smallest unpaired letter v_i with $\lceil v_i \rceil > \lceil w_j \rceil$ (if such a letter exists) and (v_i, w_j) is added to $\mathsf{pair}(v, w)$. If $v = 1, 3, 4, 5, 8, 10', 11$ and $w = 2', 6, 9, 12, 13$ (where $i' := i - \frac{1}{2}$) then $\mathsf{pair}(v, w) = \{(10', 9), (8, 6), (3, 2')\}$.

For each $i \in [n-1]$ let $f_i \colon \mathsf{Incr}_n(\mathcal{R}(\pi)) \to \mathsf{Incr}_n(\mathcal{R}(\pi)) \sqcup \{0\}$ be the operator given as follows. Let $a = (a^1, a^2, \dots, a^n) \in \mathsf{Incr}_n(\mathcal{R}(\pi))$. If every letter in a^i is the first term of an element of $\mathsf{pair}(a^i, a^{i+1})$ then $f_i(a) := 0$. Otherwise, let x be the largest unpaired letter in a^i, let y be the smallest integer not in a^{i+1} with $y \geq x$, and form $f_i(a)$ from a by removing x from a^i and adding y to a^{i+1} in the unique position that gives an increasing word.

A permutation is *vexillary* if it is 2143-avoiding. The *shape* $\lambda(\pi)$ of π is the transpose of the partition sorting the numbers $c_i := |\{j \in \mathbb{Z} : i < j \text{ and } \pi(j) < \pi(i)\}|$ for $i \in \mathbb{Z}$. The set $\mathsf{Incr}_n(\mathcal{R}(\pi))$ is nonempty if and only if $\lambda(\pi)$ has at most n parts, so is in $\mathbb{N}^n$.

Theorem 3.2 (See [16]). *For each $\pi \in S_\mathbb{Z}$, there exists a unique $\mathfrak{gl}_n$-crystal structure on $\mathsf{Incr}_n(\mathcal{R}(\pi))$ with the weight function and lowering operators f_i defined above. This $\mathfrak{gl}_n$-crystal is normal. Whenever π is vexillary and $\lambda(\pi) \in \mathbb{N}^n$, the $\mathfrak{gl}_n$-crystal $\mathsf{Incr}_n(\mathcal{R}(\pi))$ is connected with unique highest weight $\lambda(\pi)$.*

There is an extension of this result for $\mathfrak{q}_n^+$-crystals. The *Demazure product* is the unique associative product $\circ \colon S_\mathbb{Z} \times S_\mathbb{Z} \to S_\mathbb{Z}$ such that $\pi \circ s_i = \pi$ if $\pi(i) > \pi(i+1)$ and $\pi \circ s_i = \pi s_i$ if $\pi(i) < \pi(i+1)$ for each $i \in \mathbb{Z}$. Let $I_\mathbb{Z} := \{\pi \in S_\mathbb{Z} : \pi = \pi^{-1}\}$. If $z \in I_\mathbb{Z}$ and $i \in \mathbb{Z}$, then $s_i \circ z \circ s_i$ is either z when $z(i) > z(i+1)$, $zs_i = s_i z$ when $z(i) = i$ and $z(i+1) = i+1$, or $s_i z s_i$ otherwise. This implies that $I_\mathbb{Z} = \{\pi^{-1} \circ \pi : \pi \in S_\mathbb{Z}\}$.

An *involution word* for $z \in I_\mathbb{Z}$ is word $a_1 a_2 \cdots a_n$ of shortest possible length such that $z = s_{a_n} \circ \cdots \circ s_{a_2} \circ s_{a_1} \circ 1 \circ s_{a_1} \circ s_{a_2} \circ \cdots \circ s_{a_n}$. Write $\mathcal{R}_{\mathsf{inv}}(z)$ for the set of such words. A *commutation* for $a_1 a_2 \cdots a_n \in \mathcal{R}_{\mathsf{inv}}(z)$ is an index $i \in [n]$ such that both a_i and $1 + a_i$ are fixed points of the involution $s_{a_{i-1}} \circ \cdots \circ s_{a_2} \circ s_{a_1} \circ 1 \circ s_{a_1} \circ s_{a_2} \circ \cdots \circ s_{a_{i-1}}$. The number of commutations is the same for every involution word for $z \in I_\mathbb{Z}$.

A *primed word* is a word w with letters in $\frac{1}{2}\mathbb{Z} = \{\cdots < 1' < 1 < 2' < 2 < \cdots\}$, where we set $i' := i - \frac{1}{2}$. Form $\mathsf{unprime}(w)$ from a primed word w by adding $\frac{1}{2}$ to all primed letters. A *primed involution word* for $z \in I_\mathbb{Z}$ is a primed word whose unprimed form is in $\mathcal{R}_{\mathsf{inv}}(z)$ and whose primed letters occur at commutations. Write $\mathcal{R}^+_{\mathsf{inv}}(z)$ for the set of such words. For example, if $z = (1,3)(2,4)$ then $\mathcal{R}^+_{\mathsf{inv}}(z) = \{132, 13'2, 1'32, 1'3'2, 312, 31'2, 3'12, 3'1'2\}$.

Fix $z \in I_\mathbb{Z}$. For each $i \in [n-1]$ we define the lowering operator $f_i \colon \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z)) \to \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z)) \sqcup \{0\}$ as follows. Let $a = (a^1, a^2, \dots, a^n) \in \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z))$. If every letter in a^i is the first term of an element of $\mathsf{pair}(a^i, a^{i+1})$ then $f_i(a) := 0$. Otherwise, let $x \in \frac{1}{2}\mathbb{Z}$ be the largest unpaired letter in a^i, let $y \in \mathbb{Z}$ be the smallest integer not in $\mathsf{unprime}(a^{i+1})$ with $y \geq \lceil x \rceil$, and construct an n-tuple of strictly increasing words $f_i(a)$ by applying the following procedure to a:

- If x is primed then remove x from a^i and add y' to a^{i+1}:

$$a = (\dots, 13'459, 347', \dots) \mapsto (\dots, 1459, 345'7', \dots) = f_i(a).$$

- If x is unprimed then remove x from a^i, add y to a^{i+1}, and for each integer $x \leq v < y$ with $v+1 \in a^i$ and $v' \in a^{i+1}$, replace $v+1 \in a^i$ by $v+1'$ and $v' \in a^{i+1}$ by v:

$$a = (\dots, 134569, 34'58, \dots) \mapsto (\dots, 145'69, 34568, \dots) = f_i(a).$$

We also define a map $f_{\overline{1}} \colon \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z)) \to \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z)) \sqcup \{0\}$. If $a^1 = \varnothing$ or if the first letter of a^1 is not strictly smaller than every letter in a^2, then $f_{\overline{1}}(a) := 0$. If a^1 has at least two letters and the first two of these are not both primed or unprimed, then reverse the primes on these letters and move the modified first letter of a^1 to the start of a^2:

$$a = (1'34, 25, \dots) \mapsto (3'4, 125, \dots) = f_{\overline{1}}(a).$$

Otherwise, move the first letter of a^1 to the start of a^2:

$$a = (1'3'4, 25, \dots) \mapsto (3'4, 1'25, \dots) = f_{\overline{1}}(a).$$

Finally, let $f_0 \colon \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z)) \to \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z)) \sqcup \{0\}$ be the operator such that if a^1 is empty or begins with a primed letter then $f_0(a) := 0$, and otherwise $f_0(a)$ is formed from a by adding a prime to the first letter of a^1.

The *involution shape* of $z \in I_{\mathbb{Z}}$ is the transpose of the partition sorting the numbers $m_i := \{j \in \mathbb{Z} : z(j) \leq i < j \text{ and } z(j) < z(i)\}$ for $i \in \mathbb{Z}$. We denote this partition by $\mu(z)$. The set $\mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z))$ is nonempty if and only if $\mu(z)$ has at most n parts.

Theorem 3.3. *For each $z \in I_{\mathbb{Z}}$, there exists a unique $\mathfrak{q}^+_n$-crystal structure on $\mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z))$ with the weight function and lowering operators f_i defined above. This $\mathfrak{q}^+_n$-crystal is normal. The subset $\mathsf{Incr}_n(\mathcal{R}_{\mathsf{inv}}(z)) \subset \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z))$ is a union of full $\mathfrak{q}_n$-subcrystals, so inherits a $\mathfrak{q}_n$-crystal structure, which is also normal. When $z \in I_{\mathbb{Z}}$ is vexillary and $\mu(z) \in \mathbb{N}^n$, both $\mathsf{Incr}_n(\mathcal{R}_{\mathsf{inv}}(z))$ and $\mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z))$ are connected (as $\mathfrak{q}_n$- and $\mathfrak{q}^+_n$-crystals, respectively) with unique highest weight $\mu(z)$, which is always a strict partition.*

The $\mathfrak{q}_n$-crystal $\mathsf{Incr}_n(\mathcal{R}_{\mathsf{inv}}(z))$ has been studied previously in [8, 15]. The complementary subset $\mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z)) \setminus \mathsf{Incr}_n(\mathcal{R}_{\mathsf{inv}}(z))$ is another union of full $\mathfrak{q}_n$-crystals, but this is not typically normal or connected as a $\mathfrak{q}_n$-crystal.

3.2 Crystal operators on shifted tableaux

Assume $\mu = (\mu_1 > \mu_2 > \cdots \geq 0)$ is a strict partition. Let $\ell(\mu) = |\{i > 0 : \mu_i > 0\}|$. The *shifted diagram* of μ is the set $\mathsf{SD}_\mu := \{(i, i+j-1) : i \in [\ell(\mu)] \text{ and } j \in [\mu_i]\}$. A *shifted tableau* of shape μ is a map $\mathsf{SD}_\mu \to \frac{1}{2}\mathbb{Z}$. We draw tableaux in French notation, so that

$$T = \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 2' & 2 & 4' \\ \hline 1' & 1 & 1 & 4' \\ \hline \end{array} \tag{3.1}$$

is a shifted tableau of shape $\mu = (4, 3)$. The *(main) diagonal* of a shifted tableau is the set of positions (i, j) in its domain with $i = j$.

A shifted tableau is *semistandard* if its entries are all positive and its rows and columns are weakly increasing, such that no primed entry is repeated in any row and no unprimed entry is repeated in any column. The example in (3.1) is semistandard. We write $\mathsf{ShTab}^+_n(\mu)$ for the set of semistandard shifted tableaux of shape μ with all entries in $\{1' < 1 < \cdots < n' < n\}$, and $\mathsf{ShTab}_n(\mu)$ for the subset of elements in $\mathsf{ShTab}^+_n(\mu)$ with no primed entries on the diagonal.

Let $\mathrm{wt}(T) := (a_1, a_2, \ldots, a_n)$ where a_i is the number of entries in T equal to i' or i.

Definition 3.4. The *Schur P- and Schur Q-polynomials* of a strict partition $\mu \in \mathbb{N}^n$ are

$$P_\mu(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathsf{ShTab}_n(\mu)} x^{\mathrm{wt}(T)} \quad \text{and} \quad Q_\mu(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathsf{ShTab}^+_n(\mu)} x^{\mathrm{wt}(T)}.$$

These polynomials are both symmetric. As μ varies, they are linearly independent.

Fix $z \in I_{\mathbb{Z}}$. The correspondence defined below was introduced in [14]. Here, when we refer to "interchanging the primes" on two elements of $\frac{1}{2}\mathbb{Z}$, we mean the operation that adds a prime to one number while removing the prime from the other if the two are not both primed or unprimed, and otherwise leaves both numbers unchanged.

Definition 3.5. Suppose $a \in \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z))$ and $w = w_1 w_2 \cdots w_m = \mathsf{concat}(a)$. Let $\emptyset = T_0, T_1, \ldots, T_m$ be the sequence of shifted tableaux in which T_i for $i \in [m]$ is formed by inserting w_i into T_{i-1} according to the following procedure:

1. Start by inserting w_i into the first row. At each stage, an entry x is inserted into a row or column. Let y and $\tilde{y}$ be the first entries in the same row or column with $\lceil x \rceil \le \lceil y \rceil$ and $\lceil x \rceil < \lceil \tilde{y} \rceil$.

2. If no such entries exist then x is added to the end of the row or column, with the exception that if x is added to a diagonal position then its value is changed to $\lceil x \rceil$. The process to form T_i *ends in column insertion* if we are inserting into a column or if $\lceil x \rceil \neq x$ is added to the diagonal. Otherwise, the process *ends in row insertion*.

3. If $y \neq \tilde{y}$ then the primes on these entries are interchanged and $x + 1$ is inserted into the next column (if y is on the diagonal) or the next row (otherwise). If $y = \tilde{y}$ is off the diagonal then x replaces y and y is inserted into the next row. If $y = \tilde{y}$ is on the diagonal then $\lceil x \rceil$ replaces y and $y - (\lceil x \rceil - x)$ is inserted into the next column.

Define $P^{\mathsf{O}}_{\mathsf{EG}}(a) := T_m$ and construct $Q^{\mathsf{O}}_{\mathsf{EG}}(a)$ as the shifted tableau with the same shape as $P^{\mathsf{O}}_{\mathsf{EG}}(a)$ that contains j (respectively, j') in the box added to T_{i-1} to form T_i if w_i is in the jth factor of a and the insertion process ends in row (respectively, column) insertion.

Example 3.6. If $a = (4, 1'35, \varnothing, 4', \varnothing, 2)$ then $P^{\mathsf{O}}_{\mathsf{EG}}(a)$ and $Q^{\mathsf{O}}_{\mathsf{EG}}(a)$ are computed as follows:

$$\varnothing \rightsquigarrow \begin{array}{|c|}\hline 4 \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1 & 4' \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 \\ \hline 1 & 3 \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 5 \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 4 & 5 \\ \hline 1 & 3 & 4' \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 3 & 5' & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 & 5 \\ \hline\end{array} = P^{\mathsf{O}}_{\mathsf{EG}}(a),$$

$$\varnothing \rightsquigarrow \begin{array}{|c|}\hline 1 \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1 & 2' \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 2' \\ \hline 1 & 2' \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 2' & \multicolumn{1}{c}{} \\ \hline 1 & 2' & 2 \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 2' & 4 \\ \hline 1 & 2' & 2 \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 2' & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2' & 2 & 6' \\ \hline\end{array} = Q^{\mathsf{O}}_{\mathsf{EG}}(a).$$

The map $a \mapsto (P^{\mathsf{O}}_{\mathsf{EG}}(a), Q^{\mathsf{O}}_{\mathsf{EG}}(a))$ is called *orthogonal Edelman–Greene insertion* in [14]. It is a shifted version of the *Edelman–Greene correspondence* from [3], and the counterpart to a "symplectic" insertion algorithm studied in [8, 13, 15]. Restricted to $\mathsf{Incr}_n(\mathcal{R}_{\mathsf{inv}}(z)) \subsetneq \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z))$, the map is a special case of *shifted Hecke insertion* from [17].

The *row reading word* of a shifted tableau is the word given by concatenating the rows of in reverse order (so, starting with the last row). The map $a \mapsto (P^{\mathsf{O}}_{\mathsf{EG}}(a), Q^{\mathsf{O}}_{\mathsf{EG}}(a))$ is a bijection from $\mathsf{Incr}^+_n(z)$ to the set of pairs (P, Q) of shifted tableaux of the same shape, in which Q is semistandard with all entries at most n, and P has strictly increasing rows and columns, no primes on the diagonal, row reading word in $\mathcal{R}^+_{\mathsf{inv}}(z)$ [14, Corollary 3.9].

We prove some further results connecting $a \mapsto (P^{\mathsf{O}}_{\mathsf{EG}}(a), Q^{\mathsf{O}}_{\mathsf{EG}}(a))$ to crystals. To start:

Theorem 3.7. *The subsets on which $a \mapsto P^{\mathsf{O}}_{EG}(a)$ is constant are*

- *the full $\mathfrak{q}_n$-subcrystals of $\mathsf{Incr}_n(\mathcal{R}_{\mathsf{inv}}(z))$ and*
- *the full $\mathfrak{q}^+_n$-subcrystals of $\mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z))$.*

A *quasi-isomorphism* between $\mathfrak{gl}_n$-, $\mathfrak{q}_n$-, or $\mathfrak{q}^+_n$-crystals, respectively, is a map $\psi \colon \mathcal{B} \to \mathcal{C}$ such that for each full subcrystal $\mathcal{X} \subset \mathcal{B}$, the image $\mathcal{Y} := \psi(\mathcal{X})$ is a full subcrystal of $\mathcal{C}$ and the restricted map $\psi \colon \mathcal{X} \to \mathcal{Y}$ is a crystal isomorphism.

Theorem 3.8. *There exists a unique $\mathfrak{q}^+_n$-crystal structure on $\mathsf{ShTab}^+_n(\mu)$ for each strict partition $\mu \in \mathbb{N}^n$ such that the map $a \mapsto Q^{\mathsf{O}}_{EG}(a)$ is quasi-isomorphism of $\mathfrak{q}^+_n$-crystals*

$$\bigsqcup_{z \in I_{\mathbb{Z}}} \mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z)) \to \bigsqcup_{\text{strict partitions } \mu \in \mathbb{N}^n} \mathsf{ShTab}^+_n(\mu).$$

The $\mathfrak{q}^+_n$-crystal $\mathsf{ShTab}^+_n(\mu)$ is connected and normal with unique highest weight μ. For each strict partition $\mu \in \mathbb{N}^n$, the subset $\mathsf{ShTab}_n(\mu) \subset \mathsf{ShTab}^+_n(\mu)$ is a full $\mathfrak{q}_n$-subcrystal, so inherits a $\mathfrak{q}_n$-crystal structure which is also connected and normal with unique highest weight μ.

This result is highly nontrivial, since the preimage of any given $\mathsf{ShTab}^+_n(\mu)$ under $Q^{\mathsf{O}}_{\mathsf{EG}}$ will intersect $\mathsf{Incr}_n(\mathcal{R}^+_{\mathsf{inv}}(z))$ for infinitely many different $z \in I_{\mathbb{Z}}$. The $\mathfrak{q}_n$-crystal $\mathsf{ShTab}_n(\mu)$ coincides with the one studied in [1, 6, 7]. The results in [1, 6, 7] give explicit formulas for the relevant $\mathfrak{q}_n$-crystal operators. There are similarly explicit, slightly more complicated formulas for the $\mathfrak{q}^+_n$-crystal operators on $\mathsf{ShTab}^+_n(\mu)$. This information will appear in the full-length version of this extended abstract.

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