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Cluster Duality for Lagrangian and Orthgonal Grassmannians

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Abstract. In [8] Rietsch and Williams relate cluster structures and mirror symmetry for Grassmannians $\mathrm{Gr}(k,n)$, and use this to construct Newton–Okounkov bodies and associated toric degenerations. In this article we define a cluster seed for the Lagrangian Grassmannian, and prove that the associated Newton–Okounkov body agrees up to unimodular equivalence with a polytope obtained from the superpotential defined by Pech and Rietsch on the mirror Orthogonal Grassmannian in [5].

Keywords: Cluster algebra, Newton–Okounkov body, superpotential

1 Introduction

Let $\mathbb{X} = \mathrm{LGr}(n,2n)$ be the variety of n-dimensional Lagrangian subspaces of $\mathbb{C}^{2n}$ with respect to the symplectic form $\omega_{ij} = (-1)^j \delta_{i,2n+1-j}$. $\mathbb{X}$ is a *homogeneous space* of type C, *i.e.* it can be written as Sp_{2n}/P for a parabolic subgroup $P \subset \mathrm{Sp}_{2n}$. We consider its embedding as a subvariety of $\mathrm{Gr}(n,2n)$ in its Plücker embedding $\mathrm{Gr}(n,2n) \hookrightarrow \mathbb{P}(\wedge^n \mathbb{C}^{2n})$. We will index Plücker coordinates on $\mathrm{Gr}(n,2n)$ by elements of $\binom{[2n]}{n}$, the set of n-subsets of $[2n] := \{1,2,\dots,2n\}$, or alternatively by Young diagrams $\lambda \subset n \times n$ fitting inside the $n \times n$ square. $\mathbb{X}$ has dimension $N :- \binom{n+1}{2}$, and a distinguished anticanonical divisor $D_{ac} = D_0 + \cdots + D_n$ made up of the $n+1$ hyperplanes $D_i = \{p_{n\times i} = 0\} = \{p_{(n-i+1)\dots(2n-i)}\}$, where $n \times i$ denotes the corresponding Young diagram.

The *Langlands dual* $X^\vee$ is the orthogonal Grassmannian $\mathrm{OG}^{co}(n+1,2n+1)$ of co-isotropic $(n+1)$-dimensional subspaces of $\mathbb{C}^{2n+1}$ with respect to a quadratic form Q. Following [5], we consider $\mathbb{X}^\vee$ in its minimal embedding $\mathbb{X}^\vee \hookrightarrow \mathbb{P}(V^*)$, where V is the irreducible representation corresponding to the parabolic subgroup $P^\vee$ ($P^\vee$ will be a maximal parabolic subgroup since P was). As noted in [5, Section 3], because $\mathbb{X}$ is *cominuscule*, its cohomology is isomorphic (by the geometric Satake correspondence) to V.

In this article, we identify a particular seed, which we call the *co-rectangles seed*, and show that the Newton–Okounkov body corresponding to this seed is unimodularly equivalent to the superpotential polytope defined using the Landau–Ginzburg model studied in [5].

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2 The co-rectangles Seed

Seeds in a *cluster structure* of rank l for a commutative algebra are specified by a pair $(\mathbf{x}, B)$ of *cluster variables* $\mathbf{x} = (x_1, \ldots, x_m)$ and an $m \times l$ *extended exchange matrix* B, for some $m \geq l$. If the topmost $l \times l$ square submatrix of B is skew-symmetric, we can replace B with a *quiver* Q, which is a directed, oriented graph which may have parallel edges, but no 2-cycles or loops, and some vertices designated as 'frozen.' For brevity, we do not give a full definition of a cluster algebra, and instead refer to [1, Section 3.1].

The seeds for the coordinate rings of Grassmannians $\mathrm{Gr}(k, n)$ studied in [8], which were first proven to give a cluster structure in [9], can be described by quivers, and furthermore certain seeds admit an additional description in terms of certain planar, bicolored graphs called *plabic graphs*. When we wish to distinguish the vertices of a plabic graph according to the bicoloring, we will refer to them as hollow ($\circ$) or filled ($\bullet$). Roughly speaking, $\mathbf{x}$ corresponds to the set of face labels of a plabic graph G, and Q corresponds to the dual graph of G. A more thorough exposition of plabic graphs can be found in [6], where they were first introduced, and the relationship between cluster seeds and plabic graphs for Grassmannians can be found in [8, Section 5-6].

2.1 The co-rectangles Symmetric Plabic Graph

Certain plabic seeds for the Grassmannian can be used to obtain seeds for the Lagrangian Grassmannian by *quiver folding*. We first recall the notion of *symmetric plabic graphs* [2].

Definition 1 ([2, Definition 5.1])**.** A *symmetric plabic graph* for $\mathbb{X}$ is a plabic graph G with $2n$ boundary vertices, labelled clockwise by $1, \ldots, 2n$, and a distinguished diameter d of the bounding disk satisfying the following conditions:

1. d has one endpoint between vertices $2n$ and 1, and the other between n and $n+1$.
2. No vertex of G lies on d.
3. Reflecting G through d gives a graph identical to G with the colors of vertices reversed.

Our seed comes from the *co-rectangles* symmetric plabic graph $G_n^{\text{co-rect}}$ (faces are labelled by **co**mplements of **rect**angular Young diagrams in the $n \times n$ square). We define $G_n^{\text{co-rect}}$ by example for $n = 4$ (see figure 1). We note that our $G_n^{\text{co-rect}}$ is mutation equivalent to $G_{n,2n}^{\text{rec}}$ of [8], so in particular satisfies a technical assumption called *reducedness*.

2.2 The Network Parametrization ($\mathcal{X}$-cluster seed) for $\mathbb{X}$

We will now describe how to use a symmetric plabic graph to construct a *network torus* in $\mathbb{X}$. This will allow us to compute valuations associated to a seed using plabic graphs

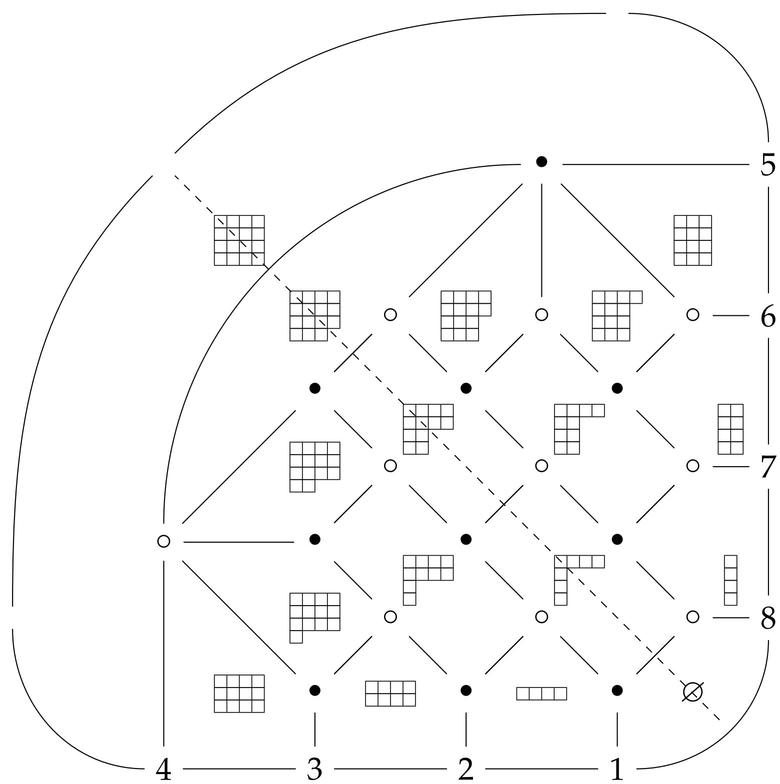


Figure 1: The co-rectangles symmetric plabic graph $G_4^{\text{co-rect}}$.

in the following section. We summarize the presentation in [8, Section 6].

Definition 2. A *perfect orientation O* of a plabic graph G is an orientation of each edge of G such that each filled internal vertex is incident to exactly one edge directed away from it, and each hollow vertex is incident to exactly one edge directed towards it. The source set I_O of O is the set of boundary vertex labels which are sources of G as a directed graph with edge directions O.

Let G denote a plabic graph with a perfect orientation O. If we need further assumptions on G, they will be stated explicitly.

Definition 3. Let J be a subset of the boundary vertices of G with $|J| = |I_O|$. A *flow* from I_O to J is a collection of pairwise vertex-disjoint paths with sources $I_O \setminus (I_O \cap J)$ and sinks $J \setminus (I_O \cap J)$.

Because each path p in a flow F begins and ends at a boundary vertex of G, p partitions the faces of G into two sets, those to the left of p and those to the right of p in the direction of the path. Let p_L denote the set of face labels to the left of p.

Definition 4. For a path p in a flow F, we define the *weight* of p to be $\mathrm{wt}(p) = \prod_{\lambda \in p_L} x_\lambda$. For a flow F, we define the *weight* to be $\mathrm{wt}(F) = \prod_{p \in F} \mathrm{wt}(p)$. Finally, for a subset J of the boundary vertices of G with $|J| = |I_O|$, let $\mathcal{F}$ denote the set of all flows from I_O to J, and define the *flow polynomial* $P_J^G = \sum_{F \in \mathcal{F}} \mathrm{wt}(F)$.

Now let $G = G_n^{\text{co-rect}}$ be the co-rectangles plabic graph. In what follows, we will need a perfect orientation $O_{\text{co-rect}}$ on G, defined as follows.

Definition 5. Set $\{1, 2, \ldots, n\}$ to be sources, and $\{n+1, n+2, \ldots, 2n\}$ to be sinks. (The edges adjacent to vertices $1 \leq i \leq n$ will be directed away from i, and the edges adjacent to vertices $n+1 \leq i \leq 2n$ will be directed towards i.) Because symmetric plabic graphs are also usual plabic graphs, then there is a unique such perfect orientation by [7, Lemma 4.5], see [8, Remark 6.4]. We call this $O_{\text{co-rect}}$.

This is the choice of perfect orientation we will use for the rest of the article. For an example of the above definitions, see example 1, where we give our perfect orientation for $n = 3$, and compute a flow polynomial.

Next, let $S = \{x_\mu \mid \mu \text{ is a face label of } G_n^{\text{co-rect}}\}$ be the set of face labels of the co-rectangles plabic graph. We think of these as coordinates on the *network torus* $\mathbb{T}_G \cong (\mathbb{C}^*)^{|S|}$, and we use the flow polynomials to define an embedding of $\mathbb{T}_G$ into $\mathrm{Gr}(n, 2n)$.

Theorem 1 ([6, Theorem 12.7], [8, Theorem 6.8]). *Let G be the co-rectangles plabic graph, and $J \in \binom{[2n]}{n}$. Consider the map $\Phi\colon \mathbb{T}_G \to \mathrm{Gr}(n, 2n)$ defined by sending $(x_\mu \mid \mu \in S) \in \mathbb{T}_G \mapsto (P_J^G(x_\mu) \mid J \in \binom{[2n]}{n}) \in \mathrm{Gr}(n, 2n)$. Then Φ is well-defined, and gives an embedding $\mathbb{T}_G \hookrightarrow \mathrm{Gr}(n, 2n)$.*

Finally, let $G = G_n^{\text{co-rect}}$ be the co-rectangles symmetric plabic graph. We define the equivalence relation $\sim$ on S given by $x_\mu \sim x_{\mu^T}$, where μ^T denotes the transpose partition to μ. In Karpman's language, this corresponds to taking a *symmetric weighting*, and Karpman shows that restricting to these weightings gives an embedding whose image lands inside of $\mathbb{X} \subset \mathrm{Gr}(n, 2n)$. We think of $S/\sim$ as coordinates on the torus $\mathbb{T}_{\text{co-rect}} :- \mathbb{T}_{G_n^{\text{co-rect}}} \cong (\mathbb{C}^*)^{|S/\sim|}$, and we use the flow polynomials to define an embedding of $\mathbb{T}_{\text{co-rect}}$ into $\mathbb{X}$.

Theorem 2 ([2, Theorem 5.15]). *Let $G = G_n^{\text{co-rect}}$ be the co-rectangles symmetric plabic graph, and $J \in \binom{[2n]}{n}$. Consider the map $\Phi\colon \mathbb{T}_{\text{co-rect}} \hookrightarrow \mathbb{X}$ which is defined by sending $(x_\mu \mid \mu \in S/\sim) \in \mathbb{T}_{\text{co-rect}} \mapsto (P_J^G(x_\mu = x_{\mu^T}) \mid J \in \binom{[2n]}{n}) \in \mathbb{X}$. Then Φ is well-defined, and gives an embedding $\mathbb{T}_{\text{co-rect}} \hookrightarrow \mathbb{X}$.*

Thus we associate to $G_n^{\text{co-rect}}$ and $O_{\text{co-rect}}$ a dense torus $\mathbb{T}_{\text{co-rect}} \hookrightarrow \mathbb{X}$. On the level of coordinate rings, this induces an injection $\mathbb{C}[\mathbb{X}] \hookrightarrow \mathbb{C}[\mathbb{T}_{\text{co-rect}}]$, so we may express polynomials in the Plücker coordinates on $\mathbb{X}$ as Laurent polynomials in the coordinates on $\mathbb{T}_{\text{co-rect}}$.

3 Polytopes

3.1 The Newton–Okounkov body $\Delta_{\text{co-rect}}$

We associate to the co-rectangles symmetric plabic graph $G_n^{\text{co-rect}}$ and ample divisor $D = D_n$ a Newton Okounkov body $\Delta_{\text{co-rect}}(D)$ by the following procedure, following [8, Definition 8.1]. First, we define a valuation using the inclusion $\mathbb{C}[\mathbb{X}] \hookrightarrow \mathbb{C}[\mathbb{T}_{\text{co-rect}}]$ obtained at the end of the previous section.

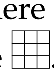
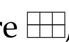
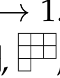
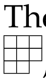
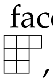
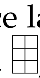
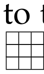
Definition 6. Fix a total order on the torus coordinates S defined at the end of the previous section. Then we define the valuation $\mathrm{val}_{\text{co-rect}} \colon \mathbb{C}[\mathbb{X}] \setminus \{0\} \to \mathbb{Z}^{|S|}$ by sending $f \in \mathbb{C}[\mathbb{X}]$ to the exponent vector of the lexicographically minimal term when f is viewed as an element of $\mathbb{C}[\mathbb{T}_{\text{co-rect}}]$, *i.e.* as a Laurent polynomial in the torus coordinates.

Now, using this valuation, we define the Newton–Okounkov body:

Definition 7. Let $\mathrm{val}_{\text{co-rect}}$ be as above. Then we define

$$\Delta_{\text{co-rect}} = \overline{\mathrm{conv}\left(\bigcup_{r=1}^{\infty} \frac{1}{r}\mathrm{val}_{\text{co-rect}}(H^0(\mathbb{X}, \mathcal{O}(rD)))\right)}.$$

Concretely, the nonzero sections in $H^0(\mathbb{X}, \mathcal{O}(rD))$ can be identified with Laurent polynomials whose numerators are degree r homogeneous polynomials in the Plücker coordinates of $\mathbb{X}$, and whose denominators are the Plücker coordinate $p_{n\times n}^r$. For $G_n^{\text{co-rect}}$ and $O_{\text{co-rect}}$, the only flow from $[n]$ to $[n]$ is the empty flow, so the expression of $p_{n\times n}$ on the torus $\mathbb{T}_{\text{co-rect}}$ is 1. Therefore, computing valuations of sections $H^0(\mathbb{X}, \mathcal{O}(rD))$ reduces to computing valuations of elements of $\mathbb{C}[\mathbb{X}]$, so we can use definition 6.

Example 1. For the flow in figure 2, there are no face labels to the left of the path $1 \to 1$. The face labels to the left of $3 \to 4$ are . The face labels to the left of $2 \to 5$ are , , , , , and , contributing a monomial $x^2_{(3,3,3)}x^2_{(3,3)}x^2_{(3,3,1)}x_{(3,3,2)}$. There is one more flow, contributing a monomial $x^2_{(3,3,3)}x^2_{(3,3)}x^2_{(3,3,1)}x_{(3,3,2)}x_{(3,1,1)}$. These are the only flows from $\{1,2,3\}$ to $\{1,4,5\}$ for $G = G_n^{\text{co-rect}}$ and $O_{\text{co-rect}}$, so the flow polynomial is the sum of these

$$P^G_{\{1,4,5\}} = (x^2_{(3,3,3)}x^2_{(3,3)}x^2_{(3,3,1)}x_{(3,3,2)})(1 + x_{(3,1,1)})$$

The minimal term is $(x^2_{(3,3,3)}x^2_{(3,3)}x^2_{(3,3,1)}x_{(3,3,2)})$, so the valuation (rearranging the coordinates) is $(0,2,0,2,1,2)$, agreeing with the coordinates given in example 2 below.

Alternatively, because symmetric plabic graphs are also plabic graphs in the usual sense, we can compute Plücker coordinate valuations for plabic seeds more directly from Young diagrams.

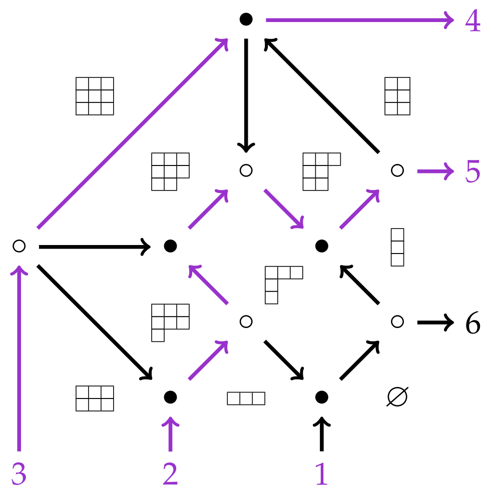


Figure 2: The co-rectangles symmetric plabic graph for $n = 3$, with acyclic perfect orientation, and (minimal) flow from $\{1,2,3\}$ to $\{1,5,4\}$ in purple.

Definition 8. For any skew partition $\nu \subset n \times n$, we define $\mathrm{maxdiag}(\nu)$ to be the maximum number of boxes along any diagonal of slope -1.

Proposition 1. *For $\mu \subset n \times n$ a face label of G and $\lambda \subset n \times n$ arbitrary , we have*

$$\mathrm{val}_{\textit{co-rect}}(p_\lambda)_\mu = \begin{cases} \mathrm{maxdiag}(\mu \backslash \lambda) + \mathrm{maxdiag}(\mu^T \backslash \lambda) & \text{if } \mu \neq \mu^T, \\ \mathrm{maxdiag}(\mu \backslash \lambda) & \text{if } \mu = \mu^T. \end{cases}$$

Example 2. For $\mathrm{LGr}(3,6)$, we have 14 Plücker coordinates with their valuations in coordinates $(156, 126, 145, 125, 124, 123)$ (or in Young diagrams (, , , , ,)):

$I \in \binom{[6]}{3}$	$\mathrm{val}_{\text{co-rect}}(p_I)$	$I \in \binom{[6]}{3}$	$\mathrm{val}_{\text{co-rect}}(p_I)$
123	$(0,0,0,0,0,0)$	$146 = 245$	$(1,2,1,2,1,2)$
124	$(0,0,0,0,0,1)$	$156 = 345$	$(1,3,1,3,2,2)$
$125 = 134$	$(0,1,0,1,1,1)$	236	$(2,2,1,2,1,1)$
$126 = 234$	$(1,1,1,2,1,1)$	246	$(2,2,1,2,1,2)$
135	$(0,2,0,2,1,1)$	$256 = 346$	$(2,3,1,3,2,2)$
$136 = 235$	$(1,2,1,2,1,1)$	356	$(2,4,1,4,2,2)$
145	$(0,2,0,2,1,2)$	456	$(2,4,1,4,2,3)$

The convex hull of these valuations has f-vector $(14, 51, 86, 78, 39, 10)$ and volume $16 = \deg \mathrm{LGr}(3,6)$. Although 456 appears as a face label of the co-rectangles plabic graph, there is no flow beginning at $\{1,2,3\}$ with this face to the left; hence every $\mathrm{val}_{\text{co-rect}}(p_\lambda)_{456} = 0$ for any λ. Thus we exclude this coordinate in order to work with a full-dimensional polytope.

The fact that the volume of the convex hull of the valuations of the Plücker coordinates is equal to the degree of $\mathrm{LGr}(3,6)$ in the example above is not an accident. In fact, as we will see in the proof of theorem 1:

Theorem 3. $\mathrm{conv}(\{\mathrm{val}_{\textit{co-rect}}(p_\lambda) \mid \lambda \subset n \times n\}) = \Delta_{\textit{co-rect}}$ *is a Newton–Okounkov body for* $\mathbb{X}$ *with respect to the valuation* $\mathrm{val}_{\textit{co-rect}}$. *(Equivalently, the Plücker coordinates form a Khovanskii basis for* $\mathbb{C}[\mathbb{X}]$ *with respect to the valuation* $\mathrm{val}_{\textit{co-rect}}$.*)*

3.2 The superpotential polytope Γ

We use the Laurent polynomial expression for the restriction of the superpotential W_q to a torus $(\mathbb{C}^*)^{\binom{n+1}{2}} \hookrightarrow \mathbb{X}^\vee$ for the Landau-Ginzburg model for $\mathbb{X}$ found by Pech and Rietsch:

Definition 9 ([5, Proposition A.1])**.** Let coordinates on the torus above be given by a_{ij} for $1 \leq i \leq j \leq n$, and let Λ denote the set of strict partitions with at least one part of size n that are contained in the maximal, right-justified staircase in the $n \times n$ square. For any $\lambda \in \Lambda$, label each box by (i,j) where i indexes the row and j the column. Then set λ_j to be the largest index such that $(\lambda_j, j) \in \lambda$. Then the restriction of the superpotential to this torus is given by

$$W_q = \sum_{i \leq j \in [n]} a_{ij} + \sum_{\lambda \in \Lambda} \frac{q}{\prod_{j \in [n]} a_{\lambda_j j}}$$

Example 3. For $n = 3$, the superpotential has $\binom{3+1}{2} + 2^{3-1} = 10$ terms:

$$a_{11} + a_{12} + a_{13} + a_{22} + a_{23} + a_{33} + \frac{q}{a_{11}a_{12}a_{13}} + \frac{q}{a_{11}a_{12}a_{23}} + \frac{q}{a_{11}a_{22}a_{23}} + \frac{q}{a_{11}a_{22}a_{33}}$$

where the last four terms correspond to the diagrams $\begin{ytableau} 11 & 12 & 13 \end{ytableau}$, $\begin{ytableau} 11 & 12 & 13 \\ \none & \none & 23 \end{ytableau}$, $\begin{ytableau} 11 & 12 & 13 \\ \none & 22 & 23 \end{ytableau}$, and $\begin{ytableau} 11 & 12 & 13 \\ \none & 22 & 23 \\ \none & \none & 33 \end{ytableau}$

In order to define the superpotential polytope Γ, we first define tropicalization for Laurent polynomial whose coefficients are all positive, real numbers.

Definition 10 ([8, Definition 10.7])**.** For any Laurent polynomial h in variables $z_1, \ldots, z_k$ with coefficients in $\mathbb{R}_{>0}$, we define $\mathrm{Trop}(h) \colon \mathbb{R}^k \to \mathbb{R}$ inductively. To begin, we set $\mathrm{Trop}(z_i)(y_1, \ldots, y_k) = y_i$, and we denote this tropicalization by a capital letter $\mathrm{Trop}(z_i) = Z_i$. Next, if h_1 and h_2 are any Laurent polynomials with positive coefficients, and c_1, c_2 are any positive real numbers, then

$$\mathrm{Trop}(c_1h_1 + c_2h_2) = \min(\mathrm{Trop}(h_1), \mathrm{Trop}(h_2)) \quad \text{and} \quad \mathrm{Trop}(h_1h_2) = \mathrm{Trop}(h_1) + \mathrm{Trop}(h_2).$$

This inductively defines $\mathrm{Trop}(h)$.

Following [8, Definition 10.14], we make the following definition for the Γ.

Definition 11. Consider $W_q \colon \mathbb{R}^{\binom{n}{2}} \times \mathbb{R} \to \mathbb{R}$ as a Laurent polynomial with positive coefficients in the variables a_{ij} (corresponding to the first factor of $\mathbb{R}^{\binom{n}{2}}$ and q corresponding to the second factor of $\mathbb{R}$. Then the *superpotential polytope* Γ is defined by

$$\Gamma = \{y \in \mathbb{R}^{\binom{n}{2}} \mid \operatorname{Trop}(W_q)(y, 1) \geq 0\}.$$

Implicitly, we are "tropicalizing" q^i to i by the evaluation $\operatorname{Trop}(W_q)(y, 1)$.

Example 4. The superpotential polytope for $n = 3$ corresponding to the potential in example 3 is a polytope in $\mathbb{R}^{\binom{3+1}{2}}$, with coordinates indexed by the A_{ij} ordered lexicographically, defined by the inequalities:

$$\begin{aligned} A_{ij} &\geq 0, \\ 1 - A_{11} - A_{12} - A_{13} &\geq 0, \\ 1 - A_{11} - A_{12} - A_{23} &\geq 0, \\ 1 - A_{11} - A_{22} - A_{23} &\geq 0, \\ 1 - A_{11} - A_{22} - A_{33} &\geq 0. \end{aligned}$$

3.3 Poset polytope combinatorics

In [10], Stanley associated two polytopes to a poset P: the *order polytope* and the *chain polytope*. The chain polytope lives in $\mathbb{R}^{|P|}$, and is defined by the inequalities $e_b \geq 0$ for any $b \in P$ and for any chain $b_1 < b_2 < \cdots < b_k$ of P, we have $e_{b_1} + e_{b_2} + \cdots + e_{b_k} \leq 1$. In particular, because of the positivity inequalities, it is enough to consider the chain inequalities $e_{b_1} + e_{b_2} + \cdots + e_{b_k} \leq 1$ when $b_1 < b_2 < \cdots < b_k$ is any maximal chain of P.

Let $\mathcal{P}_n$ be the poset on the elements $\{b_{ij} \mid 1 \leq i \leq j \leq n\}$, with the cover relations $b_{ij} > b_{i+1j+1}, b_{ij+1}$. The superpotential polytope Γ produced above is the chain polytope of $\mathcal{P}_n$: the terms a_{ij} correspond to the positivity inequalities, and the terms $\frac{q}{\prod_{j \in [n]} a_{i_j j}}$ correspond to maximal chain inequalities.

$$\begin{tikzcd}
 & & b_{11} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & b_{22} \arrow[dl, dash] \arrow[dr, dash] & & b_{12} \arrow[dl, dash] \arrow[dr, dash] & \\
b_{33} & & b_{23} & & b_{13}
\end{tikzcd}$$

Figure 3: Hasse diagram of $\mathcal{P}_3$

Example 5. The four maximal chains of $\mathcal{P}_3$ are $b_{33} \le b_{22} \le b_{11}$, $b_{23} \le b_{22} \le b_{11}$, $b_{23} \le b_{12} \le b_{11}$, and $b_{13} \le b_{12} \le b_{11}$. These correspond exactly to the 'q' terms of the superpotential above, so the chain polytope of $\mathcal{P}_3$ coincides with the superpotential polytope.

By [10, Theorem 2.2 and Corollary 4.2], the superpotential polytope has as many vertices as antichains in $\mathcal{P}_n$, and volume equal to the number of linear extensions of $\mathcal{P}_n$.

Lemma 1. *The number of antichains of $\mathcal{P}_n$ is $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$.*

Hence the superpotential polytope has C_{n+1} many vertices. This is also the number of Young diagrams contained in the $n \times n$ square up to transpose, and hence the number of distinct Plücker coordinates for the Lagrangian Grassmannian $\mathbb{X}$. This bijection is described in more detail later as part of the proof that the superpotential polytope coincides with the Newton–Okounkov body.

Example 6. We illustrate the bijection between Dyck paths and antichains.

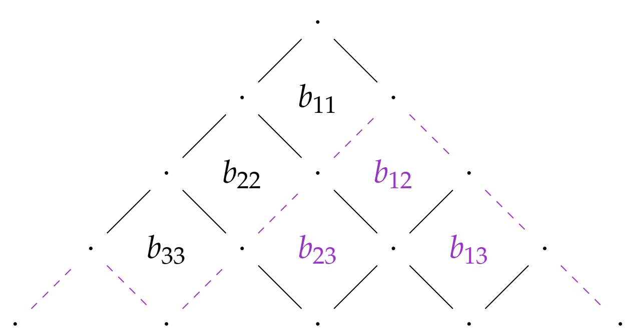


Figure 4: Dyck path (dashed, purple) corresponding to the antichain $\{a_{12}\}$, with corresponding order filter marked in purple.

Lemma 2. *The number of linear extensions of $\mathcal{P}_n$ is equal to* $\deg(\mathbb{X} = \mathrm{LGr}(n, 2n))$.

Hence the superpotential polytope has volume equal to $\deg(\mathbb{X})$. By our choice of $D = D_n$ (sections of $\mathcal{O}(rD)$ correspond to degree r homogeneous polynomials) and G (the valuation is full rank, for example from proposition 2), the volume of the Newton–Okounkov bodies constructed above should also be equal to $\deg(\mathbb{X})$ (see, *e.g.*, [3, Corollary 3.2] or [4], noting that we are not using the normalized volume, so should disregard the normalizing factor of $\dim(\mathbb{X})!$).

4 $\Delta_{\text{co-rect}} \cong \Gamma$

Now we show that for the seed $G = G_n^{\text{co-rect}}$ and corresponding valuation $\mathrm{val}_{\text{co-rect}}$, the Newton–Okounkov body $\Delta_{\text{co-rect}}$ and superpotential polytope Γ defined above are unimodularly equivalent, *i.e.* that there is a lattice isomorphism sending one polytope to the other. Our strategy is as follows. First, we define a linear map $M_n\colon \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}^{\binom{n+1}{2}}$ with integer entries. We show next that M_n is unimodular, and finally that $M_R(\Gamma) = \Delta_{\text{co-rect}}$.

Consider the Plücker coordinates p_λ such that $(n-1)\times(n-1) \subseteq \lambda \subsetneq n \times n$ and there are at least as many boxes to the right of the main diagonal (upper left to lower right) as there are below. Note that there are $\binom{n+1}{2}$ such Plücker coordinates: for each pair $(0 \le i \le j \le n-1)$, associate the Young diagram containing the $(n-1)\times(n-1)$ rectangle with j additional boxes to the right of the diagonal, and i additional boxes below the diagonal.

Form the $\binom{n+1}{2} \times \binom{n+1}{2}$ matrix M_n whose columns are the valuations of the above Plücker coordinates, ordered as follows: order first by *decreasing* order (*i.e.* $2n \le 2n-1 \le \cdots \le 1$) in the last entry, then break ties by *increasing* order in the first to last, second to last, *etc.* entries. On the level of Young diagrams, this corresponds to ordering first by increasing number of additional boxes below the main diagonal, and then by dexreasing number of additional boxes right of the main diagonal. When these diagrams are indexed by pairs (i,j), the ordering is $(i,j) \le (i',j')$ if $i < i'$ or $i = i'$ and $j \ge j'$.

4.1 Unimodularity

Lemma 3. *The upper left $n \times n$ block of M_n has the form:*

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 2 \\ 1 & 1 & \cdots & 1 & 2 & 2 \\ \vdots & & \ddots & & & \vdots \\ 1 & 2 & \cdots & 2 & 2 & 2 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } i = n, \\ 1 & \text{if } j \le n-i, i \neq, n \\ 2 & \text{if } j > n-i, i \neq n, \end{cases}$$

and is thus unimodular.

Lemma 4. *The lower right $\binom{n}{2} \times \binom{n}{2}$ block of M_n is M_{n-1}.*

Furthermore, since all of the Plücker coordinates indexing the last $\binom{n}{2}$ columns of M_n contain the hook $(n, 1^{n-1})$, then in particular $\mathrm{val}_{\text{co-rect}}(p_{I_j})_{(n,1^{n-1})} = 0$ for any I_j in these columns. In other words, the bottom row of the upper right $n \times \binom{n}{2}$ submatrix of M_n is the 0 vector.

Lemma 5. *The lower left $\binom{n}{2} \times n$ block of M_n has all columns equal and nonzero.*

Proposition 2. *M_n is unimodular.*

4.2 Surjectivity

It remains to prove that $M_n(\Gamma) = \Delta_{\text{co-rect}}$. To aid in our proof, we define an auxiliary polytope δ_G in the same ambient space as $\Delta_{\text{co-rect}}$:

Definition 12. Define $\delta_G = \text{conv}(\{\text{val}_{\text{co-rect}}(p_\lambda) \mid \lambda \in \binom{[2n]}{n}\})$.

A priori $\delta_G \subset \Delta_{\text{co-rect}}$ (for our choice of G and D, $\delta_G = \text{conv}(\text{val}_{\text{co-rect}}(H^0(\mathbb{X}, \mathcal{O}(D))))$ is the $r = 1$ part of the Newton–Okounkov body). For general G, $\delta_G \subsetneq \Delta_{\text{co-rect}}$ (*i.e.* the Plücker coordinates may not form a Khovanskii basis), but in our case, we will show that $\delta_G = \Delta_{\text{co-rect}}$ by showing that $M_n(\Gamma) = \delta_G$, and computing volumes.

Recall that the vertices of Γ are characteristic functions of antichains of $\mathcal{P}_n$. For each singleton antichain $\{a_{ij}\}$ of $\mathcal{P}_n$, we associate the vertex $v_{ij} \in \Gamma$, and the hook partition $\nu_{ij} = (n+1-i, 1^{j-i})$. Because $j \leq n$, then $n+1-i > j-i$.

Lemma 6. *The map $\{a_{ij}\} \to \nu_{ij}$ between singleton antichains of $\mathcal{P}_n$ and nonempty hook partitions $(a, 1^b) \subset n \times n$ with $a > b$ described above is a bijection.*

We first identify where these vertices are sent under M_n, and then use that to identify which antichains correspond to which Plücker coordinate valuations.

Lemma 7. *$M_n(v_{ij}) = \text{val}_{\text{co-rect}}(p_\lambda)$, where λ is the complement of ν_{ij} in the $n \times n$ square, and the complement is taken by right-justifying ν_{ij} in the bottom right corner.*

Any partition $\lambda \subset n \times n$ has a right-justified complement partition $\lambda^c \subset n \times n$. We decompose λ^c into a union of k nonempty (right-justified) hooks $\lambda^c = \nu_1 + \cdots + \nu_k$, where the decomposition comes from taking the hooks from the boxes of λ^c along the main diagonal.

Example 7. For $\ydiagram{1} \subset \ydiagram{3,3,3}$, the complement is $\ydiagram{1+2,3,3}$, which decomposes into the hooks $\ydiagram{2+1,2+1,3} + \ydiagram{1+1,2}$. For an asymmetric example, take $\ydiagram{2} \subset \ydiagram{3,3,3}$. The complement is $\ydiagram{2+1,3,3}$, and decomposes into the hooks $\ydiagram{2} + \ydiagram{2+1,2+1,3}$.

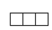
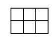
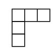
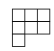
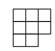
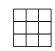
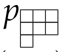
Lemma 8. *Let $\lambda \subset n \times n$ be a partition with at least as many boxes above the main diagonal as below. Then the hook decomposition of the transpose of the complement corresponds to an antichain of $\mathcal{P}_n$ under the bijection lemma 6.*

Example 8. For $\ydiagram{1} \subset \ydiagram{3,3,3}$, the complement is $\ydiagram{1+2,3,3}$, which decomposes into the hooks $\ydiagram{2+1,2+1,3} + \ydiagram{1+1,2}$, corresponding to the elements a_{13} and a_{23}, respectively. For $\ydiagram{2} \subset \ydiagram{3,3,3}$, the complement is $\ydiagram{2+1,3,3}$, and decomposes into the hooks $\ydiagram{2} + \ydiagram{2+1,2+1,3}$, corresponding to the elements a_{13}, a_{22}.

Lemma 9. *Let $\lambda \subset n \times n$ be any partition with at least as many boxes above the main diagonal as below. Let $\lambda^c = \nu_1 + \cdots + \nu_k$ be the hook decomposition of the complement. Then* $\text{maxdiag}(\mu \backslash \lambda) = \sum_i \text{maxdiag}(\mu \backslash \lambda_i)$*, where λ_i is the complement of ν_i (right-justified in the bottom right corner of $n \times n$), for any μ labelling a face of $G_{\text{co-rect}}$.*

Corollary 1. *Let $\lambda \subset n \times n$ be any partition with at least as many boxes above the main diagonal as below. Let $\lambda^c = \nu_1 + \cdots + \nu_k$ be the hook decomposition with corresponding antichain $\{x_i\}$, using the bijection from Lemma 8. Then M_n sends the vertex of Γ corresponding to the antichain $\{x_i\}_{i=1}^k$ to $\mathrm{val}_{co\text{-}rect}(p_\lambda)$. Hence $\Delta_{co\text{-}rect} \cong \Gamma$.*

Example 9. Adding the bottom two rows gives the top row, as desired:

μ						
$\mathrm{val}_{\text{co-rect}}(p$$)_\mu$	2	3	1	3	2	2
$\mathrm{val}_{\text{co-rect}}(p$$)_\mu$	0	1	0	1	1	1
$\mathrm{val}_{\text{co-rect}}(p$$)_\mu$	2	2	1	2	1	1

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