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A q-Deformation of Enriched P-Partitions

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Abstract. We introduce a q-deformation that generalises in a single framework previous works on classical and enriched P-partitions. In particular, we build a new family of power series with a parameter q that interpolates between Gessel's fundamental ($q = 0$) and Stembridge's peak quasisymmetric functions ($q = 1$) and show that it is a basis of QSym when $q \notin \{-1, 1\}$. Furthermore we build their corresponding monomial bases parametrised with q that cover our previous work on enriched monomials and the essential quasisymmetric functions of Hoffman.

Résumé. Nous introduisons une q-déformation qui généralise dans un cadre unique les travaux antérieurs sur les P-partitions classiques et enrichies. En particulier, nous construisons une famille de séries formelles avec un paramètre q qui interpole entre les fonctions quasisymétriques fondamentales de Gessel ($q = 0$) et les fonctions de pic de Stembridge ($q = 1$) et montrons qu'il s'agit d'une base de QSym quand $q \notin \{-1, 1\}$. De plus, nous construisons leur bases de monômes associées paramétrées par q qui généralisent nos travaux sur les monôes enrichis et les fonctions essentielles de Hoffman.

Keywords: quasisymmetric functions, enriched P-partitions, peak functions

1 Introduction

Introduced by Stanley in [6], P-partitions are order preserving maps from a partially ordered set P to the set of positive integers with many significant applications in algebraic combinatorics. In particular, they are the building block of Gessel's ring of quasisymmetric functions (QSym) in [1]. Replacing positive integers by signed ones, Stembridge introduces in [8] an enriched version of P-partitions to build the algebra of peaks, a subalgebra of QSym. The generating functions of classical (enriched) P-partitions on labelled chains are the fundamental (peak) quasisymmetric functions, an important basis of QSym (the algebra of peaks) related to the descent (peak) statistic on permutations. More recently, in [3], we redefine these generating functions on weighted posets to extend their nice properties to the monomial and enriched monomial bases of

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QSym. However the classical and enriched frameworks remained so far separated. We merge them into one via a new q-deformation of the generating function for enriched P-partitions that interpolates between Gessel's and Stembridge's works.

1.1 Posets and enriched P-partitions

We recall the main definitions regarding posets and (enriched) P-partitions. The reader is referred to [1, 7, 8] for further details.

Definition 1 (Labelled posets)**.** Let $[n] = \{1, 2, \ldots, n\}$. A *labelled poset* $P = ([n], <_P)$ is an arbitrary partial order $<_P$ on the set $[n]$.

Definition 2 (P-partition)**.** Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ and let $P = ([n], <_P)$ be a labelled poset. A *P-partition* is a map $f\colon [n] \longrightarrow \mathbb{P}$ that satisfies the two following conditions:

1. If $i <_P j$, then $f(i) \leq f(j)$.
2. If $i <_P j$ and $i > j$, then $f(i) < f(j)$.

The relations $<$ and $>$ stand for the classical order on $\mathbb{P}$. Let $\mathcal{L}_{\mathbb{P}}(P)$ denote the set of P-partitions.

Definition 3 (Enriched P-partition)**.** Let $\mathbb{P}^{\pm}$ be the set of positive and negative integers totally ordered by $-1 < 1 < -2 < 2 < -3 < 3 < \cdots$. We embed $\mathbb{P}$ into $\mathbb{P}^{\pm}$ and let $-\mathbb{P} \subseteq \mathbb{P}^{\pm}$ be the set of all $-n$ for $n \in \mathbb{P}$. Given a labelled poset $P = ([n], <_P)$, an *enriched P-partition* is a map $f\colon [n] \longrightarrow \mathbb{P}^{\pm}$ that satisfies the two following conditions:

1. If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in \mathbb{P}$.
2. If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -\mathbb{P}$.

Further, let $\mathcal{L}_{\mathbb{P}^{\pm}}(P)$ be the set of enriched P-partitions.

Finally recall the weighted variants of posets introduced in [3].

Definition 4 ([3])**.** A *labelled weighted poset* is a triple $P = ([n], <_P, \epsilon)$ where $([n], <_P)$ is a labelled poset and $\epsilon\colon [n] \longrightarrow \mathbb{P}$ is a map (called the *weight function*).

Each node of a labelled weighted poset is marked with its label and weight (Figure 1).

1.2 Quasisymmetric functions

Consider the set of indeterminates $X = \{x_1, x_2, x_3, \ldots\}$, the ring $\mathbf{k}[[X]]$ of formal power series on X where $\mathbf{k}$ is a commutative ring, and let $\mathcal{Z} \in \{\mathbb{P}, \mathbb{P}^{\pm}\}$. Given a labelled weighted poset $([n], <_P, \epsilon)$, define its generating function $\Gamma_{\mathcal{Z}}([n], <_P, \epsilon) \in \mathbf{k}[[X]]$ by

$$\Gamma_{\mathcal{Z}}([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_{\mathcal{Z}}([n], <_P)} \prod_{1 \leq i \leq n} x_{|f(i)|}^{\epsilon(i)}, \tag{1.1}$$

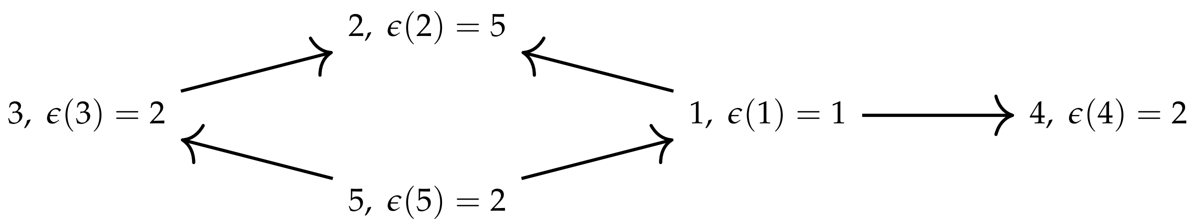

Figure 1: A 5-vertex labelled weighted poset. Arrows show the covering relations.

where $|f(i)| = -f(i)$ (resp. $= f(i)$) for $f(i) \in -\mathbb{P}$ (resp. $\mathbb{P}$). Let S_n be the symmetric group on $[n]$. Given a *composition*, *i.e.* a sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with n entries, and a permutation $\pi = \pi_1 \ldots \pi_n$ of S_n, we let $P_{\pi,\alpha} = ([n], <_\pi, \alpha)$ be the labelled weighted poset on the set $[n]$, where the order relation $<_\pi$ is such that $\pi_i <_\pi \pi_j$ if and only if $i < j$ and α is the weight function sending the vertex labelled π_i to α_i (see Figure 2). For $\mathcal{Z} \in \{\mathbb{P}, \mathbb{P}^{\pm}\}$, its generating function $U^{\mathcal{Z}}_{\pi,\alpha} = \Gamma_{\mathcal{Z}}([n], <_\pi, \alpha)$ is called the *universal quasisymmetric function* ([3]) indexed by π and α.

Figure 2: The labelled weighted poset $P_{\pi,\alpha}$.

Definition 5. Let $[1^n]$ denote the composition with n entries equal to 1. For each $\pi \in S_n$, let $L_\pi = U^{\mathbb{P}}_{\pi,[1^n]}$ and $K_\pi = U^{\mathbb{P}^{\pm}}_{\pi,[1^n]}$. The power series L_π (resp. K_π) are *Gessel's fundamental (resp. Stembridge's peak) quasisymmetric functions* indexed by the permutation π.

The power series L_π and K_π belong to the subalgebra of $\mathbf{k}[[X]]$ called the ring of *quasisymmetric functions* (QSym), *i.e.* for any strictly increasing sequence of indices $i_1 < i_2 < \cdots < i_p$ the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$ is equal to the coefficient of $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_p}^{k_p}$. Furthermore they are related to two major statistics on permutations. Given $\pi \in S_n$, define its *descent set* $\mathrm{Des}(\pi) = \{1 \le i \le n-1 | \pi(i) > \pi(i+1)\}$ and its *peak set* $\mathrm{Peak}(\pi) = \{2 \le i \le n-1 | \pi(i-1) < \pi(i) > \pi(i+1)\}$. The peak set of a permutation is *peak-lacunar*, *i.e.* it neither contains 1 nor contains two consecutive integers.

Proposition 1 ([1, 8])**.** *For any permutation $\pi \in S_n$, the fundamental quasisymmetric function L_π and the peak quasisymmetric function K_π satisfy*

$$L_\pi = \sum_{\substack{i_1 \le \cdots \le i_n; \\ j \in \mathrm{Des}(\pi) \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}, \qquad K_\pi = \sum_{\substack{i_1 \le \cdots \le i_n; \\ j \in \mathrm{Peak}(\pi) \Rightarrow i_{j-1} < i_{j+1}}} 2^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

As a result L_π (K_π) depends only on n and $\mathrm{Des}(\pi)$ ($\mathrm{Peak}(\pi)$) *and we may use set indices and write $L_{n,\mathrm{Des}(\pi)}$ ($K_{n,\mathrm{Peak}(\pi)}$) instead of L_π (K_π). Furthermore $(L_{n,I})_{n\geq 0, I\subseteq[n-1]}$ is a basis of* QSym *(we assume $[-1]=[0]=\emptyset$), and $(K_{n,I})_{n\geq 0,I}$ is a basis of a subalgebra of* QSym *called the algebra of peaks when I runs over all peak-lacunar subsets of $[n-1]$ for all integers n.*

Definition 6. Let id_n and $\overline{id_n}$ denote the permutations in S_n given by $id_n = 1\,2\,3\cdots n$ and $\overline{id_n} = n\; n-1 \cdots 1$. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_n)$ of n entries, define the *monomial* M_α ([1]), *essential* E_α ([4]) and *enriched monomial* η_α ([3, 5]) quasisymmetric functions

$$M_\alpha = U^{\mathbb{P}}_{\overline{id_n},\alpha} = \sum_{i_1<\cdots<i_n} x_{i_1}^{\alpha_1}\cdots x_{i_n}^{\alpha_n}, \qquad E_\alpha = U^{\mathbb{P}}_{id_n,\alpha} = \sum_{i_1\leq\cdots\leq i_n} x_{i_1}^{\alpha_1}\cdots x_{i_n}^{\alpha_n},$$
$$\eta_\alpha = U^{\mathbb{P}^\pm}_{id_n,\alpha} = \sum_{i_1\leq\cdots\leq i_n} 2^{|\{i_1,\ldots,i_n\}|} x_{i_1}^{\alpha_1}\cdots x_{i_n}^{\alpha_n}.$$

Compositions $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 + \cdots + \alpha_n = s$ are in bijection with subsets of $[s-1]$. For $I \subseteq [s-1]$, we also use the following alternative indexing for monomial, essential and enriched monomials. References to s in indices are removed for clarity.

$$M_I = \sum_{\substack{i_1\leq\cdots\leq i_s \\ j\in I \Leftrightarrow i_j = i_{j+1}}} x_{i_1}\cdots x_{i_s}, \quad E_I = \sum_{\substack{i_1\leq\cdots\leq i_s \\ j\in I \Rightarrow i_j = i_{j+1}}} x_{i_1}\cdots x_{i_s}, \quad \eta_I = \sum_{\substack{i_1\leq\cdots\leq i_s \\ j\in I \Rightarrow i_j = i_{j+1}}} 2^{|\{i_1,\ldots,i_s\}|} x_{i_1}\cdots x_{i_s}.$$

Proposition 2. *Let $s \geq 0$. Let I and J be a subset and a peak-lacunar subset of $[s-1]$. Then,*

$$L_I = \sum_{U\subseteq I} (-1)^{|U|} E_U, \qquad K_J = \sum_{V\subseteq J} (-1)^{|V|} \eta_{(V-1)\cup V}, \tag{1.2}$$

where for V peak-lacunar, we set $V - 1 = \{v - 1 | v \in V\}$.

2 A q-deformed generating function for P-partitions

Equation (1.1) and Propositions 1 and 2 exhibit the strong similarities between enriched and classical P-partitions. As we will see, both are special cases of a more general theory. Looking at Equation (1.1), one may notice that the generating function does not depend on the sign of $f(i)$. Let ω be the map that sends the element i of a labelled weighted poset $([n], <_P, \epsilon)$ and an enriched P-partition f to the contributing monomial in Γ. That is, $\omega(i, f) = x^{\epsilon(i)}_{|f(i)|}$. As proposed by Stembridge, the value of ω does not depend on the sign of f. We break this assumption and write for an additional parameter q:

$$\omega(i, f, q) = x^{\epsilon(i)}_{f(i)} \text{ if } f(i) \in \mathbb{P}, \qquad \omega(i, f, q) = q x^{\epsilon(i)}_{-f(i)} \text{ if } f(i) \in -\mathbb{P}.$$

Definition 7. Let $q \in \mathbf{k}$ (the base ring of the power series). The *q-generating function* for enriched P-partitions on the weighted poset $([n], <_P, \epsilon)$ is

$$\Gamma_q([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_{\mathbb{P}^\pm}([n], <_P)} \prod_{1 \le i \le n} \omega(i, f, q) = \sum_{f \in \mathcal{L}_{\mathbb{P}^\pm}([n], <_P)} \prod_{1 \le i \le n} q^{[f(i)<0]} x_{|f(i)|}^{\epsilon(i)}, \tag{2.1}$$

where $[f(i) < 0] = 1$ if $f(i) < 0$ and 0 otherwise.

This definition covers the case of Gessel ($q = 0$) with no negative numbers allowed and the one of Stembridge ($q = 1$) where the sign of f is ignored in the generating function. Define also the *q-universal quasisymmetric function*

$$U^q_{\pi,\alpha} = \Gamma_q([n], <_\pi, \alpha). \tag{2.2}$$

Proposition 3. *Let $q \in \mathbf{k}$, $\pi \in S_n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a composition with n entries. Then,*

$$U^q_{\pi,\alpha} = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n; \\ j \in \mathrm{Peak}(\pi) \Rightarrow i_{j-1} < i_{j+1}}} q^{|\{j \in \mathrm{Des}(\pi) | i_j = i_{j+1}\}|} (q+1)^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_n}^{\alpha_n}. \tag{2.3}$$

Proof. Let $([n], <_\pi, \alpha)$ be the weighted chain poset associated to $\pi \in S_n$ and to the composition α with n entries. Consider an enriched P-partition $f \in \mathcal{L}_{\mathbb{P}^\pm}([n], <_\pi)$ and an $a \in \mathbb{P}$. All the $i \in [n]$ satisfying $|f(\pi_i)| = a$ form an interval $[j, k] = \{j, j+1, \ldots, k\}$ for some positive integers j and k. By Definition 3, we have $[j, k] \cap \mathrm{Peak}(\pi) = \emptyset$. As a result, there exists l such that $\pi_j > \cdots > \pi_l < \cdots < \pi_k$. We have $f(\pi_j) = \cdots = f(\pi_{l-1}) = -a$, $f(\pi_{l+1}) = \cdots = f(\pi_k) = a$ and $f(\pi_l) \in \{-a, a\}$. The two contributions in x_a are

$$x_a^{\alpha_j + \alpha_{j+1} + \cdots + \alpha_k} [q^{l-j} + q^{l-j+1}] = (q+1) q^{l-j} x_a^{\alpha_j + \alpha_{j+1} + \cdots + \alpha_k}.$$

Note that $l - j = \{i \in \mathrm{Des}(\pi) || f(\pi_i)| = a\}$ to complete the proof. □

The nice formula for the product of two generating functions of chain posets extends naturally to this q-deformation. Recall the definition of coshuffle from [3]:

Definition 8. Let $\pi \in S_n$ and $\sigma \in S_m$ be two permutations. Let α and β be two compositions with n and m entries, respectively. The *coshuffle* of (π, α) and (σ, β), denoted $(\pi, \alpha) \sqcup (\sigma, \beta)$, is the set of pairs (τ, γ) where

- $\tau \in S_{n+m}$ is a shuffle of π and $n + \sigma = (n + \sigma_1, n + \sigma_2, \ldots, n + \sigma_m)$, and
- γ is a composition with $n + m$ entries, obtained by shuffling the entries of α and β using the *same shuffle* used to build τ from the letters of π and $n + \sigma$.

Example 1. $(1\mathbf{3}2, (2, \mathbf{1}, 2))$ is a coshuffle of $(12, (2, 2))$ and $(1, (1))$.

Proposition 4. *Let $q \in \mathbf{k}$, let π and σ be two permutations in S_n and S_m, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be two compositions with n and m entries. The product of two q-universal quasisymmetric functions is given by*

$$U^q_{\pi,\alpha} U^q_{\sigma,\beta} = \sum_{(\tau,\gamma) \in (\pi,\alpha) \sqcup\!\sqcup (\sigma,\beta)} U^q_{\tau,\gamma}. \tag{2.4}$$

Proof. The proof is similar to [3, Theorem 3]. □

3 Enriched q-monomials

3.1 Definition, relation to q-universal quasisymmetric functions and product formula

We introduce a new basis of QSym that generalises the essential and enriched monomial quasisymmetric functions in Definition 6.

Definition 9 (Enriched q-monomials)**.** Let $q \in \mathbf{k}$ and α be a composition with n entries. The *enriched q-monomial* indexed by α is defined as

$$\eta^{(q)}_\alpha = U^q_{id_n,\alpha}. \tag{3.1}$$

As an immediate consequence of Definition 9, one has $\eta^{(0)}_\alpha = E_\alpha$ and $\eta^{(1)}_\alpha = \eta_\alpha$.

Proposition 5. *With the notation of Definition 9, one has*

$$\eta^{(q)}_\alpha = \sum_{i_1 \le i_2 \le \cdots \le i_n} (q+1)^{|\{i_1, i_2, \ldots, i_n\}|} x^{\alpha_1}_{i_1} x^{\alpha_2}_{i_2} \cdots x^{\alpha_n}_{i_n}. \tag{3.2}$$

Proof. This is a direct consequence of Proposition 3. □

Interestingly, one may express general q-universal quasisymmetric functions in terms of the $\eta^{(q)}_\alpha$. To state this result we need the following definition.

Definition 10. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a composition with n entries. For any integer $1 \le i \le n-1$, we let $\alpha^{\downarrow i}$ denote the following composition with $n-1$ entries:

$$\alpha^{\downarrow i} = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n).$$

Furthermore, for any subset $I \subseteq [n-1]$, we set

$$\alpha^{\downarrow I} = \left(\left(\cdots \left(\alpha^{\downarrow i_k} \right) \cdots \right)^{\downarrow i_2} \right)^{\downarrow i_1},$$

where $i_1, i_2, \ldots, i_k$ are the elements of I in increasing order. Finally, if I and J are two subsets of $[n-1]$, with J being peak-lacunar, then we set $\alpha^{\downarrow I \downarrow\downarrow J} = \alpha^{\downarrow K}$, where $K = I \cup J \cup (J-1)$.

Theorem 1. *Let $\pi \in S_n$ be a permutation and α be a composition with n entries. The q-universal quasisymmetric function $U^q_{\pi,\alpha}$ may be expressed as a combination of the enriched q-monomials:*

$$U^q_{\pi,\alpha} = \sum_{\substack{I\subseteq \mathrm{Des}(\pi)\\ J\subseteq \mathrm{Peak}(\pi)\\ I\cap J=\varnothing}} (-q)^{|J|}(q-1)^{|I|}\eta^{(q)}_{\alpha\downarrow I\downarrow\downarrow J}. \tag{3.3}$$

Proof. For any subset $V \subseteq [n-1]$, set $\overline{V} = [n-1] \setminus V$. Then, (2.3) becomes[1]

$$\begin{aligned} U^q_{\pi,\alpha} &= \sum_{K\subseteq \mathrm{Des}(\pi)} \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n\\ j\in \mathrm{Des}(\pi)\setminus K\Rightarrow i_{j-1}\leq i_j<i_{j+1}\\ j\in K\cap \mathrm{Peak}(\pi)\Rightarrow i_{j-1}<i_j=i_{j+1}\\ j\in K\cap\overline{\mathrm{Peak}(\pi)}\Rightarrow i_{j-1}\leq i_j=i_{j+1}}} q^{|K|}(q+1)^{|\{i_1,i_2,\ldots,i_n\}|}x^{\alpha_1}_{i_1}x^{\alpha_2}_{i_2}\cdots x^{\alpha_n}_{i_n} \\ &= \sum_{\substack{K\subseteq \mathrm{Des}(\pi)\\ U\subseteq \mathrm{Des}(\pi)\setminus K\\ J\subseteq K\cap \mathrm{Peak}(\pi)}} q^{|K|}(-1)^{|U|+|J|} \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n\\ j\in U\cup K\cap\overline{\mathrm{Peak}(\pi)}\cup K\cap \mathrm{Peak}(\pi)\setminus J\Rightarrow i_{j-1}\leq i_j=i_{j+1}\\ j\in J\Rightarrow i_{j-1}=i_j=i_{j+1}}} (q+1)^{|\{i_1,i_2,\ldots,i_n\}|}x^{\alpha_1}_{i_1}x^{\alpha_2}_{i_2}\cdots x^{\alpha_n}_{i_n} \\ &= \sum_{\substack{K\subseteq \mathrm{Des}(\pi)\\ U\subseteq \mathrm{Des}(\pi)\setminus K\\ J\subseteq K\cap \mathrm{Peak}(\pi)}} q^{|K|}(-1)^{|U|+|J|} \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n\\ j\in U\cup K\setminus J\Rightarrow i_{j-1}\leq i_j=i_{j+1}\\ j\in J\Rightarrow i_{j-1}=i_j=i_{j+1}}} (q+1)^{|\{i_1,i_2,\ldots,i_n\}|}x^{\alpha_1}_{i_1}x^{\alpha_2}_{i_2}\cdots x^{\alpha_n}_{i_n}. \end{aligned}$$

If we set $I = U \cup K \setminus J$ and $U' = I \setminus U = K \setminus J$, then $|U'| = |K| - |J|$ and $I \subseteq \mathrm{Des}(\pi) \setminus J$. Thus, the above computation becomes

$$U^q_{\pi,\alpha} = \sum_{\substack{U'\subseteq I\\ I\subseteq \mathrm{Des}(\pi)\\ J\subseteq \mathrm{Peak}(\pi)\\ I\cap J=\varnothing}} q^{|U'|+|J|}(-1)^{|U'|+|I|+|J|} \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n\\ j\in I\Rightarrow i_{j-1}\leq i_j=i_{j+1}\\ j\in J\Rightarrow i_{j-1}=i_j=i_{j+1}}} (q+1)^{|\{i_1,i_2,\ldots,i_n\}|}x^{\alpha_1}_{i_1}x^{\alpha_2}_{i_2}\cdots x^{\alpha_n}_{i_n}.$$

Summing over U' yields formula (3.3). □

Corollary 1. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be two compositions. Let $\alpha \sqcup\!\sqcup \beta$ be the multiset of compositions obtained by shuffling α and β. As in [3], given $\gamma \in \alpha \sqcup\!\sqcup \beta$, let $S_\beta(\gamma)$ be the set of the positions of the entries of β in γ. Set furthermore $S_\beta(\gamma) - 1 = \{i-1 | i \in S_\beta(\gamma)\}$. Then,*

$$\eta^{(q)}_\alpha \eta^{(q)}_\beta = \sum_{\substack{\gamma\in\alpha\sqcup\!\sqcup\beta;\\ I\subseteq S_\beta(\gamma)\\ J\subseteq \left(S_\beta(\gamma)\setminus(S_\beta(\gamma)-1)\right)\setminus\{1\}\\ I\cap J=\varnothing}} (q-1)^{|I|}(-q)^{|J|}\eta^{(q)}_{\gamma\downarrow I\downarrow\downarrow J}. \tag{3.4}$$

Proof. Corollary 1 is a consequence of Theorem 1, Equation (3.1) and Proposition 4. □

[1] We understand i_{j-1} to be 0 whenever $j = 1$.

3.2 Relation to the monomial and fundamental bases

We consider the alternative indexing with sets proposed at the end of Section 1.2. Given a set of positive integers $I \subseteq [s-1]$, the enriched q-monomial may be written as

$$\eta_I^{(q)} = \sum_{\substack{i_1 \leq \cdots \leq i_s \\ j \in I \Rightarrow i_j = i_{j+1}}} (q+1)^{|\{i_1,\ldots,i_s\}|} x_{i_1} \cdots x_{i_s}.$$

Proposition 6. *Let $I \subseteq [s-1]$ be a set of positive integers. One has*

$$\eta_I^{(q)} = \sum_{I \subseteq J} (q+1)^{s-|J|} M_J. \tag{3.5}$$

Theorem 2. *Let $q \in \mathbf{k}$ be such that $q+1$ is invertible. The family of enriched q-monomial quasisymmetric functions $\left(\eta_{s,I}^{(q)}\right)_{s \geq 0, I \subseteq [s-1]}$ is a basis of* QSym. *Furthermore*

$$(q+1)^{s-|J|} M_J = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \eta_I^{(q)}. \tag{3.6}$$

Proof. Follows from Equation (3.5) by Möbius inversion. □

We develop further the properties of the enriched q-monomial basis of QSym.

Proposition 7. *Let s be a positive integer and $I \subseteq [s-1]$. One may expand the enriched q-monomials in the fundamental basis as*

$$\eta_I^{(q)} = (q+1) \sum_{J \subseteq [s-1]} (-1)^{|J|} (-q)^{|J \setminus I|} L_J. \tag{3.7}$$

Proof. The expression above is a consequence of Equation (3.5) and the expansion of monomial quasisymmetric functions in the fundamental basis (see, *e.g.*, [1]). □

Proposition 8. *Let s be a positive integer, $J \subseteq [s-1]$ and let $q \in \mathbf{k}$. Then,*

$$(q+1)^s L_J = \sum_{I \subseteq [s-1]} (-1)^{|I|} (-q)^{|I \setminus J|} \eta_I^{(q)}. \tag{3.8}$$

Equations (3.7) and (3.8) expand the fundamental and enriched q-monomial bases in terms of one another, and thus suggest a duality relation between the two. Let QSym_s be the vector subspace of QSym containing the homogeneous quasisymmetric functions of degree s. Define $f \colon \mathrm{QSym}_s \to \mathrm{QSym}_s$ as the $\mathbf{k}$-linear map that sends each L_I to $\eta_I^{(q)}$ for $I \subseteq [s-1]$. Then f^2 is a scaling by $(q+1)^{s+1}$ (that is, $f^2 = (q+1)^{s+1}\,\mathrm{id}$). Moreover,

$$f(M_I) = (q+1)^{|I|+1} M_{[s-1] \setminus I} \qquad \text{for any } I \subseteq [s-1].$$

3.3 Antipode

For an integer s and a subset $I \subseteq [s-1]$, we set $s - I = \{s - i | i \in I\}$. The *antipode* of QSym (see [2, Chapter 5]) can be defined as the unique $\mathbf{k}$-linear map $S : \mathrm{QSym} \to \mathrm{QSym}$ that satisfies

$$S(M_I) = (-1)^{s-|I|} \sum_{(s-I) \subseteq J} M_J.$$

Proposition 9. *Assume that q is invertible in $\mathbf{k}$, and let $p = \frac{1}{q}$. Then, for $I \subseteq [s-1]$,*

$$S\left(\eta_I^{(q+1)}\right) = (-q)^{s-|I|} \, \eta_{s-I}^{(p)}. \tag{3.9}$$

Proof. This can be derived from Equation (3.7). □

4 A q-interpolation between Gessel and Stembridge quasisymmetric functions

4.1 q-fundamental quasisymmetric functions

We introduce a new family of quasisymmetric functions that interpolate between Gessel's fundamental and Stembridge peak quasisymmetric functions and show that it is a basis of QSym in all but the Stembridge case.

Definition 11 (q-fundamental quasisymmetric functions)**.** Let π be a permutation in S_n and $q \in \mathbf{k}$. Define the *q-fundamental quasisymmetric function* indexed by $\mathrm{Des}(\pi)$ as

$$L^{(q)}_{n,\mathrm{Des}(\pi)} = U^q_{\pi,[1^n]}. \tag{4.1}$$

Let I be a subset of $[n-1]$. Set $I + 1 = \{i + 1 | i \in I\}$, and let $\mathrm{Peak}(I) = I \setminus (I + 1) \setminus \{1\}$ the peak-lacunar subset obtained from I (so $\mathrm{Peak}(I) = \mathrm{Peak}(\pi)$ for every $\pi \in S_n$ satisfying $\mathrm{Des}(\pi) = I$). One recovers immediately that for $q = 0$, $L^{(0)}_{n,I} = L_{n,I}$ is the Gessel fundamental quasisymmetric function indexed by the set I. For $q = 1$, $L^{(1)}_{n,I} = K_{n,\mathrm{Peak}(I)}$ is the Stembridge peak function indexed by the relevant peak-lacunar set. In the sequel we remove the reference to n in indices when it is clear from context. Proposition 2 admits a nice generalisation to this q-deformation.

Theorem 3. *Let $I \subseteq [n-1]$ and $q \in \mathbf{k}$. The q-fundamental quasisymmetric functions may be expressed in the enriched q-monomial basis as*

$$L_I^{(q)} = \sum_{\substack{J \subseteq I \\ K \subseteq \mathrm{Peak}(I) \\ J \cap K = \emptyset}} (-q)^{|K|} (q-1)^{|J|} \eta^{(q)}_{J \cup (K-1) \cup K}. \tag{4.2}$$

Proof. This a consequence of Equation (3.3). □

Proposition 10. *Recall the antipode S of Section 3.3. Let $q \in \mathbf{k}$ be invertible, and set $p = \frac{1}{q}$. Let $I \subseteq [n-1]$, and set $n - I = \{n - i \mid i \in I\}$. Then,*

$$S(L_I^{(q)}) = (-q)^n L_{n-I}^{(p)}. \tag{4.3}$$

Proof. This a consequence of Equations (4.2) and (3.9). □

To know whether $(L_{n,I}^{(q)})_{n\geq 0, I\subseteq[n-1]}$ is a basis of QSym for some value of q appears as a natural question. For example, for $n = 3$, we can invert Equation (4.2) as follows:

- $\eta_{\varnothing}^{(q)} = L_{\varnothing}^{(q)}$;
- $(q-1)\eta_{\{1\}}^{(q)} = L_{\{1\}}^{(q)} - L_{\varnothing}^{(q)}$;
- $(q-1)\eta_{\{2\}}^{(q)} = \frac{(q-1)^2}{(q-1)^2+q}(L_{\{2\}}^{(q)} - L_{\varnothing}^{(q)}) + \frac{q}{(q-1)^2+q}(L_{\{1,2\}}^{(q)} - L_{\{1\}}^{(q)})$;
- $\eta_{\{1,2\}}^{(q)} = \frac{1}{(q-1)^2+q}\left(L_{\{1,2\}}^{(q)} - L_{\{2\}}^{(q)} - L_{\{1\}}^{(q)} + L_{\varnothing}^{(q)}\right)$.

We see that except for the case of Stembridge $q = 1$ (and the degenerate case $q = -1$), $\left(L_{2,I}^{(q)}\right)_{I\subseteq[2]}$ seems to be a basis of QSym. We state one of our main theorems:

Theorem 4. *Let $\mathbf{k}$ be the set $\mathbb{R}$ of real numbers. The family of q-fundamental quasisymmetric functions $(L_{n,I}^{(q)})_{n\geq 0, I\subseteq[n-1]}$ is a basis of* QSym *for $q \notin \{-1, 1\}$.*

Remark 1. We set $\mathbf{k} = \mathbb{R}$ for the sake of simplicity. For a more general field, $(L_{n,I}^{(q)})_{n,I\subseteq[n-1]}$ is a basis if and only if $q \notin \{\rho | \rho^k = 1 \text{ for some integer } k > 0\}$.

4.2 Proof of Theorem 4

To prove Theorem 4 we characterise the transition matrix between the q-fundamental and enriched q-monomial quasisymmetric functions and show it is invertible for $q \neq -1, 1$.

Definition 12. Let B_n be the transition matrix between $(L_I^{(q)})_{I\subseteq[n-1]}$ and $(\eta_J^{(q)})_{J\subseteq[n-1]}$ with coefficients given by Equation (4.2). Columns and rows are indexed by subsets I of $[n-1]$ sorted in reverse lexicographic order. A subset I is before subset J if and only if the word obtained by writing the elements of I in decreasing order is before the word obtained from J for the lexicographic order.

Example 2. For $n = 4$, let us show the transition matrix B_4 between $(L_I^{(q)})_{I\subseteq[3]}$ and $(\eta_J^{(q)})_{J\subseteq[3]}$. The entry at row index I and column index J is the coefficient in $\eta_J^{(q)}$ of $L_I^{(q)}$ in Equation (4.2).

$$B_4 = \begin{array}{c|cccccccc} & \varnothing & \{1\} & \{2\} & \{2,1\} & \{3\} & \{3,1\} & \{3,2\} & \{3,2,1\} \\ \hline \varnothing & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \{1\} & 1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \{2\} & 1 & 0 & q-1 & -q & 0 & 0 & 0 & 0 \\ \{2,1\} & 1 & q-1 & q-1 & (q-1)^2 & 0 & 0 & 0 & 0 \\ \{3\} & 1 & 0 & 0 & 0 & q-1 & 0 & -q & 0 \\ \{3,1\} & 1 & q-1 & 0 & 0 & q-1 & (q-1)^2 & -q & -q(q-1) \\ \{3,2\} & 1 & 0 & q-1 & -q & q-1 & 0 & (q-1)^2 & -q(q-1) \\ \{3,2,1\} & 1 & q-1 & q-1 & (q-1)^2 & q-1 & (q-1)^2 & (q-1)^2 & (q-1)^3 \end{array}$$

Using Definition 12 and Equation (4.2), one can deduce the following lemmas.

Lemma 1. *The matrix B_n is block triangular. To be more specific:*

For each $k \in [n]$, let A_k denote the transition matrix from $(L_I^{(q)})_{I\subseteq[n-1],\ \max(I)=k-1}$ to $(\eta_J^{(q)})_{J\subseteq[n-1],\ \max(J)=k-1}$ (where $\max\varnothing := 0$); this actually does not depend on n. Note that A_k is a $2^{k-2}\times 2^{k-2}$-matrix if $k \geq 2$, whereas A_1 is a 1×1-matrix. We have

$$B_n = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ * & A_2 & 0 & \dots & 0 \\ * & * & A_3 & \dots & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & A_n \end{pmatrix}.$$

Lemma 2. *The matrices $(B_n)_n$ and $(A_n)_n$ satisfy the following recurrence relations (for $n \geq 1$ and $n \geq 2$, respectively):*

$$B_n = \begin{pmatrix} B_{n-1} & 0 \\ B_{n-1} & A_n \end{pmatrix}, \qquad A_n = \begin{pmatrix} (q-1)B_{n-2} & -qB_{n-2} \\ (q-1)B_{n-2} & (q-1)A_{n-1} \end{pmatrix}.$$

Thanks to Lemmas 1 and 2, we are ready to state and show the main proposition of this section and prove Theorem 4.

Proposition 11. *The matrix B_n is invertible for $q \neq 1$.*

Proof. For any square matrix M, let $|M|$ denote its determinant. We want to show that for all n, $|B_n| \neq 0$ or equivalently that $|A_n| \neq 0$. To this end we compute for any rational functions in q, α and β:

$$|\alpha A_n + \beta B_{n-1}| = ((q-1)\alpha + \beta)|B_{n-2}||((q-1)\alpha + \beta)A_{n-1} + q\alpha B_{n-2}|. \tag{4.4}$$

Equation (4.4) exhibits a recurrence relation on the determinants that we solve by defining the sequence of coefficients:

$$\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}, \qquad \begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix} = \begin{pmatrix}q-1 & 1\\ q & 0\end{pmatrix}\begin{pmatrix}\alpha_{i-1}\\ \beta_{i-1}\end{pmatrix} = \begin{pmatrix}q-1 & 1\\ q & 0\end{pmatrix}^i \begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix}.$$

We have:

$$|A_n| = \left[\prod_{i=0}^{n-3} |B_{n-2-i}| \begin{pmatrix}q-1 & 1\end{pmatrix}\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}\right] \left|\begin{pmatrix}A_2 & B_1\end{pmatrix}\begin{pmatrix}\alpha_{n-2}\\ \beta_{n-2}\end{pmatrix}\right|.$$

But $A_2 = (q-1)$, $B_1 = (1)$ and one may compute that (left to the reader):

$$\begin{pmatrix}q-1 & 1\end{pmatrix}\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix} = \frac{1}{q+1}\left(q^{i+2} - (-1)^{i+2}\right) = (-1)^{(i+1)}[i+2]_{-q},$$

where for integer p, $[p]_q$ is the *q-number*, $[p]_q = 1 + q + q^2 + \cdots + q^{p-1}$. Define the *q-factorial* $[p]_q! = [1]_q \cdot [2]_q \cdots [p]_q$. We find

$$|A_n| = (-1)^{n(n-1)/2}[n]_{-q}! \prod_{i=1}^{n-2} |B_i|.$$

Then, notice that $[n]_{-q}!$ is 0 if and only if $q = 1$ and $n > 1$ (when q runs over real numbers). Finish the proof with a simple recurrence argument on $|B_i|$. □

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