

Product-Coproduct Prographs and Triangulations of the Sphere

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Abstract. In this paper, we explain how the classical Catalan families of objects involving paths, tableaux, triangulations, parentheses configurations and more generalize canonically to a three-dimensional version. In particular, we present product-coproduct prographs as central objects explaining the combinatorics of the triangulations of the sphere. Then we expose a natural way to extend the Tamari lattice to the product-coproduct prographs.

Résumé. Dans ce papier, nous expliquons comment les familles d'objets combinatoires comptées par les nombres de Catalan, comme les chemins, tableaux, triangulations, parenthésages, ou autre, se généralisent dans le monde des objets comptés par les nombres de Catalan tridimensionnels. En particulier, nous présentons les prographes produit-coproduit comme centraux pour expliquer la combinatoire des triangulations de la sphère. Nous exposons alors une manière naturelle d'étendre le treillis de Tamari aux prographes produit-coproduit.

Keywords: Tamari lattice, Catalan numbers, prographs, triangulation, sphere

1 Introduction

This paper is about a generalization of the famous combinatorial problem defining Catalan numbers and involving parentheses configurations, binary trees and triangulations of the n -gon. This paper is a sequel of [1] where product-coproduct prographs (PC prographs) were introduced to study the combinatorics of the three-dimensional Catalan world. The work presented in this paper follows the results obtained in FPSAC 2017 [1], whose statements can be summarized in Table 1.

In this paper, we complete these first statements to the dual side of planar objects (triangulations of the sphere) and the combinatorics of the planar objects.

Mireille Bousquet-Mélou did expose in [2] that the three-dimensional Catalan numbers count the number of triangulations of the bipolar sphere. By embedding PC prographs on the sphere and linking the only global output to the global input, we present

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PC prographs as Voronoï diagrams of the triangulations of the bipolar sphere. This implies that the original bijection from rectangular Young Tableaux to PC prographs can be extended to triangulations.

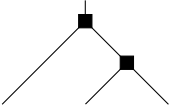
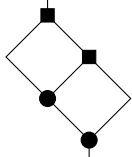
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Table 1: Classical Catalan world and its three-dimensional counterpart.

By duality, since rotations in prographs correspond to triangle flipping, we define four rotations rules on PC-prographs. We explain that these rotations correspond exactly to all admissible triangulation flips on the triangulations side. Among these four rotations rules, two are extensions of the classical rotations balancing binary search trees and the other two are suggested by the geometry of the triangulations of the sphere. Finally, we present some structural properties of the set of PC prographs of size n extended with the four rotations viewed as oriented rewriting rules. We argue that, in some cases, PC prographs can be viewed as a gluing of two binary trees; endowing this subfamily with two of the four rotations makes it a lattice that is a product of the Tamari lattice by itself. However, although adding the other two rotations does generate all PC-prographs with faithful action of the rotations, this does not provide a lattice.

This paper is organised as follows. In the next section, we first recall what a PC prograph is and describe the duality between PC prographs and rooted triangulations

of the sphere. We exploit the duality to introduce and orient flips and rotations. In Section 4, we investigate structures that can be defined over the set of PC prographs.

2 Duality and triangulations

PC prographs are connected oriented planar graphs without cycles whose vertices are products or coproducts. Coproducts have one input and two outputs and products have two inputs and one output. Just one coproduct (resp. product) has its input (resp. output) unconnected, which is the input (resp. output) of the prograph (see Figure 2 for examples). Since each vertex has degree three, the dual graph consists of triangles.

Now, embed a prograph on the sphere and connect its output to its input going through the exterior non visible side of the sphere. The prograph becomes a Voronoï diagram of some triangulation of the sphere. Since [2, 6] prove that bipolar oriented triangulations of the sphere are counted by the three-dimensional Catalan numbers, this procedure provides an alternative and explicit bijection.

For example, consider the following three-row rectangular standard Young tableau:

7	8	11	12
3	4	9	10
1	2	5	6

This tableau is stable by the Schützenberger involution (see [1, 5]) (rotate 180° the tableau then complement its values sending i to $3n + 1 - i$). Using the central bijection of [1] from tableaux to prographs, we obtain the black prograph of Figure 1, with coproducts represented as circles and products as squares. Considering this prograph as a Voronoï diagram, we build the red oriented triangulation of the sphere with a single edge passing through its dark side. This red dashed edge (its midpoint could have been sent to infinity) splits the dark side of the sphere into two triangles: one containing the North pole of the sphere and the other one containing its South pole.

Remark 1. Switching the context from prographs to bipolar oriented triangulations of the sphere, the Schützenberger involution consists in rotating the sphere by 180 degrees, swapping the poles, and flipping the orientation of the edges of the triangulation. It is also equivalent to the antipodal map of the sphere with renamed poles since they have been swapped.

The reader can check that the triangulation obtained in Figure 1 is stable by the Schützenberger involution.

Theorem 1. *With PC prographs as an intermediate object, graph duality provides an explicit and constructive bijection from 3-row rectangular standard Young tableaux onto bipolar-oriented triangulations of the sphere. This bijection commutes with the Schützenberger involution.*

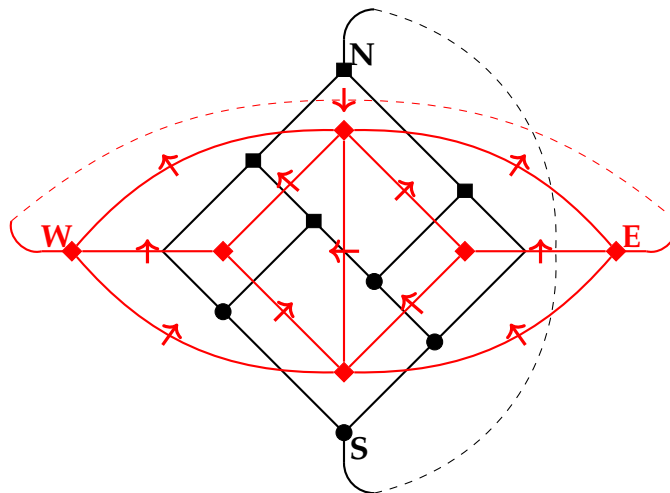


Figure 1: A self-dual prograph (black) and its bipolar-oriented triangulation (red).

Since there exists numerous algorithms for exhaustive enumeration and (possibly random) generation of Young tableaux, this bijection using PC prographs enables a lot of calculations and computations on the triangulations side. For example, we believe that the easiest way to check if two oriented triangulations of the bipolar sphere are homotopic is to build their associated prographs with graph duality and check whether they are associated with the same Young tableau.

Remark 2. The world of prographs meets here the world of oriented planar maps. Indeed, prographs are planar assemblies of operators with identified inputs and outputs; if one puts reasonable operators on the vertices of a planar oriented map, with respect to the number of incoming and outgoing edges for each vertex, one builds a prograph. Since, for PC prographs, a single entry means the vertex is a coproduct and two entries mean it is a product, there is no choice to be made and everyone works on the same objects.

3 Flips and rotations

The Tamari lattice has numerous different realizations: using parentheses configurations, binary trees, ordered forests, triangulations and more. Our first attempts to uncover a structure on PC prographs were by focusing on the prographs and Young tableaux side. Unfortunately, there were too many choices when trying to establish a set of transforms and give them orientations, thereby exhibiting a partial order (or more).

On the other hand, the geometric approach revealed the needed rigidity. When one considers triangulations, the reasonable flip consists in focusing on an edge, check that it is the border of two triangles; if the union of these two triangles forms a quadrilateral and the edge is a diagonal of it, then one flips this edge by removing it and drawing the

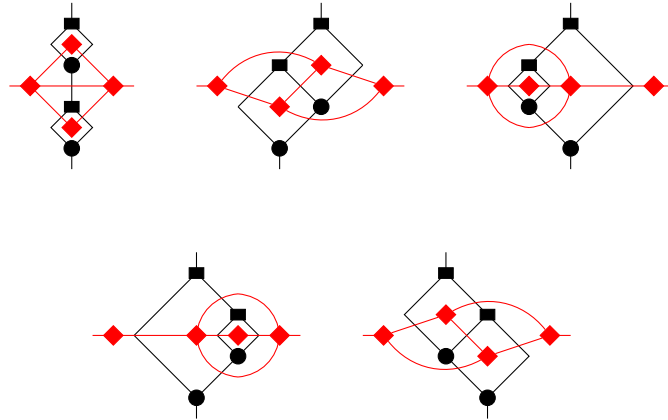


Figure 2: The five prographs with two coproducts, two products (in black), and their associated triangulations (in red).

type of edges	input of coproduct	left input of product	right input of product
left output of coproduct	Type I reduced	Type II no possible flip	Type III reducible to type VII
right output of coproduct	Type IV reducible to type I	Type V reduced	Type VI no possible flip
output of product	Type VII reducible to type V	Type VIII reducible to type IX	Type IX reduced

Figure 3: An edge is flippable depending on the type of its entry and its outgoing.

other diagonal instead. Using this rule alone and by exhaustion of cases, we now show that there is only one reasonable way to consider flips and rotations.

By graph duality, each triangle contains a single operator, a product or a coproduct. Pairs of neighboring triangles (sharing at least an edge) can be enumerated by the entries inside the associated Young tableau since, if triangles are neighbors, there exists an edge between both involved operators. We forget 1 which virtually connects the northernmost product with the southernmost coproduct *via* the dark side of the sphere. This pointed edge in PC prographs correspond to a pointed edge in triangulations that we will not flip (it plays the same role as the first side of the n -gon in the classical Catalan world.)

In PC prographs, there are three types of outputs : outputs of products, left outputs of coproducts and right outputs of coproducts. Similarly, there are three types of inputs : inputs of coproducts, left inputs of products and right inputs of products. Since edges in PC prographs connect an output of operator to an input, we should consider nine cases.

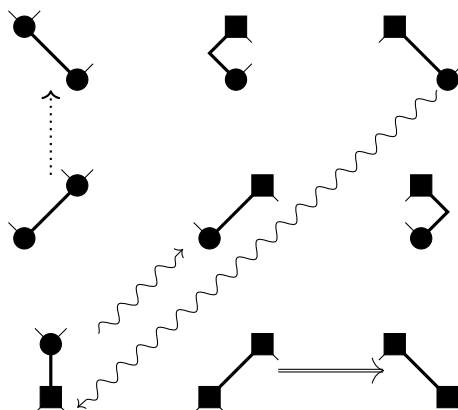


Figure 4: The 9 types of edges and their reductions via oriented rotations

The only choice we made is the orientation of operations. Here, types I, V and IX are reduced by choice, and, since we want to be coherent with the Schützenberger involution, the only other possible choice is to reverse all orientations simultaneously. Reversing all rules, types III, IV and VI would have been the reduced edges. This corresponds to exchanging the left and right everywhere and flipping the east and west on the sphere.

The reader can check exhaustively that a flip of types II and VI would create a loop inside the prograph, whatever the choice of orientation. Therefore flipping these two types of edges is technically impossible. It so happens that these edges correspond exactly to pathological organizations of triangles on the sphere. For example, on the sphere, there exist valid triangulations in which two neighboring triangles have two edges in common. The union of two such triangles does not form a quadrilateral and no flip is possible.

Theorem 2. *The four rotations rules in PC prographs shown in Figure 5 correspond to all valid edge flippings in rooted oriented triangulations of the sphere.*

The reader can check that the first two rotations are directly derived from the classical rotation rule in binary trees.

The four rotations rules, without their orientations, are completely rigid as far as geometry is concerned. Anyway, the reader could argue that choices of orientation were made. Since we still want to be compatible with the Schützenberger involution, the choice of minimum and maximum must be reversed for product and coproduct. But exactly as the classical Tamari lattice implies a choice (one can set either the left comb tree or the right comb tree to be the maximum), there is an equivalent choice with PC prographs. This will result in the same kind of symmetries for the classical Tamari lattice and our new structure on PC prographs.

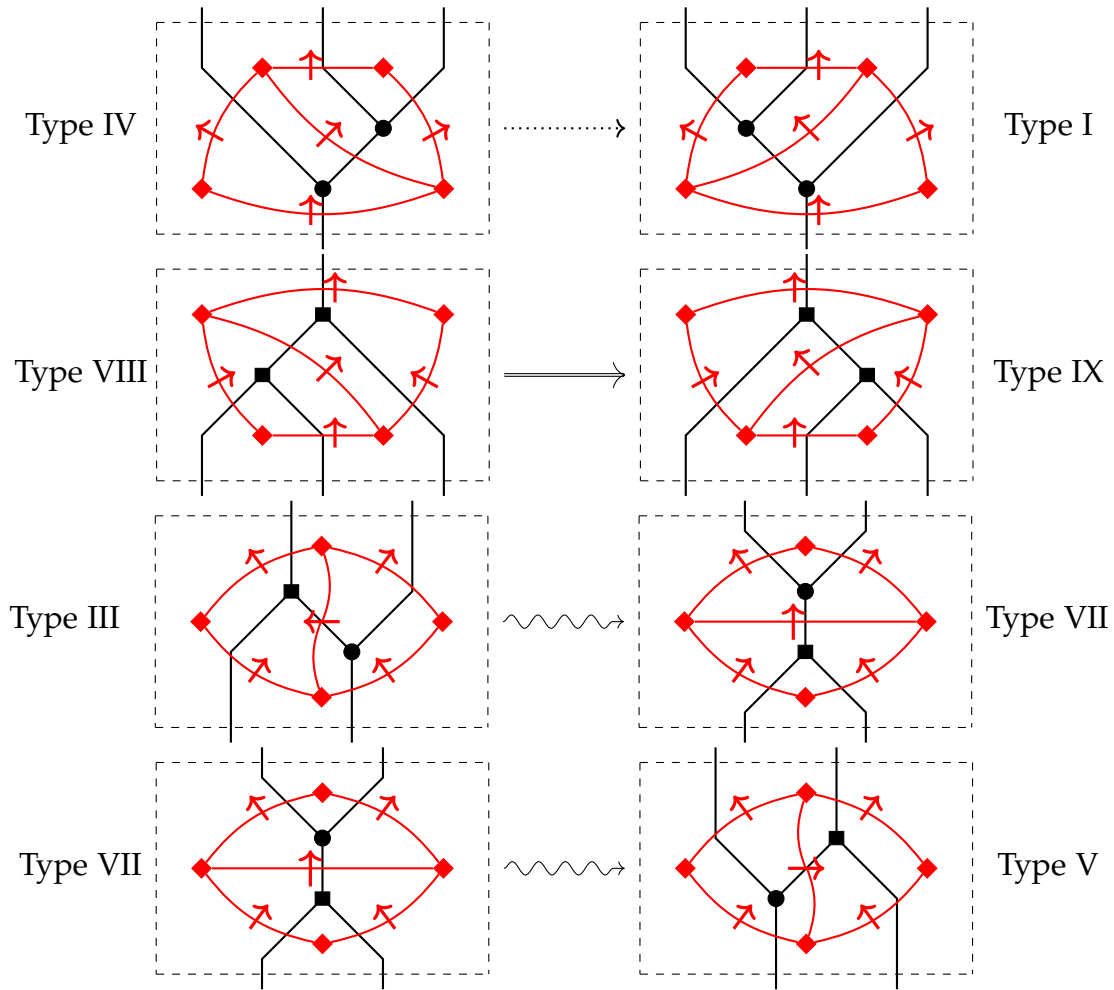


Figure 5: The four local rotations over PC prographs shown with their dual.

4 Poset structure over PC prographs

In this section, we investigate what type of structure can be placed over the set of PC prographs.

4.1 Gluing two trees at their canopies

The orientation in rooted triangulations of the sphere come directly from orientations of the classical case, the triangulations of the rooted n -gon. Choosing a root side in the n -gon amounts to setting an orientation to a chosen side (incoming or outgoing, the flow

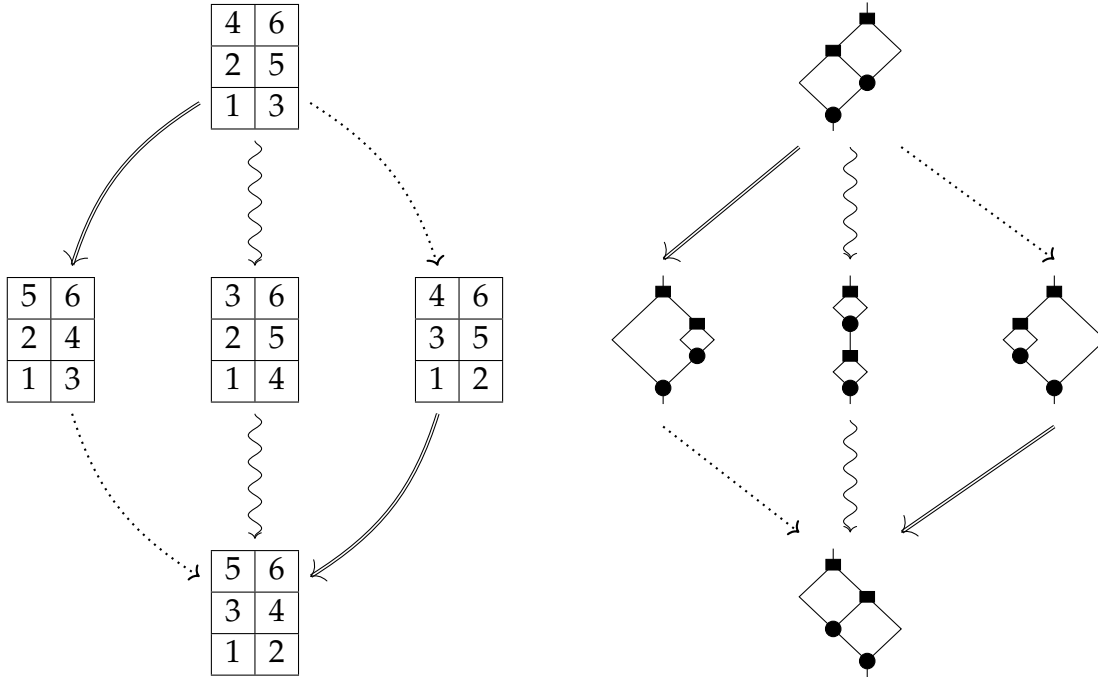


Figure 6: The poset (lattice for this size) of PC prographs of size 2.

being perpendicular to this side) and setting the reverse orientation to all other sides. So there are only two isomorphic choices. On the one hand, one can set the root edge to be outgoing. The other sides are thus incoming, hence inputs, and the dual of the triangulation ends up being a rooted tree of products. On the other hand, one can set the root edge to be incoming, all other sides become outputs and the dual becomes a rooted tree of coproducts.

Now, consider two trees with the same number of nodes. Build their associated triangulations of the n -gon and set opposite orientations on each of them (as illustrated in Figure 7). By deformation, we can imagine these two n -gons are two hemispheres that can be glued along their equator, the root sides being glued together. This way, considering the underlying trees, the $n - 1$ outputs of the tree of coproducts end up connected to the $n - 1$ inputs of the tree of products. Call the product of the outgoing root side the North and the coproduct of the incoming root side the South, and set them at the top and bottom by convention, respectively. This operation produces a PC prograph having its products connected up to the North point and its coproducts connected up from the South. The North and the South are also connected by the only upside-down edge of the equator, on the dark side of the sphere. We obtain a figure similar to Figure 1.

An immediate property is that PC prographs having coproducts all connected from

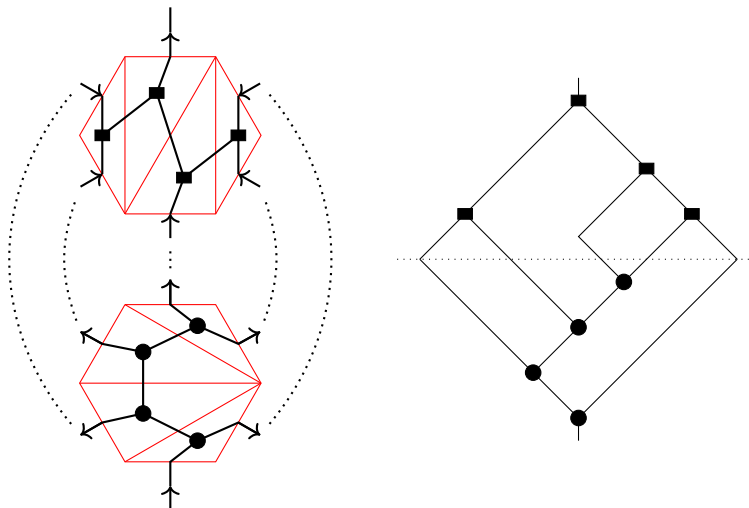


Figure 7: Build the sphere and a PC prograph from two triangulations of the n -gon.

the South and products all connected up to the North are exactly PC prographs avoiding a pattern: the output of a product grafted as input of a coproduct.

Theorem 3. *The subset of PC prographs avoiding edges of types VII (connecting a product output to a coproduct input) endowed with the two classical flip types forms a lattice which is exactly the product lattice of the classical Tamari lattice with itself.*

4.2 PC prographs whose coproducts show up first

In Section 5 of [1], we exposed a process labelling the operators: from bottom to top and from left to right for operators whose entries lean on already labelled operators. This natural process (called Borification in [4]) is recalled in Figure 8.

Among all PC prographs, a small portion produce labels $1, 2, \dots, n$ to coproducts and labels $n + 1, n + 2, \dots, 2n$ to products. In some sense, the depth-left-first transversal of operators enumerate all coproducts before the first product can go out. Regarding standard tableaux, it is equivalent to check that the last entry of the first row (indexing the last coproduct) is smaller than the first entry of the last row (indexing the first product to have its two entries labelled). This subfamily of PC prographs is counted by Sequence A274969 of OEIS [8]. It is also a subfamily of that described in Section 4.1. Such prographs can be obtained as a gluing of two binary trees with a reattachment condition.

Labelling nodes of binary trees T with the depth-left-first search algorithm, we can refine the enumeration of binary trees by counting leaves to the left of the last node. Denoting by $size(T)$ the number of nodes in T and by $dg(T)$ (disabled grafting sites) the

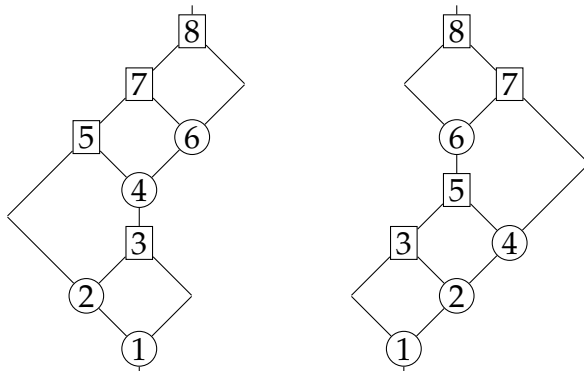


Figure 8: Labelling operators of a prograph and its image by the Schützenberger involution.

number of leaves to the left of the last node, we define $Cat_i(q)$ to be the following formal sum over all binary trees:

$$Cat_i(q) = \sum_{size(T)=i} q^{dg(T)}. \quad (4.1)$$

$Cat_i(q)$ deploys one very famous Catalan triangle. Here are the first values.

$$\begin{aligned} Cat_0(q) &= 1, \\ Cat_1(q) &= 1, \\ Cat_2(q) &= 1 + q, \\ Cat_3(q) &= 1 + 2q + 2q^2, \\ Cat_4(q) &= 1 + 3q + 5q^2 + 5q^3, \\ Cat_5(q) &= 1 + 4q + 9q^2 + 14q^3 + 14q^4, \\ Cat_6(q) &= 1 + 5q + 14q^2 + 28q^3 + 42q^4 + 42q^5. \end{aligned}$$

It is known that the coefficients of this triangle satisfy

$$C(n, k) = \binom{n+k}{k} - \binom{n+k}{k-1}.$$

Proposition 1. *PC prographs of size n whose operators are labelled by indices $1, 2, \dots, n$ for coproducts and indices $n+1, n+2, \dots, 2n$ for products are counted by pairs of binary trees of size n with d_1 and d_2 disabled grafting sites and $d_1 + d_2 \leq n+1$. Equivalently, we have*

$$\sum_{d_1+d_2 \leq n+1} C(n, d_1)C(n, d_2) = \binom{3n}{n} - 2\binom{3n}{n-1} + \binom{3n}{n-2}.$$

This formula is more complicated than the compact combination of three binomial due to Janis Stipins A274969 of OEIS [8]. Anyway, it provides an alternative way to

compute OEIS[8, A274969]: 1, 1, 4, 21, 121, 728, 4488, . . . Furthermore, it exposes an alternative description of pushall stack words of length $3n$ [7] as pairs of binary trees. The first values are the sum of the first coefficients of the square of $Cat_i(q)$.

$$\begin{aligned}
 Cat_0(q)^2 \bmod q &= 1 =_{|q=1} 1 \\
 Cat_1(q)^2 \bmod q^2 &= 1 =_{|q=1} 1 \\
 Cat_2(q)^2 \bmod q^3 &= 1 + 2q + q^2 =_{|q=1} 4 \\
 Cat_3(q)^2 \bmod q^4 &= 1 + 4q + 8q^2 + 8q^3 =_{|q=1} 21 \\
 Cat_4(q)^2 \bmod q^5 &= 1 + 6q + 19q^2 + 40q^3 + 55q^4 =_{|q=1} 121 \\
 Cat_5(q)^2 \bmod q^6 &= 1 + 8q + 34q^2 + 100q^3 + 221q^4 + 364q^5 =_{|q=1} 728 \\
 Cat_6(q)^2 \bmod q^7 &= 1 + 10q + 53q^2 + 196q^3 + 560q^4 + 1288q^5 + 2380q^6 =_{|q=1} 4488
 \end{aligned}$$

4.3 A poset that is not a lattice on PC prograph

Proposition 2. *The set of all PC prographs with the four rotations rules does not form a lattice.*

The generalization of the classical Catalan world stops here. The smallest pair of uncomparable elements are prographs of size 3. There are 3 elements higher in the poset than both prographs but no join can be defined (there are two uncomparable minima among these 3 higher elements).

7	8	9
2	4	6
1	3	5

5	8	9
2	6	7
1	3	4

Figure 9 displays the poset over the 42 PC prographs of size 3 (as standard tableaux).

This research was driven by computer exploration using the open-source mathematical software SAGEMATH [9] and its algebraic combinatorics features developed by the SAGE-COMBINAT community [3].

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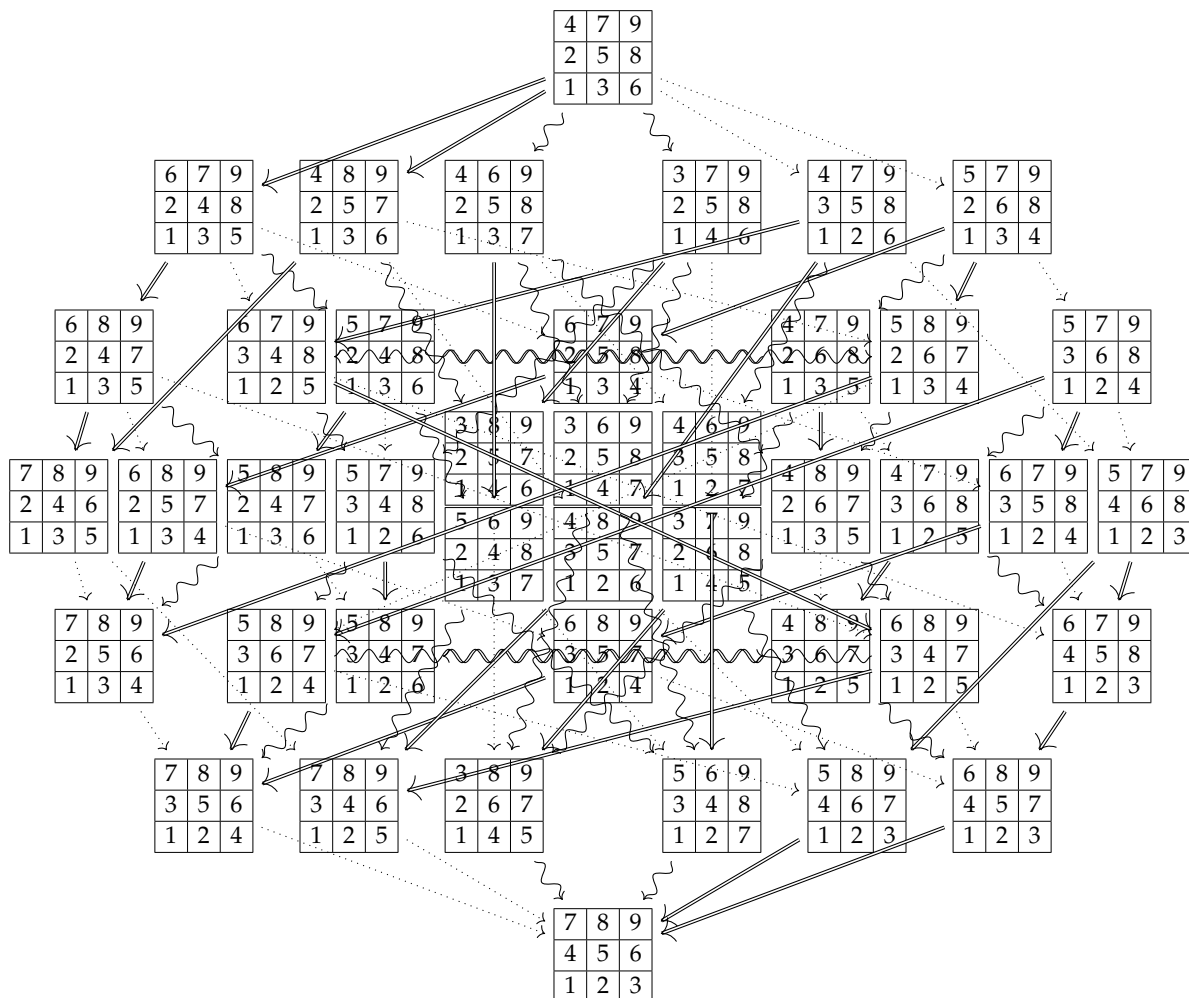


Figure 9: The poset formed by the 42 PC progaphs of size 3.

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