Séminaire Lotharingien de Combinatoire **86B** (2022)
Article #84, 12 pp.

Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)

Bumpless Pipe Dreams Encode Gröbner Geometry of Schubert Polynomials

Patricia Klein*[1] and Anna Weigandt†[2]

[1] *School of Mathematics, University of Minnesota, Minneapolis, MN 55455*
[2] *Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02142*

Abstract. In their study of infinite flag varieties, Lam, Lee, and Shimozono (2021) introduced bumpless pipe dreams in a new combinatorial formula for double Schubert polynomials. These polynomials are the $T \times T$-equivariant cohomology classes of matrix Schubert varieties and of their flat degenerations. We give diagonal term orders with respect to which bumpless pipe dreams index the irreducible components of diagonal Gröbner degenerations of matrix Schubert varieties, counted with scheme-theoretic multiplicity.

This indexing was conjectured by Hamaker, Pechenik, and Weigandt (2022). We also give a generalization to equidimensional unions of matrix Schubert varieties. This result establishes that bumpless pipe dreams are dual to and as geometrically natural as classical pipe dreams, for which an analogous anti-diagonal theory was developed by Knutson and Miller (2005).

Keywords: bumpless pipe dreams, Gröbner bases, Gröbner degenerations, matrix Schubert varieties, alternating sign matrix varieties

Introduction: The **complete flag variety** $\mathcal{F}(\mathbb{C}^n) = B_- \backslash \mathsf{GL}(\mathbb{C}^n)$ is the quotient of the general linear group by the Borel subgroup B_- of lower triangular matrices. There is a natural action of the Borel subgroup of upper triangular matrices B_+ on $\mathcal{F}(\mathbb{C}^n)$ by matrix multiplication. The orbits Ω_w of this action, called **Schubert cells**, are indexed by permutations w in the symmetric group S_n. The closures $\mathfrak{X}_w = \overline{\Omega_w}$ of these orbits are called **Schubert varieties**. Schubert varieties emerged in the study of the enumerative geometry problems posed by Schubert and his contemporaries.

Each Schubert variety gives rise to a **Schubert class** σ_w in the integral cohomology ring $H^*(\mathcal{F}(\mathbb{C}^n))$. Indeed, these Schubert classes form a $\mathbb{Z}$-linear basis for $H^*(\mathcal{F}(\mathbb{C}^n))$. Borel [4] showed that $H^*(\mathcal{F}(\mathbb{C}^n))$ is isomorphic to $\mathbb{Z}[x_1, \ldots, x_n]/I^{S_n}$, where I^{S_n} is the ideal generated by the nonconstant *elementary symmetric polynomials*. Geometric properties of $\mathcal{F}(\mathbb{C}^n)$ are readily expressed in terms of Schubert classes. For instance, the coefficients $c_{u,v}^w$ in the product $\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{u,v}^w \sigma_w$ are nonnegative integers; $c_{u,v}^w$ counts

*klein847@umn.edu. Travel was partially supported by an AMS-Simons Travel Grant
†weigandt@mit.edu. Partially supported by Bill Fulton's Oscar Zariski Distinguished University Professor Chair funds

points in the intersection of three Schubert varieties that depend on u, v, and w generically translated by elements of $\mathsf{GL}(\mathbb{C}^n)$. Something that for decades hindered the study of $\mathbb{Z}[x_1, \ldots, x_n]/I^{S_n}$ was that there was no known choice of desirable coset representatives.

Motivated by earlier work of Bernšteĭn, Gel'fand, and Gel'fand as well as Demazure, Lascoux and Schützenberger [21] proposed one such choice: the *Schubert polynomials* $\mathfrak{S}_w(\mathbf{x})$. Schubert polynomials have many desirable combinatorial properties. Importantly, if $u, v \in S_n$, then, for N sufficiently large with respect to n, the coefficients in the product $\mathfrak{S}_u(\mathbf{x})\mathfrak{S}_v(\mathbf{x}) = \sum_{w \in S_N} c_{u,v}^w \mathfrak{S}_w(\mathbf{x})$ agree with those arising from the multiplication of the corresponding Schubert classes in $\mathcal{F}(\mathbb{C}^N)$. Moreover, Schubert polynomials expand positively in the monomial basis, allowing for numerous combinatorial interpretations for these coefficients. Of particular importance are the *pipe dream formula* of [2, 3, 7, 8] and a recent formula due to Lam, Lee, and Shimozono [19] in terms of *bumpless pipe dreams*, which they introduced in their study of back stable Schubert calculus.

Bumpless pipe dreams had also appeared earlier in a different form in work related to study of the *six-vertex model*. In this context, they are called *osculating lattice paths* (see, *e.g.*, [1]). In an unpublished preprint, Lascoux [20] used the six-vertex model to give a formula for Grothendieck polynomials, which can be used to recover the formula of [19] for Schubert polynomials (see [26]). By interpreting bumpless pipe dreams as planar histories for permutations, Lam, Lee, and Shimozono gave a formula for *double Schubert polynomials* that is analogous to (but distinct from) the traditional pipe dream formula. Double Schubert polynomials represent classes of Schubert varieties in the Borel-equivariant cohomology of $\mathcal{F}(\mathbb{C}^n)$. Lam, Lee, and Shimozono's innovation has inspired a great deal of further exploration of the combinatorics of Schubert polynomials.

Despite the combinatorial desirability of Schubert polynomials, there was for many years skepticism over whether they were really the right choice. It was profoundly unclear whether Schubert polynomials reflected any of the geometric content of Schubert varieties. Progress on this front came by way of understanding torus-equivariant classes of *matrix Schubert varieties*.

The torus T of diagonal matrices acts on $\mathsf{Mat}(\mathbb{C}^n)$ by matrix multiplication on the right, and so we can study the ring $H_T^*(\mathsf{Mat}(\mathbb{C}^n)) \cong \mathbb{Z}[x_1, \ldots, x_n]$ of T-equivariant cohomology. There is a projection map $\pi\colon \mathsf{GL}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n) = B_- \backslash \mathsf{GL}(\mathbb{C}^n)$ and an inclusion map $\iota\colon \mathsf{GL}(\mathbb{C}^n) \to \mathsf{Mat}(\mathbb{C}^n)$ taking elements of the general linear group into the space of $n \times n$ matrices. The **matrix Schubert variety** of w, introduced by Fulton [9], is $X_w = \overline{\iota(\pi^{-1}(\mathfrak{X}_w))}$, which is an orbit closure for the natural $B_- \times B_+$ action on $\mathsf{Mat}(\mathbb{C}^n)$. Because X_w is stable under the action of T, it gives rise to a class $[X_w]_T \in H_T^*(\mathsf{Mat}(\mathbb{C}^n))$. Furthermore, this class is a polynomial representative for the Schubert class σ_w in $H^*(\mathcal{F}(\mathbb{C}^n))$. Remarkably, $[X_w]_T = \mathfrak{S}_w(\mathbf{x})$, *i.e.*, the coset representative for σ_w that was singled out by Lascoux and Schützenberger is the same one identified by the theory of T-equivariant cohomology (see [9], [6, Theorem 4.2], [14, Theorem A]). In this sense, Schubert polynomials are canonical representatives for Schubert classes.

Analogously, double Schubert polynomials represent classes of matrix Schubert varieties in $H^*_{T\times T}(\mathsf{Mat}(\mathbb{C}^n))$, and so double Schubert polynomials are identified as natural representatives for Schubert classes in $H^*_{B_+}(\mathcal{F}(\mathbb{C}^n))$.

Furthermore, Knutson and Miller [14] were able to use Gröbner geometry to explain the appearance of the traditional pipe dream formula for Schubert polynomials. Fixing an *anti-diagonal* term order σ on the coordinate ring of $\mathsf{Mat}(\mathbb{C}^n)$, one can degenerate X_w to $\mathtt{in}_\sigma(X_w)$ (the scheme defined by the σ-initial ideal of the defining ideal of X_w), which Knutson and Miller showed to be a union of coordinate subspaces indexed by pipe dreams.

In this way, the pipe dream formula is a canonical choice of expression for Schubert polynomials, but only insofar as anti-diagonal term orders would be considered canonical term orders. Perhaps the most natural term order is lexicographic order on reading order, which is diagonal. Indeed, several years after [14], Knutson, Miller, and Yong [16] studied an arbitrary diagonal term order σ, but their results were restricted to the special case of *vexillary* matrix Schubert varieties. They showed that, in this case, the irreducible components of $\mathtt{in}_\sigma(X_w)$ are indexed by *flagged tableaux* (or, equivalently, *diagonal pipe dreams*). One challenge of the diagonal degenerations of X_w is that they are not always reduced. For this reason, the complete story of the diagonal degenerations must include a count on the irreducible components with *multiplicity*. Outside of the vexillary setting, there was no combinatorial candidate to index components of $\mathtt{in}_\sigma(X_w)$.

Recently, Hamaker, Pechenik, and Weigandt [10] extended [16] to a wider class of matrix Schubert varieties. They showed that in this larger special case the irreducible components of $\mathtt{in}_\sigma(X_w)$ are indexed by the bumpless pipe dreams of [19]. The main theorem of the present work was previously conjectured by Hamaker, Pechenik, and Weigandt ([10, Conjecture 1.2]).

Main Theorem. *There exist diagonal term orders with respect to which the irreducible components, counted with multiplicity, of the initial scheme* $\mathtt{in}(X_w)$ *of the matrix Schubert variety* X_w *naturally correspond to the bumpless pipe dreams for the permutation* w.

In fact, we prove a more general statement for arbitrary unions of matrix Schubert varieties of a fixed dimension (see Theorem 20). Hamaker, Pechenik, and Weigandt's conjecture remains open for arbitrary diagonal term orders. Our theorem holds over an arbitrary field κ. When $\kappa = \mathbb{C}$, one recovers T-equivariant classes from the *multidegrees* of [11, 24]. Indeed, when S is a multigraded polynomial ring over $\mathbb{C}$ and I is a multihomogenous ideal, then the multidegree of S/I is the class of $\mathrm{Spec}(S/I)$ in the T-equivariant Chow ring of $\mathrm{Spec}(S)$ (see [15, Proposition 1.19]).

In order to prove Theorem 20, we give an algebro-geometric recurrence on unions of matrix Schubert varieties which mirrors both the recurrence of Lascoux and Schützenberger's transition equations and also a corresponding transition on bumpless pipe dreams (Lemma 7 and [26, Section 5]). This situation is (projectively) dual to that of [13], which involves co-transition, pipe dreams, and unions of matrix Schubert varieties.

Background and preliminaries: Throughout this paper, we will take κ to be an arbitrary field. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}_+ = \{1, 2, 3 \ldots\}$. Given $m, n \in \mathbb{Z}_+$, let $[n] = \{1, 2, \ldots, n\}$ and $[m, n] = \{i \in \mathbb{Z}_+ : m \leq i \leq n\}$. The **symmetric group** S_n is the group of permutations of n letters. We often represent permutations in one-line notation. It will sometimes also be convenient to represent permutations as **permutation matrices**. We identify the permutation $w \in S_n$ with the matrix that has 1's in positions $(i, w(i))$ for all $i \in [n]$ and 0's in all other positions. The transposition $t_{i,j}$ is the 2-cycle (ij), and we write s_i for the simple reflection $(i\, i+1)$. We use $\ell(w) = \#\{(i, j) : i < j \text{ and } w(i) > w(j)\}$ to denote the **Coxeter length** of $w \in S_n$.

The **(strong) Bruhat order** on S_n is the transitive closure of covering relations of the form $w < wt_{i,j}$ if $\ell(w) + 1 = \ell(wt_{i,j})$. There is another characterization of Bruhat order we will use: Define the **rank function** of w to be $\mathtt{rk}_w(a, b) = \#\{(i, j) \in [a] \times [b] : w(i) = j\}$. Then $w \leq v$ if and only if $\mathtt{rk}_w(i, j) \geq \mathtt{rk}_v(i, j)$ for all $i, j \in [n]$.

Given $w \in S_n$, the **Rothe diagram** of w is $D(w) = \{(i, j) : i, j \in [n], w(i) > j, \text{ and } w^{-1}(j) > i\}$. The Coxeter length of w satisfies $\ell(w) = \#D(w)$. The **essential set** of w is $\mathtt{Ess}(w) = \{(i, j) \in D(w) : (i + 1, j), (i, j + 1) \notin D(w)\}$, *i.e.*, the maximally southeast corners of the connected components of $D(w)$. A permutation $\pi \in S_n$ is **bigrassmannian** if $\#\mathtt{Ess}(\pi) = 1$. A bigrassmannian permutation is uniquely determined by the position of its essential cell and the value of its rank function at this position.

Alternating sign matrices: An **alternating sign matrix** (ASM) is a square matrix with entries in $\{-1, 0, 1\}$ so that the entries in each row (and column) sum to 1 and the nonzero entries in each row (and column) alternate in sign. An ASM with no negative entries is a permutation matrix. Write $\mathtt{ASM}(n)$ for the set of $n \times n$ ASMs.

The **corner sum function** of $A = (A_{i,j}) \in \mathtt{ASM}(n)$ is defined by

$$\mathtt{rk}_A(a, b) = \sum_{i=1}^{a} \sum_{j=1}^{b} A_{i,j} \text{ for } (a, b) \in [n] \times [n].$$

It will also be useful to define $\mathtt{rk}_A(i, j) = 0$ whenever $i = 0$ or $j = 0$. If $A \in S_n$, then $\mathtt{rk}_A$ agrees with the definition of the rank function of a permutation. We may also use $\mathtt{rk}_A$ to denote the **corner sum matrix** of A, the $n \times n$ matrix whose $(i, j)^{th}$ entry is $\mathtt{rk}_A(i, j)$: that is, $\mathtt{rk}_A = ((\mathtt{rk}_A)_{i,j}) = (\mathtt{rk}_A(i, j))$. When $A \in S_n$, the corner sum matrix is commonly called the **rank matrix** of A.

Corner sum functions induce a lattice structure on $\mathtt{ASM}(n)$ defined by $A \geq B$ if and only if $\mathtt{rk}_A(i, j) \leq \mathtt{rk}_B(i, j)$ for all $i, j \in [n]$. Restricting to permutations recovers the (strong) Bruhat order on S_n; indeed, the ASM poset is the smallest lattice with this property [22, Lemme 5.4]. One computes the **join** (least upper bound) $A \vee B$ by taking entry-wise minima of $\mathtt{rk}_A$ and $\mathtt{rk}_B$ and the **meet** (greatest lower bound) $A \wedge B$ by taking entry-wise maxima of $\mathtt{rk}_A$ and $\mathtt{rk}_B$. The bigrassmannian permutations are the joinirreducibles of the lattice of ASMs. To compare a bigrassmannian with an ASM, it is enough to compare a single value of their corresponding corner sum functions.

Let $\texttt{Perm}(A) = \{w \in S_n : w \geq A, \text{ and, if } w \geq v \geq A \text{ for some } v \in S_n, \text{ then } w = v\}$. We define $\deg(A) = \min\{\ell(w) : w \in \texttt{Perm}(A)\}$. If $\ell(w) = \deg(A)$ for all $w \in \texttt{Perm}(A)$, then we say that A is **equidimensional**.

Given a matrix M, let $M_{[i],[j]}$ be the submatrix of M consisting of the first i rows and j columns. Given $A \in \texttt{ASM}(n)$, we define the **ASM variety** of A to be $X_A = \{M \in \mathsf{Mat}(n) : \texttt{rk}(M_{[i],[j]}) \leq \texttt{rk}_A(i,j) \text{ for all } i,j \in [n]\}$. When $w \in S_n$, we say X_w is a **matrix Schubert variety**. For background on matrix Schubert varieties, see [9, 23].

Fix an $n \times n$ generic matrix $Z = (z_{i,j})$, and let $R = \kappa[z_{1,1}, \dots, z_{n,n}]$. We write $I_k(Z_{[i],[j]})$ for the ideal of R generated by the k-minors in $Z_{[i],[j]}$. As a convention, if $i = 0$ or $j = 0$, then define $I_k(Z_{[i],[j]}) = (0)$. The **ASM ideal** of A is $I_A = \sum_{i,j=1}^{n} I_{\texttt{rk}_A(i,j)+1}(Z_{[i],[j]})$.

We call the union of the $\texttt{rk}_A(i,j) + 1$-minors in $Z_{[i].[j]}$, as i and j range from 1 to n, the **natural generators** of I_A. If $w \in S_n$, I_w is also called a **Schubert determinantal ideal**.

Proposition 1 ([9, Proposition 3.3])**.** *If $w \in S_n$, then I_w is prime and* $\mathrm{ht} I_w = \ell(w)$.

By [9, Lemma 3.10], a Schubert determinantal ideal can be generated by a (usually proper) subset of its natural generators: $I_w = \sum_{(i,j) \in \texttt{Ess}(w)} I_{\texttt{rk}_w(i,j)+1}(Z_{[i],[j]})$. We call these generators the **Fulton generators**. There is a generalization of the Fulton generators for ASM ideals (see [25, Lemma 5.9]).

Given a simplicial complex Δ on vertex set $[n]$, we define the **Stanley–Reisner ideal** $I_\Delta \subseteq \kappa[z_1, \dots, z_n]$ of Δ to be $I_\Delta = (\prod_{i \in U} z_i : U \subseteq [n], U \notin \Delta)$. This map $\Delta \mapsto I_\Delta$ is a bijection from simplicial complexes on $[n]$ to squarefree monomial ideals of $\kappa[z_1, \dots, z_n]$. Let $\Delta(I)$ denote the simplicial complex associated to a squarefree monomial ideal I. For a subset $F \subseteq [n]$, observe that the prime ideal $P = (z_i : i \notin F)$ is a minimal prime of I if and only if F is a facet of $\Delta(I)$. For background, we refer the reader to [23, Chapter 1].

For general background on term orders, initial ideals, and Gröbner bases, we refer the reader to [5, Chapter 15]. For a term order σ on the polynomial ring R and an ideal I of R, we will use $\texttt{in}_\sigma(I)$ to denote the initial ideal of I with respect to σ. We say that a term order is **diagonal** (respectively, **anti-diagonal**) if the lead term of the determinant of a generic matrix is the product of the entries along the main diagonal (respectively, along the anti-diagonal).

Lemma 2. *Let $A \in \texttt{ASM}(n)$, and fix an anti-diagonal term order σ on $\kappa[z_{1,1}, \dots, z_{n,n}]$. Then the following hold: (i) If $w_1, \dots, w_r \in S_n$ such that $A = \vee\{w_1, \dots, w_r\}$, then $\sum_{i=1}^{r} \texttt{in}_\sigma(I_{w_i}) = \texttt{in}_\sigma(I_A) = \bigcap_{u \in \texttt{Perm}(A)} \texttt{in}_\sigma(I_u)$. (ii) If $w_1, \dots, w_r \in S_n$ such that $A = \vee\{w_1, \dots, w_r\}$, then $I_A = \sum_{i=1}^{r} I_{w_i}$. (iii) I_A is radical. (iv) I_A has the irredundant prime decomposition $I_A = \bigcap_{w \in \texttt{Perm}(A)} I_w$. (v)* $\mathrm{ht} I_A = \deg(A)$. *(vi) A is equidimensional if and only if Spec(R/I_A) is equidimensional.*

Hilbert functions and multidegrees: Fix a $\mathbb{Z}^d$-grading on the finitely generated κ-algebra S, and fix a finitely generated, $\mathbb{Z}^d$-graded S-module M. For $\mathbf{t} = (t_1, \dots, t_d)$

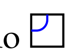
and $\mathbf{a} = (a_1, \ldots, a_d)$, let $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d}$ and $\langle \mathbf{a}, \mathbf{t} \rangle = a_1 t_1 + \cdots + a_d t_d$. Let $M_{\mathbf{a}}$ denote the $\mathbf{a}^{th}$ graded piece of M. If $\dim_{\kappa\text{-vect}}(M_{\mathbf{a}}) < \infty$ for all $\mathbf{a} \in \mathbb{Z}^d$, we define the **multigraded Hilbert series** of M, denoted $\mathrm{Hilb}(M; \mathbf{t})$, as $\mathrm{Hilb}(M; \mathbf{t}) = \sum_{\mathbf{a}} \dim_{\kappa\text{-vect}}(M_{\mathbf{a}}) \cdot \mathbf{t}^{\mathbf{a}}$. Note that $\mathrm{Hilb}(M; \mathbf{t})$ is an element of the formal Laurent series ring $\mathbb{Z}((t_1, \ldots, t_d))$. We refer the reader to [23, Chapter 8] for general background on multigraded Hilbert series.

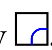
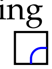
We will often want to consider Hilbert series of finitely generated modules over polynomial rings, especially polynomial rings equipped with term orders. We now restrict to that case. Let $S = \kappa[z_1, \ldots, z_n]$, and assume that the degrees $\deg(z_i) = (a_1, \ldots, a_d)$ of the algebra generators of S all lie in a single open half-space of $\mathbb{Z}^d$. This assumption guarantees that, for each finitely generated S-module M, $\dim_{\kappa\text{-vect}}(M_{\mathbf{a}}) < \infty$ for all $\mathbf{a} \in \mathbb{Z}^d$. Recall that if S is equipped with a term order σ and I is an ideal of S, then $\mathrm{Hilb}(S/I; \mathbf{t}) = \mathrm{Hilb}(S/\mathtt{in}_\sigma(I); \mathbf{t})$. If M is any finitely generated, $\mathbb{Z}^d$-graded S-module, then there is a unique polynomial $\mathcal{K}(M; \mathbf{t}) \in \mathbb{Z}[t_1, \ldots, t_d]$ so that $\mathrm{Hilb}(M; \mathbf{t}) = (\mathcal{K}(M; \mathbf{t}))/(\prod_{i=1}^n (1 - \mathbf{t}^{\deg(z_i)}))$. We call this polynomial $\mathcal{K}(M; \mathbf{t})$ the **K-polynomial** of M. The **multidegree** $\mathcal{C}(M; \mathbf{t})$ consists of the lowest degree terms of $\mathcal{K}(M; \mathbf{1} - \mathbf{t})$. For further background, see [14].

Bumpless pipe dreams and transition equations: A **bumpless pipe dream** (BPD) is a tiling of the $n \times n$ grid with the pictures in ($\star$) so that (1) there are n total pipes, (2) pipes start at the bottom edge of the grid and end at the right edge, and (3) pairwise, pipes cross at most one time.

 ($\star$)

Given a BPD $\mathcal{B}$, label its pipes $1, \ldots, n$ from left to right according to their starting columns. We obtain a permutation $w_{\mathcal{B}}$ by defining $w_{\mathcal{B}}(i)$ to be the label of the pipe that terminates in row i. Let $\mathsf{BPD}(w)$ be the set of BPDs of $w \in S_n$.

The **diagram** of $\mathcal{B}$ is the set $D(\mathcal{B}) = \{(i,j) : \text{there is a blank tile in row } i \text{ and column } j \text{ of } \mathcal{B}\}$. We associate to $\mathcal{B}$ the weight $\mathtt{wt}(\mathcal{B}) = \prod_{(i,j) \in D(P)} (x_i - y_j)$. The **Rothe** BPD for w is the (unique) BPD that has  tiles in cell $(i, w(i))$ for all $i \in [n]$ and has no  tiles. Notice that if $\mathcal{B}$ is the Rothe BPD for w, then $D(w) = D(\mathcal{B})$.

Fix a BPD $\mathcal{B}$. Suppose the tile in cell (i,j) is a downward elbow . Take $(a,b) \in D(\mathcal{B})$ with $i < a$ and $j < b$. Suppose further that the only  or  tile in the region $[i,a] \times [j,b]$ occurs in cell (i,j). Then we can take the pipe passing through (i,j) and bend it within the rectangle so that there are downward elbows  in cells (i,b) and (a,j) and an upward elbow  in cell (a,b). This move is called a **droop move**. Applying a droop move to $\mathcal{B} \in \mathsf{BPD}(w)$ produces another element of $\mathsf{BPD}(w)$. Furthermore, $\mathsf{BPD}(w)$ is connected by such moves:

Proposition 3 ([19, Proposition 5.3]). *Every $\mathcal{B} \in \mathsf{BPD}(w)$ can be reached from the Rothe BPD for w by a sequence of droop moves.*

Schubert polynomials are traditionally defined using divided difference operators. The content of [19, Theorem 5.13] is that those definitions are equivalent to the definitions we give here. The **double Schubert polynomial** of $w \in S_n$ is the sum $\mathfrak{S}_w(\mathbf{x},\mathbf{y}) = \sum_{\mathcal{B}\in\mathsf{BPD}(w)} \mathtt{wt}(\mathcal{B})$. We also at times consider the **single Schubert polynomial** $\mathfrak{S}_w(\mathbf{x}) = \sum_{\mathcal{B}\in\mathsf{BPD}(w)} \mathtt{wt}_{\mathbf{x}}(\mathcal{B})$, where $\mathtt{wt}_{\mathbf{x}}(\mathcal{B}) = \prod_{(i,j)\in D(\mathcal{B})} x_i$. Notice $\mathfrak{S}_w(\mathbf{x}) = \mathfrak{S}_w(\mathbf{x},\mathbf{0})$.

Write $\mathfrak{S}_w(\mathbf{1})$ for the result of substituting $x_i \mapsto 1$ for all $i \in [n]$ in $\mathfrak{S}_w(\mathbf{x})$. From the BPD definition of $\mathfrak{S}_w(\mathbf{x})$ given above, it is immediate that $\mathfrak{S}_w(\mathbf{1}) = \#\mathsf{BPD}(w)$. For the remainder of this abstract, we take $R = \kappa[z_{1,1},\ldots,z_{n,n}]$. We will be interested in two gradings on R. The first is the $\mathbb{Z}^n$ grading that assigns generators the degrees $\deg(z_{i,j}) = e_i$, the i^{th} standard basis vector. The second is the $\mathbb{Z}^{2n}$ grading for which $\deg(z_{i,j}) = e_i - e_{n+j}$. Let $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$. When writing Hilbert functions with respect to the $\mathbb{Z}^n$ grading, we will have $\mathbf{t} = \mathbf{x}$ and, when with respect to the $\mathbb{Z}^{2n}$ grading, $\mathbf{t} = (\mathbf{x},\mathbf{y})$.

Theorem 4 **([6, 14]).** *If I_w is the Schubert determinantal ideal of $w \in S_n$, then, with respect to the $\mathbb{Z}^n$ and $\mathbb{Z}^{2n}$ gradings on R given above, $\mathcal{C}(R/I_w;\mathbf{x}) = \mathfrak{S}_w(\mathbf{x})$ and $\mathcal{C}(R/I_w;\mathbf{x},\mathbf{y}) = \mathfrak{S}_w(\mathbf{x},\mathbf{y})$.*

Transition equations: Let $w \in S_n$. Pick a lower outside corner (a,b) of $D(w)$. Set $v = wt_{a,w^{-1}(b)}$. Observe that $D(w) = D(v) \cup \{(a,b)\}$ and, in particular, that $\ell(w) = \ell(v)+1$.

Notation 5. Let $\phi(w,z_{a,b}) = \{i \in [a-1] : vt_{i,a} > v \text{ and } \ell(vt_{i,a}) = \ell(v)+1 = \ell(w)\}$. Write $\Phi(w,z_{a,b}) = \{vt_{i,a} : i \in \phi(w,z_{a,b})\}$.

Theorem 6 **([18, Proposition 4.1]).** *Keeping the above notation, $\mathfrak{S}_w(\mathbf{x},\mathbf{y}) = (x_a - y_b)\cdot \mathfrak{S}_v(\mathbf{x},\mathbf{y}) + \sum_{u\in\Phi(w,z_{a,b})} \mathfrak{S}_u(\mathbf{x},\mathbf{y})$.*

If one takes the BPD formula of [19] as a definition for Schubert polynomials, Theorem 6 may be proved by appealing to the combinatorics of BPDs. Indeed, there is a bijective explanation.[1] Alternatively, one may define Schubert polynomials using transition equations and then recover the BPD formula as a consequence. The following combinatorial lemma shows the two definitions are equivalent.

Lemma 7. *Let I_w be a Schubert determinantal ideal and (a,b) a lower outside corner of $D(w)$. Let $v = wt_{a,w^{-1}(b)}$. There is a bijection $\psi\colon \mathsf{BPD}(w) \to \mathsf{BPD}(v) \cup \bigcup_{u\in\Phi(w,y)} \mathsf{BPD}(u)$ that respects the diagrams of the bumpless pipe dreams. Specifically, if $\psi(\mathcal{B}) \in \mathsf{BPD}(v)$, then $D(\mathcal{B}) = D(\psi(\mathcal{B})) \sqcup \{(a,b)\}$. Otherwise, $D(\mathcal{B}) = D(\psi(\mathcal{B}))$.*

Bumpless pipe dreams and geometric vertex decomposition: Geometric vertex decomposition, which was introduced by Knutson, Miller, and Yong [16] in the study of vexillary matrix Schubert varieties, will be one of our main tools for understanding Schubert determinantal ideals in the context of BPDs.

[1] See also [17] for a diagrammatic interpretation of transition.

Fix one of the algebra generators $z_{a,b}$ of R, and set $y = z_{a,b}$. For a polynomial $f = \sum_{i=0}^{m} \alpha_i y^i \in R$ with each $\alpha_i \in \kappa[z_{1,1}, \ldots, \widehat{y}, \ldots, z_{n,n}]$ and $\alpha_m \neq 0$, define the **initial y-form of f** to be $\mathtt{in}_y(f) = \alpha_m y^m$. Given an ideal I, define $\mathtt{in}_y(I)$ to be the ideal generated by the initial y-forms of I; that is, $\mathtt{in}_y(I) = (\mathtt{in}_y(f) : f \in I)$. We say that a term order σ on R is **y-compatible** if it satisfies $\mathtt{in}_\sigma(f) = \mathtt{in}_\sigma(\mathtt{in}_y(f))$ for every $f \in R$. Notice that whenever a term order σ is y-compatible, $\mathtt{in}_\sigma(I) = \mathtt{in}_\sigma(\mathtt{in}_y(I))$.

Definition 8 ([16, Section 2.1]). Suppose that I is an ideal of a polynomial ring that is equipped with the y-compatible term order σ and that I has a Gröbner basis of the form $\mathcal{G} = \{yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_\ell\}$ where y does not divide any q_i, r_i, or h_i. We define the ideals $C_{y,I} = (q_1, \ldots, q_k, h_1, \ldots, h_\ell)$ and $N_{y,I} = (h_1, \ldots, h_\ell)$. Then $\mathtt{in}_y(I) = C_{y,I} \cap (N_{y,I} + (y))$, and this decomposition is called a **geometric vertex decomposition of I with respect to y**.

Lemma 9. *Fix $w \in S_n$ and take (a,b) to be a lower outside corner of $D(w)$ corresponding to the variable $y = z_{a,b}$ of R. Write the Fulton generators of I_w as $\{yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_\ell\}$, where y does not divide any q_i, r_i, or h_j. If $N = (h_1, \ldots, h_\ell)$, then N is the Schubert determinantal ideal I_v for $v = wt_{a,w^{-1}(b)}$.*

Corollary 10. *Fix $w \in S_n$ with some lower outside corner (a,b) of $D(w)$ corresponding to the variable $y = z_{a,b}$ of R, and assume $\mathtt{rk}_w(a,b) \geq 1$. Write $v = wt_{a,w^{-1}(b)}$, and let π be the bigrassmannian permutation so that $\mathtt{Ess}(\pi) = \{(a-1, b-1)\}$ and $\mathtt{rk}_\pi(a-1, b-1) = \mathtt{rk}_w(a,b) - 1$. Then $C_{y,I_w} = I_v + I_\pi = I_{v\vee\pi}$ and $\mathtt{in}_y(I_w) = I_{v\vee\pi} \cap (I_v + (y))$.*

We now give a combinatorial lemma that will allow us to use Corollary 10 to identify the associated primes of C_{y,I_w}.

Proposition 11. *Fix $w \in S_n$ with some lower outside corner (a,b) of $D(w)$ corresponding to the variable $y = z_{a,b}$ of R, and assume $\mathtt{rk}_w(a,b) \geq 1$. Set $v = wt_{a,w^{-1}(b)}$, and let π be the bigrassmannian permutation so that $\mathtt{Ess}(\pi) = \{(a-1, b-1)\}$ and $\mathtt{rk}_\pi(a-1, b-1) = \mathtt{rk}_w(a,b) - 1$. Then $\mathtt{Perm}(v \vee \pi) = \Phi(w,y)$ and $\deg(v \vee \pi) = \ell(w)$.*

In light of Proposition 11, we may interpret the transition equations using ASMs:

Corollary 12. *Fix a lower outside corner (a,b) of $D(w)$. Suppose $\mathtt{rk}_w(a,b) \geq 1$ and let u be the bigrassmannian permutation so that $\mathtt{Ess}(\pi) = \{(a-1, b-1)\}$ and $\mathtt{rk}_\pi(a-1, b-1) = \mathtt{rk}_w(a,b) - 1$. Let $v = wt_{a,w^{-1}(b)}$. Then*

$$\mathfrak{S}_w(\mathbf{x}, \mathbf{y}) = (x_a - y_b) \cdot \mathfrak{S}_v(\mathbf{x}, \mathbf{y}) + \sum_{u \in \mathtt{Perm}(v\vee\pi)} \mathfrak{S}_u(\mathbf{x}, \mathbf{y}).$$

Proposition 13. *Fix $w \in S_n$. Let (a,b) be a lower outside corner of $D(w)$ corresponding to the variable $y = z_{a,b}$ of R and $\mathtt{in}_y(I_w) = C_{y,I_w} \cap (N_{y,I_w} + (y))$ be the geometric vertex decomposition of I_w at y. Then $C_{y,I_w} = \bigcap_{u\in\Phi(w,y)} I_u$, and $N_{y,I_w} = I_v$ where $v = wt_{a,w^{-1}(b)}$.*

The main result: For the remainder of this abstract, let σ be the lexicographic order on the ordering of the variables $z_{ij} > z_{i'j'}$ if $i > i'$ or if $i = i'$ and $j > j'$ unless otherwise stated. We will use $\min(I)$ to denote the set of minimal primes of the ideal I and $\ell(M)$ to denote the length of a finite length module M. When working over an algebraically closed field $\mathcal{F}$, $\ell(M)$ is the $\mathcal{F}$-vector space dimension of M. Suppose P is a minimal prime of I, in which case $\mathrm{Spec}(R/P)$ is an irreducible component of $\mathrm{Spec}(R/I)$. Recall that the multiplicity of $\mathrm{Spec}(R/P)$ along $\mathrm{Spec}(R/I)$ is defined to be the length $\ell(R_P/IR_P)$ (equivalently, $\ell((R/I)_P)$) and is denoted $\mathtt{mult}_P(R/I)$.

We will use geometric vertex decomposition to develop a recurrence on unions of matrix Schubert varieties that mirrors the recurrence on bumpless pipe dreams discussed earlier. Our tool for tracking multiplicities will be multidegrees.

Lemma 14. *Let σ be any term order on R. Let $w_1, \ldots, w_r$ for some $r \geq 1$ be distinct permutations of the same Coxeter length, and set $J = \bigcap_{i\in[r]} I_{w_i}$. Let $\min(\mathtt{in}_\sigma(J)) = \{P_1, \ldots, P_m\}$. Then $e(R/J) = \sum_{i=1}^{m} \mathtt{mult}_{P_i}(\mathtt{in}_\sigma(J)) = \sum_{i=1}^{r} \#\mathsf{BPD}(w_i)$.*

Lemma 15. *Fix $w_1, \ldots, w_r \in S_n$, and set $J = \bigcap_{i\in[r]} I_{w_i}$ for some $r \geq 1$. Fix a maximally southeast cell (a,b) among elements of $\bigcup_{i\in[r]} D(w_i)$, with (a,b) corresponding to the variable $y = z_{a,b}$ of R. Then $N_{y,J} = \bigcap_{i\in[r]} N_{y,I_{w_i}}$.*

Notation 16. Let $w_1, \ldots, w_r \in S_n$ be distinct permutations of Coxeter length h and $J = \bigcap_{i\in[r]} I_{w_i}$. Suppose that (a,b) is a maximally southeast cell among elements of $\bigcup_{i\in[r]} D(w_i)$ and that $y = z_{a,b}$ is involved in some Fulton generator of I_{w_i} for $i \in [q]$ but not for $i \in [q+1, r]$ for some $q \in [r]$. Set $N_{y,J}^{\mathrm{ht}} = \bigcap_{i\in[q]} N_{y,w_i}$. By Lemma 9, the N_{y,w_i} with $i \in [q]$ have height $\mathrm{ht}J - 1 = \mathrm{ht}N_{y,J}$ while the $N_{y,w_i} = I_{w_i}$ with $i \in [q+1, r]$ have height $\mathrm{ht}J$. Hence, $N_{y,J}^{\mathrm{ht}}$ is the intersection of the minimal primes of $N_{y,J}$ realizing its height.

Lemma 17. *Fix $w_1, \ldots, w_r \in S_n$ all of Coxeter length h, and set $J = \bigcap_{i\in[r]} I_{w_i}$ for some $r \geq 1$. Fix a maximally southeast cell (a,b) among elements of $\bigcup_{i\in[r]} D(w_i)$, with (a,b) corresponding to the variable $y = z_{a,b}$ of R. Then the minimal primes of $\mathtt{in}_y(J) = C_{y,J} \cap (N_{y,J} + (y))$ that do not contain y are exactly minimal primes of $C_{y,J}$, and those that do contain y are exactly the minimal primes of $N_{y,J}^{\mathrm{ht}} + (y)$. Moreover,*

$$\sum_{\substack{\mathcal{P}\in\min(\mathtt{in}_y(J))\\ y\notin\mathcal{P}}} \mathtt{mult}_{\mathcal{P}}(R/\mathtt{in}_y(J)) \cdot e(R/\mathcal{P}) = \sum_{i=1}^{r} \#\left\{\mathcal{B} \in \mathsf{BPD}(w_i) : (a,b) \notin D(\mathcal{B})\right\}$$

and

$$\sum_{\substack{\mathcal{P}\in\min(\mathtt{in}_y(J))\\ y\in\mathcal{P}}} \mathtt{mult}_{\mathcal{P}}(R/\mathtt{in}_y(J)) \cdot e(R/\mathcal{P}) = \sum_{i=1}^{r} \#\left\{\mathcal{B} \in \mathsf{BPD}(w_i) : (a,b) \in D(\mathcal{B})\right\}.$$

Having completely understood $N_{y,J}$ and, by induction, $\mathtt{in}_\sigma(N_{y,J}^{\mathrm{ht}})$, our next goal is to show that each irreducible component of $\mathrm{Spec}(R/C_{y,J})$ appears with multiplicity governed by the $\mathsf{BPD}(w_i)$ as a step towards understanding multiplicity of each irreducible component of $\mathrm{Spec}(R/\mathtt{in}_\sigma(C_{y,J}))$.

Following [16], for $w \in S_n$, we will call a lower outside corner (a,b) of $D(w)$ an **accessible** cell if $\mathtt{rk}_w(a,b) \geq 1$. For $w_1,\ldots,w_r \in S_n$, call (a,b) the **maximal accessible cell** of $\{w_1,\ldots,w_r\}$ if (1) (a,b) is an accessible cell of some w_i, (2) if (a',b') is an accessible cell of some w_i, then $a' \leq a$, and (3) if (a',b') is an accessible cell of some w_i satisfying $a' = a$, then $b' \leq b$.

Lemma 18. *Fix distinct permutations $w_1,\ldots,w_r$, $r \geq 1$, all of Coxeter length h. Suppose $\{w_1,\ldots,w_r\}$ has maximal accessible cell (a,b) corresponding to the variable $y = z_{a,b}$ and that y is involved in the Fulton generators of $I_{w_1},\ldots,I_{w_q}$ but not of $I_{w_{q+1}},\ldots,I_{w_r}$. Suppose further that (a,b) is a lower outside corner of every $D(w_i)$ in which it appears. Set $J = \bigcap_{i\in[r]} I_{w_i}$, with geometric vertex decomposition $\mathtt{in}_y(J) = C_{y,J} \cap (N_{y,J} + (y))$. Then the minimal primes of $C_{y,J}$ are $\mathcal{A} = \{I_u : u \in \Phi(w_i,y), i \in [q]\} \cup \{I_{w_i} : i \in [q+1,r]\}$. Moreover, for each $I_w \in \mathcal{A}$,*

$$\mathtt{mult}_{I_w}(R/\mathtt{in}_y(J)) = \begin{cases} \#\{i \in [q] : w \in \Phi(w_i,y)\} & \text{if } I_w \neq I_{w_i} \text{ for any } i \in [q+1,r] \\ \#\{i \in [q] : w \in \Phi(w_i,y)\} + 1 & \text{if } I_w = I_{w_i} \text{ for some } i \in [q+1,r]. \end{cases}$$

The following lemma facilitates an inductive argument in our main theorem by allowing us to track multiplicities along primary components and, by replacing $C_{y,J}$ by its radical, to return to the case of a union of matrix Schubert varieties.

Lemma 19. *Taking the standard grading on R, let J be a homogeneous ideal defining an equidimensional scheme with $\min(J) = \{P_1,\ldots,P_r\}$. Let σ be any term order. If, for all $\mathcal{P} \in \min(\mathtt{in}_\sigma(J))$,*

$$\mathtt{mult}_{\mathcal{P}}(R/\mathtt{in}_\sigma(J)) = \sum_{i=1}^{r} \mathtt{mult}_{\mathcal{P}}(R/\mathtt{in}_\sigma(P_i)),$$

then

$$\mathtt{mult}_{\mathcal{P}}(R/\mathtt{in}_\sigma(J)) = \sum_{i=1}^{r} \mathtt{mult}_{P_i}(R/J) \cdot \mathtt{mult}_{\mathcal{P}}(R/\mathtt{in}_\sigma(P_i)).$$

We now give our main result. The case $r = 1$ is the situation asked about by [10, Conjecture 1.2].

Theorem 20. *Fix distinct permutations $w_1,\ldots,w_r$, $r \geq 1$, all of Coxeter length h. If $J = \bigcap_{i\in[r]} I_{w_i}$, then the irreducible components of Spec$(R/\mathtt{in}_\sigma(J))$, counted with multiplicity, are indexed by $\bigcup_{i\in[r]} \mathsf{BPD}(w_i)$. Precisely, the multiplicity of Spec(R/P) along Spec$(R/\mathtt{in}_\sigma(J))$ is $\#\left\{\mathcal{B} \in \bigcup_{i\in[r]} \mathsf{BPD}(w_i) : P = I_{D(\mathcal{B})}\right\}$.*

The situation of Theorem 20 is especially nice when $\mathrm{Spec}(R/\mathtt{in}_\sigma(J))$ is reduced. In that case, for all $D \subseteq [n] \times [n]$, $\#\{\mathcal{B} \in \bigcup_{i=1}^{r} \mathsf{BPD}(w_i) : D(\mathcal{B}) = D\} = 1$; that is, there are no repeated diagrams occurring among the BPDs of the w_i.

Acknowledgements

The authors thank Zach Hamaker, Oliver Pechenik, and Jenna Rajchgot for their previous collaboration with the authors on [10] and [12], respectively, and for valuable discussions during the writing of the paper on which this extended abstract is based. They also thank Anders Buch, Allen Knutson, David Speyer, and Alexander Yong for helpful conversations. They express their gratitude to Allen Knutson, Ezra Miller, Oliver Pechenik, and Alexander Yong for comments on an earlier version of this manuscript. The first author thanks Alain Lascoux for email correspondence concerning [22].

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