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Torsors From Toggling Independent Sets

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Abstract. In this paper, we consider the problem of toggling independent sets in cycle graphs. In each orbit, we find an infinite abelian "snake group" that acts simply transitively on the "live entries". This allows us to characterize a number of combinatorial properties of the dynamics by studying the topological covering maps between this torsor and finite quotients. We also characterize the orbits via solutions to a three-variable Diophantine equation. Preliminary work has found other toggle actions where the live entries are a torsor for a group, suggesting that this work is a special case of a more general framework, and posing the question of when this phenomenon arises and why.

Keywords: dynamical algebraic combinatorics, Diophantine equation, group action, independent set, orbits, ouroboros, snake group, toggling, torsor

1 Introduction

Toggling combinatorial objects has gained considerable interest over the past decade, and has elevated the subfield now known as *dynamical algebraic combinatorics*. Let $\mathcal{C}_n$ be the cycle graph with vertex set $v(\mathcal{C}_n) = [n] = \{1, \ldots, n\}$ and edges $\{i, i+1\}$ with the indices taken modulo n. Denote the set of *independent sets* of $\mathcal{C}_n$ by $\mathcal{I}_n$. We will write these as cyclic binary strings $v_1, \ldots, v_n$ such that no two adjacent entries[1] are 1. The *toggle operation* at position k is the function $\tau_k \colon \mathcal{I}_n \to \mathcal{I}_n$ that "attempts to flip" the k^{th} bit. Specifically, if $v_k = 1$, then τ_k flips it to 0. On the other hand, if $v_k = 0$, then it flips it to 1 if doing so does not introduce consecutive 1s; otherwise, it fixes the k^{th} bit. In this paper, we will consider the action of iteratively toggling the bits of our binary string in the order $\tau_1, \ldots, \tau_n$, and we will denote this by the bijection $\tau = \tau_n \circ \cdots \circ \tau_1$. Given an initial cyclic binary string $x^{(0)}$, let $x^{(1)} = \tau(x^{(0)})$, $x^{(2)} = \tau(x^{(1)})$, and so on. Eventually, after some $m \geq 1$ number of steps, we will return to our original string. That is, $x^{(i+m)} = x^{(i)}$

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[1] By "cyclic string," we mean that v_1 and v_n are also adjacent.

for all $i \in \mathbb{Z}$. The dynamics of this action can be represented by $\mathbb{Z} \times \mathbb{Z}_n$ binary tables called *scrolls*, each of which is naturally embedded on a bi-infinite cylinder. An example on $n = 12$ vertices that repeats every $m = 15$ rows is shown in Figure 1.

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x^{(0)}$	**1**	0	**1**	0	**1**	0	0	0	**1**	0	**1**	0
$x^{(1)}$	0	0	0	0	0	**1**	0	0	0	0	0	**1**
$x^{(2)}$	0	**1**	0	**1**	0	0	**1**	0	**1**	0	0	0
$x^{(3)}$	0	0	0	0	**1**	0	0	0	0	**1**	0	**1**
$x^{(4)}$	0	**1**	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(5)}$	0	0	**1**	0	0	0	0	0	**1**	0	**1**	0
$x^{(6)}$	**1**	0	0	**1**	0	**1**	0	0	0	0	0	0
$x^{(7)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	**1**	0
$x^{(8)}$	0	0	**1**	0	**1**	0	0	0	0	0	0	**1**
$x^{(9)}$	0	0	0	0	0	**1**	0	**1**	0	**1**	0	0
$x^{(10)}$	**1**	0	**1**	0	0	0	0	0	0	0	**1**	0
$x^{(11)}$	0	0	0	**1**	0	**1**	0	**1**	0	0	0	**1**
$x^{(12)}$	0	**1**	0	0	0	0	0	0	**1**	0	0	0
$x^{(13)}$	0	0	**1**	0	**1**	0	**1**	0	0	**1**	0	**1**
$x^{(14)}$	0	0	0	0	0	0	0	**1**	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Sum:	**3**	**4**	**5**	**3**	**4**	**5**	**3**	**4**	**5**	**3**	**4**	**5**

Figure 1: A single period of an orbit from toggling independent sets over $\mathcal{C}_n$, for $n = 12$. This bi-infinite table, naturally embedded on a cylinder, is called a *scroll*.

In this paper, we will see some simple questions that lead to surprising structural properties of the dynamics, which seem to be a special case of a more general unexplored framework. As as gentle introduction, let's pause to make a few innocuous observations about Figure 1. As we've noted, this scroll repeats every 15 rows, which we call its *period*. If we view the dynamics as one reads a book – from left to right and down the rows, this bi-infinite sequence, called the *ticker tape*, has period 45. Finally, the column sums over one period, written as a cyclic string, have period 3. It turns out that the period of the column sums is odd in any orbit, and the relationships between these various periods can be characterized algebraically by an abelian "*snake group*" that acts simply transitively on the set of "*live entries*" in the scroll (the positions with a 1). This action endows the live entries with a Cayley diagram structure, *i.e.*, they are a torsor for the snake group. Specifically, some of the fundamental combinatorial properties of these scrolls, and thus of the dynamics generated by toggling independent sets, can be explained by interpreting the scroll as a covering space and using some basic algebraic topology. We can also enumerate the orbit tables for any n via a Diophantine equation in three variables. Preliminary work has suggested that other toggle actions admit such a torsor structure – not only just toggling independent sets over other graphs, but other toggle actions as well.

2 Scrolls and snakes

There are two formats for viewing the dynamics that result from toggling independent sets of $\mathcal{C}_n$: the scroll and the ticker tape, which we informally described above, and will formalize now. Recall that for an independent set $x = x^{(0)} = (x_1, \dots, x_n) \in \mathcal{I}_n$ the result of iterating τ exactly i times from x is denoted $x^{(i)}$. Since τ is bijective on $\mathcal{I}_n$, we can define this for all $i \in \mathbb{Z}$. Formally, the *scroll* of x is the bi-infinite table with n columns, indexed by $j = 1, \dots, n$, and rows indexed by $i \in \mathbb{Z}$, reading downward. We denote this by $\mathcal{S} = (X_{i,j}) = \mathsf{Scroll}(x)$, and so the (i, j)-entry $X_{i,j}$ is the state of vertex v_j in $x^{(i)}$. Since the scroll is naturally embedded on a cylinder, with the end of one line wrapping around to the beginning of the next, we will define $X_{i,n+1} = X_{i+1,1}$. As an example, the scroll of $x = 101010001010 \in \mathbb{F}_2^{12}$ from Figure 1 is simply the infinite table with the $m = 15$ rows shown repeated indefinitely, both above and below.

The *ticker tape* of x, denoted $\mathcal{X} = (X_k) = \mathsf{Tape}(x)$, is a bi-infinite sequence defined by reading off the scroll as one reads from a book: across each row from left to right, and then the rows downward, starting with $X_1 = X_{0,1}$, $X_2 = X_{0,2}$, $X_3 = X_{0,3}$, and so on.

In both a scroll and a ticker tape, positions that have a value of 1 are said to be *live*. Formally, the sets of live entries, in both formats, are

$$\mathsf{Live}(\mathcal{S}) = \big\{(i, j) \in \mathbb{Z} \times \mathbb{Z}_n \mid X_{i,j} = 1\big\}, \qquad \mathsf{Live}(\mathcal{X}) = \big\{k \in \mathbb{Z} \mid X_k = 1\big\}.$$

For a second example, and henceforth our "*running example*," a repeating portion of the scroll of $x = 00001010000 \in \mathbb{F}_2^{11}$ is shown twice in Figure 2. The ticker tape is

$$\dots, \underbrace{X_{-6}, X_{-5}, X_{-4}, X_{-3}, X_{-2}, X_{-1}, X_0}_{0,0,0,0,1,0,1}, \underbrace{X_1, X_2, X_3, X_4, X_5, X_6, X_7}_{0,0,0,0,1,0,1}, \underbrace{X_8, X_9, X_{10}, X_{11}, X_{12}, X_{13}, X_{14}}_{0,0,0,0,1,0,1} \dots.$$

Notice that in this example, the period of the scroll is $m = 7$ (rows), and the period of the ticker tape is 7 (bits).

It is elementary to check that for each live entry (i, j) in the scroll, exactly one of the entries $(i, j+2)$ and $(i+1, j+1)$ is live, as is exactly one of the entries $(i+2, j-2)$ and $(i+2, j-1)$. These define canonical bijections on the live entries, that we call the *successor* and *co-successor*, respectively. These are illustrated in Figure 2, and the formal definition follows.

Definition 1. Given a scroll $\mathcal{S}$, the *successor* and *co-successor* functions are the bijections $s, c \colon \mathsf{Live}(\mathcal{S}) \to \mathsf{Live}(\mathcal{S})$ that send (i, j) to the unique element of

$$\big\{(i, j+2), (i+1, j+1)\big\} \cap \mathsf{Live}(\mathcal{S}) \quad \text{and} \quad \big\{(i+2, j-2), (i+2, j-1)\big\} \cap \mathsf{Live}(\mathcal{S}),$$

respectively. If desired, we can view these as functions $s, c \colon \mathsf{Live}(\mathcal{X}) \to \mathsf{Live}(\mathcal{X})$ of ticker tapes.

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(1)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(2)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(3)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(4)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(5)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(6)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

$\cdots\ 0\ 0\ \cdots$
$\cdots\ 0\ \mathbf{1}\ 0\ \mathbf{a}\ \cdots$
$\cdots\ 0\ 0\ \bar{\mathbf{a}}\ 0\ \cdots$

"successor of (i,j)"

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(1)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(2)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(3)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(4)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(5)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(6)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

$\cdots\ 0\ 0\ \cdots$
$\cdots\ 0\ \mathbf{1}\ 0\ \cdots$
$\cdots\ 0\ 0\ \cdots$
$\cdots\ \mathbf{b}\ \bar{\mathbf{b}}\ \cdots$

"co-successor of (i,j)"

Figure 2: At left: a scroll consists of the seven rows $x^{(0)}, \dots, x^{(6)}$, repeated indefinitely. This is shown twice, with different color schemes, to emphasize the snakes and co-snakes separately. There are two snakes and six co-snakes. At right: a visualization of the successor and co-successor functions; the 1 in each figure is in position (i, j).

The successor and co-successor define equivalence classes on the live entries called *snakes* and *co-snakes*, respectively, and denoted (in scroll notation)

$$\mathsf{Snake}(i,j) = \{s^k(i,j) \mid k \in \mathbb{Z}\}, \qquad \mathsf{CoSnake}(i,j) = \{c^k(i,j) \mid k \in \mathbb{Z}\}.$$

These are highlighted by color in Figure 2. It is elementary to show that the successor and co-successor functions commute. Therefore, they define an abelian group called the *snake group*, denoted $G(\mathcal{S})$. This group acts simply transitively on the live entries. Throughout this paper, we will assume that a scroll $\mathcal{S}$ has α snakes and β co-snakes.

Theorem 1. *The set* $\mathsf{Live}(\mathcal{S})$ *is a torsor for the snake group, which has presentation*

$$G(\mathcal{S}) = \langle s, c \mid sc = cs,\ s^\beta = c^\alpha \rangle.$$

It follows that there is a bijective correspondence between snakes and cosets of $\langle s \rangle$, and between co-snakes and cosets of $\langle c \rangle$. That is,

$$\alpha = [G(\mathcal{S}) : \langle s \rangle] \qquad \text{and} \qquad \beta = [G(\mathcal{S}) : \langle c \rangle]$$

are the smallest positive integers for which $s^\beta \in \langle c \rangle$ and $c^\alpha = \langle s \rangle$.

Since snakes are cosets, they have the same algebraic structure in a Cayley diagram. The live entries in a (co-)snake are also embedded in the scroll via the (co-)successor function, but it is not guaranteed that these necessarily have the same "shape". However, it turns out that this is indeed the case. To establish this, we need to formalize the notion of the "shape" of a snake.

From a fixed $(i, j) \in \mathsf{Live}(\mathcal{S})$, consider the next live entry reached when applying the successor or co-successor function. There are two cases for each, as was shown back in Figure 2. We will annotate a step of $s(i, j) = (i+1, j+1)$ by "D" for "diagonal" and a step of $s(i, j) = (i, j+2)$ by "2." Similarly, we will annotate a step of $c(i, j) = (i+2, j-1)$ by "S" for "short", and $c(i, j) = (i+2, j-2)$ by "L" for "long." Allowing inverses and concatenations, it is straightforward to annotate any path in the Cayley diagram of $G(\mathcal{S})$. We will call this the *shape* of a path.

The elements of the cyclic quotient group $G(\mathcal{S})/\langle c\rangle \cong \mathbb{Z}_\beta$ correspond to the co-snakes in $\mathcal{S}$. Thus, starting at any live entry and iterating the successor function β times defines an ordering of the co-snakes. The shape of this path (a length-β word over $\{D, 2\}$), up to cyclic shift, is independent of the starting live entry. We will call any such sequence a *slither* of $\mathsf{Snake}(i, j)$. For ease of notation, we can use exponents to write a slither. For example, $(D2)^3 = D2D2D2$ is a slither of the snakes in Figure 2. Since any cyclic shift of a slither is also a slither, when we speak of "the slither," we mean up to cyclic shift.

There is a similar construction for co-snakes. The elements of the cyclic quotient group $G(\mathcal{S})/\langle s\rangle \cong \mathbb{Z}_\alpha$ are the snakes in $\mathcal{S}$. Starting at any live entry and iterating the co-successor function α times defines an ordering of the snakes. The shape of this path (a length-α sequence over $\{S, L\}$), up to cyclic shift, is independent of the starting live entry. We will call any such sequence a *co-slither* of $\mathsf{CoSnake}(i, j)$. The co-slithers of the co-snakes in Figure 2 are all $S^2 = SS$.

Proposition 1. *In any scroll, all (co-)snakes have the same (co-)slither.*

By Proposition 1, we can define the slither and co-slither of a scroll (or ticker tape), which we will denote by $\mathsf{Slither}(\mathcal{S})$ and $\mathsf{CoSlither}(\mathcal{S})$, respectively. Our next definition is meant to capture the exponent that we used when writing $(D2)^3$ and S^2 above.

Definition 2. The *degree* of a scroll, denoted $\deg(\mathcal{S})$, is the length of the co-slither of any co-snake divided by the period of the co-slither as a cyclic word. The *co-degree*, denoted $\mathrm{codeg}(\mathcal{S})$, is the length of the slither of any snake divided by the period of the slither as a cyclic word.

The scroll from our initial example Figure 1 has degree and co-degree 1, because $\mathsf{Slither}(\mathcal{S}) = 22DD2D$ and $\mathsf{CoSlither}(\mathcal{S}) = SL$. In contrast, the scroll in our running example from Figure 2 has degree 2 because its co-slither is S^2, and co-degree 3 because its slither is $(D2)^3$. Note that $\deg(\mathcal{S})$ must divide α, the number of snakes, and $\mathrm{codeg}(\mathcal{S})$ must divide β, the number of co-snakes.

3 Orbit tables and ouroboroi

Thus far, we have viewed the dynamics generated by toggling independent sets using infinite scrolls and ticker tapes. However, sometimes it is convenient to restrict our attention to a repeating sequence of rows and identify the top and bottom by a quotient map, thereby allowing snakes and co-snakes to "wrap around" from bottom-to-top. Inspired by the ancient symbol of a snake swallowing its tail, we will call such a finite circular snake an *ouroboros*. A *co-ouroboros* is defined similarly.

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(1)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(2)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(3)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(4)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(5)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(6)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(1)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(2)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(3)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(4)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(5)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(6)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

Figure 3: The fundamental orbit table $\mathcal{T}_1$ from our running example in Figure 2. When we allow snakes and co-snakes to wrap from bottom-to-top, the two snakes merge into one ouroboros with slither $\overline{D}\,\overline{2}$ (left), and the six co-snakes merge into two co-ouroboroi, with co-slither $\overline{S}$ (right).

Let $x \in \mathbb{F}_2^n$, and consider a sequence $x = x^{(0)}, \dots, x^{(m-1)}, x^{(m)} = x$. Let k be the *fundamental period*, which we define to be the minimum number of rows before the scroll repeats. The *frequency* is $\omega := m/k$. Define the *ω-fold orbit table* of x, denoted $\mathcal{T}_\omega = \mathsf{Table}_\omega(x)$, to be the $m \times n$ table with top row $x^{(0)}$ and bottom row $x^{(m-1)}$. We will refer to the 1-fold orbit table as the *fundamental orbit table*. Returning to our running example, the fundamental orbit table appears in Figure 3, and the 2-fold orbit table in Figure 4. These are both shown twice, with the live entries colored to highlight the (co-)ouroboroi, which will be formalized soon.

At times it is useful to work with a finite version of the ticker tape. If $m = \omega k$ as above, then define the *ω-fold orbit vector* to be the length-mn subsequence of the ticker tape that has x as an initial sequence—the result of reading the ω-fold orbit table across each column, downward row-by-row. We denote this as

$$\mathcal{V}_\omega = \mathsf{Vector}_\omega(x) = \big(X_{0,1}, \dots, X_{0,n}, X_{1,1}, \dots, X_{1,n}, \dots, X_{m-1,1}, \dots, X_{m-1,n}\big) \in \mathbb{F}_2^{nm}. \quad (3.1)$$

If $\mathcal{T}$ is an orbit table and $\mathcal{V}$ an orbit vector, we define their live entries as the sets

$$\mathsf{Live}(\mathcal{T}) = \big\{(i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n \mid X_{i,j} = 1\big\}, \qquad \mathsf{Live}(\mathcal{V}) = \big\{k \in \mathbb{Z}_{mn} \mid X_k = 1\big\}.$$

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(1)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(2)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(3)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(4)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(5)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(6)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**
$x^{(7)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(8)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(9)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(10)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(11)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(12)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(13)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(1)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(2)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(3)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(4)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(5)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(6)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**
$x^{(7)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(8)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(9)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(10)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(11)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(12)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(13)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

Figure 4: In the 2-fold orbit table $\mathcal{T}_2$ from our running example in Figure 2, there are two ouroboroi with slither $\overline{D}\,\overline{2}$ and two co-ouroboroi with co-slither $\overline{S}^2$.

Though it makes no difference either way, we will continue with the convention of numbering the columns $1, \ldots, n$ and the rows $0, \ldots, m-1$. As such, we will harmlessly take $\mathbb{Z}_n = \{1, \ldots, n\}$ and $\mathbb{Z}_m = \{0, \ldots, m-1\}$ in the orbit table and $\mathbb{Z}_{mn} = \{1, \ldots, mn\}$ in the orbit vector.

The live entries in an orbit table are simply the images of the live entries in the corresponding scroll under the natural quotient map $p_\omega \colon \mathsf{Live}(\mathcal{S}) \to \mathsf{Live}(\mathcal{T}_\omega)$ that reduces the first coordinate of each entry modulo m. Under this map, the successor and co-successor functions descend to bijections on $\mathsf{Live}(\mathcal{T}_\omega)$ that we call the *ω-successor function* $\bar{s}_\omega$ and *ω-co-successor function* $\bar{c}_\omega$. The relationship between the successor and ω-counterparts is illustrated by the following commutative diagrams.

$$\begin{tikzcd} \mathsf{Live}(\mathcal{S}) \arrow[r, "s"] \arrow[d, "p_\omega"'] & \mathsf{Live}(\mathcal{S}) \arrow[d, "p_\omega"] \\ \mathsf{Live}(\mathcal{T}_\omega) \arrow[r, "\bar{s}_\omega"] & \mathsf{Live}(\mathcal{T}_\omega) \end{tikzcd} \qquad \begin{tikzcd} (i+km, j) \arrow[r, mapsto, "s"] \arrow[d, mapsto, "p_\omega"'] & s(i+km, j) \arrow[d, mapsto, "p_\omega"] \\ (i, j) \arrow[r, mapsto, "\bar{s}_\omega"] & \bar{s}_\omega(i, j) \end{tikzcd}$$

Naturally, there is an analogous diagram relating c and $\bar{c}_\omega$. The functions $\bar{s}_\omega$ and $\bar{c}_\omega$ generate a finite abelian group $G(\mathcal{T}_\omega) := \langle \bar{s}_\omega, \bar{c}_\omega \rangle$ that we call the *ouroboros group* of $\mathcal{T}_\omega$, or the *ω-fold ouroboros group* of $\mathcal{S}$. Since p_ω is a topological covering map, there is an induced homomorphism $p_\omega^* \colon G(\mathcal{S}) \to G(\mathcal{T}_\omega)$ sending $s \mapsto \bar{s}_\omega$ and $c \mapsto \bar{c}_\omega$. The ouroboros group is the quotient

$$G(\mathcal{T}_\omega) \cong G(\mathcal{S}) / \ker p_\omega^*,$$

and it acts simply transitively on $\mathsf{Live}(\mathcal{S}) / \ker p_\omega$, which can be canonically identified with $\mathsf{Live}(\mathcal{T}_\omega)$. We get a bijective correspondence between the orbits under $\bar{s}_\omega$ and $\bar{c}_\omega$

and the cosets of $\langle \bar{s}_\omega \rangle$ and $\langle \bar{c}_\omega \rangle$. These are the images of the snakes and the co-snakes under the quotient map p_ω.

Definition 3. Given a live entry (i,j) in an orbit table $\mathcal{T}_\omega$, the *ouroboros* and *co-ouroboros* containing it are the sets

$$\mathsf{Ouro}_\omega(i,j) = \{\bar{s}_\omega^k(i,j) \mid k \in \mathbb{Z}\}, \qquad \mathsf{CoOuro}_\omega(i,j) = \{\bar{c}_\omega^k(i,j) \mid k \in \mathbb{Z}\}.$$

Throughout the rest of this paper, we will continue to assume that a scroll $\mathcal{S}$ has α snakes and β co-snakes. If $\mathcal{T} = \mathcal{T}_\omega$ is the ω-fold orbit table, then we will say that it has $\overline{\alpha}_\omega$ ouroboroi and $\overline{\beta}_\omega$ co-ouroboroi. If ω is clear from the context, which it usually will be, then we will typically drop it as a subscript. Similarly, we will often write $\bar{s}$ and $\bar{c}$ rather than $\bar{s}_\omega$ and $\bar{c}_\omega$, because there should be no ambiguity about ω.

Theorem 2. *The set* $\mathsf{Live}(\mathcal{T})$ *is a torsor of the ouroboros group, which has presentation*

$$G(\mathcal{T}) = \left\langle \bar{s}, \bar{c} \;\middle|\; \bar{s}\,\bar{c} = \bar{c}\,\bar{s},\ \bar{s}^\beta = \bar{c}^\alpha,\ \bar{s}^{\lambda/\overline{\alpha}} = \bar{c}^{\lambda/\overline{\beta}} = 1 \right\rangle \cong \mathbb{Z}_{\overline{\alpha}} \times \mathbb{Z}_{\lambda/\overline{\alpha}} \cong \mathbb{Z}_{\overline{\beta}} \times \mathbb{Z}_{\lambda/\overline{\beta}},$$

where λ is the number of live entries of the orbit table $\mathcal{T}$.

It follows that for each ω, there are bijective correspondences between ouroboroi and cosets of $\langle \bar{s} \rangle$, and between co-ouroboroi and cosets of $\langle \bar{c} \rangle$. Since $G(\mathcal{T}) = \langle \bar{s}, \bar{c} \rangle \cong G(\mathcal{S}) / \ker p_\omega^*$ is a finite abelian group of order λ, the first two relations hold, and

$$\overline{\alpha} = [G(\mathcal{T}) : \langle \bar{s} \rangle] = \lambda / \overline{\beta}, \qquad \text{and} \qquad \overline{\beta} = [G(\mathcal{T}) : \langle \bar{c} \rangle] = \lambda / \overline{\alpha}. \tag{3.2}$$

In other words, $G(\mathcal{T}) / \langle \bar{s} \rangle \cong \mathbb{Z}_{\overline{\alpha}}$ and $G(\mathcal{T}) / \langle \bar{c} \rangle \cong \mathbb{Z}_{\overline{\beta}}$.

Definition 4. The *(co-)ouroboros degree* of an orbit table $\mathcal{T}_\omega$ is the number of (co-)snakes in the p_ω-preimage of each (co-)ouroboros. We denote these as

$$\deg(p_\omega^*) := \frac{[G(\mathcal{S}) : \langle s \rangle]}{[G(\mathcal{T}_\omega) : \langle \bar{s}_\omega \rangle]} = \alpha / \overline{\alpha}_\omega, \qquad \operatorname{codeg}(p_\omega^*) := \frac{[G(\mathcal{S}) : \langle c \rangle]}{[G(\mathcal{T}_\omega) : \langle \bar{c}_\omega \rangle]} = \beta / \overline{\beta}_\omega.$$

The case when $\omega = 1$ is called the *fundamental (co-)ouroboros degree*.

Returning to our running example, in the fundamental orbit table (*i.e.*, $\omega = 1$) the $\alpha = 2$ snakes in $\mathcal{S}$ merge into $\overline{\alpha}_1 = 1$ ouroboros, and the $\beta = 6$ co-snakes merge into $\overline{\beta}_1 = 2$ co-ouroboroi. Thus, the fundamental ouroboros degree is $\deg(p_1^*) = 2/1 = 2$, and the fundamental co-ouroboros degree is $\operatorname{codeg}(p_1^*) = 6/2 = 3$. In contrast, the 2-fold orbit table has $\overline{\alpha}_2 = 2$ ouroboroi and $\overline{\beta}_2 = 2$ co-ouroboroi. Its ouroboros degree is $\deg(p_2^*) = 2/2 = 1$, and its co-ouroboros degree is $\operatorname{codeg}(p_2^*) = 6/2 = 3$.

Slithers and co-slithers naturally descend to orbit tables via the canonical quotient map $p_\omega \colon \mathsf{Live}(\mathcal{S}) \to \mathsf{Live}(\mathcal{T}_\omega)$. The slither of $\mathcal{S}$ is a length-β sequence of Ds and 2s, and it defines a cyclic ordering $\langle c\rangle, s\langle c\rangle, \dots, s^{\beta-1}\langle c\rangle$ of co-snakes. If we apply the quotient map $p_\omega^*\colon G(\mathcal{S}) \to G(\mathcal{T}_\omega)$, we get a cyclic ordering of the $\overline{\beta}_\omega$ co-ouroboroi. Each co-ouroboros appears in the sequence $\langle\overline{c}_\omega\rangle, \overline{s}_\omega\langle\overline{c}_\omega\rangle, \dots, \overline{s}_\omega^{\beta-1}\langle\overline{c}_\omega\rangle$ exactly $\operatorname{codeg}(p_\omega^*) = \beta/\overline{\beta}_\omega$ times, and this must be a divisor of $\operatorname{codeg}(\mathcal{S})$ (the exponent in the slither). Define the *slither* of $\mathcal{T}_\omega$, or *ω-slither* of $\mathcal{S}$, denoted $\mathsf{Slither}(\mathcal{T}_\omega)$, to be any length-$\overline{\beta}$ subsequence of a slither of $\mathcal{S}$.

Everything we just said above has a natural analogue for co-slithers. That is, a co-slither of $\mathcal{S}$ is a length-α sequence of Ss and Ls that defines a cyclic ordering $\langle s\rangle, c\langle s\rangle, \dots, c^{\alpha-1}\langle s\rangle$ of snakes. Via the quotient map $p_\omega^*\colon G(\mathcal{S}) \to G(\mathcal{T}_\omega)$, we get a cyclic ordering of the $\overline{\alpha}_\omega$ ouroboroi. Each ouroboros appears in the sequence $\langle\overline{s}\rangle, \overline{c}\langle\overline{s}\rangle, \dots, \overline{c}^{\alpha-1}\langle\overline{s}\rangle$ exactly $\deg(p_\omega^*) = \alpha/\overline{\alpha}$ times, and this must be a divisor of the degree of $\mathcal{S}$ (the exponent that appears in the co-slither). We define the *co-slither* of $\mathcal{T}_\omega$, or the *ω-co-slither* of $\mathcal{S}$, as any length-$\overline{\alpha}$ subsequence of a co-slither of $\mathcal{S}$, and we denote this as $\mathsf{CoSlither}(\mathcal{T}_\omega)$. The preceding two paragraphs have established the following.

Lemma 1. *For any scroll $\mathcal{S}$, we have*

$$\big(\mathsf{Slither}(\mathcal{T}_\omega)\big)^{\operatorname{codeg}(p_\omega^*)} = \mathsf{Slither}(\mathcal{S}) \qquad \text{and} \qquad \big(\mathsf{CoSlither}(\mathcal{T}_\omega)\big)^{\deg(p_\omega^*)} = \mathsf{CoSlither}(\mathcal{S}).$$

We will refer to the (co-)slither of the fundamental orbit table (*i.e.*, $\omega = 1$) as the *fundamental (co-)slither*. To emphasize that we are taking the slither in an orbit table rather than in the scroll, we will sometimes write $\overline{2}$ and $\overline{D}$ rather than 2 and D, and similarly use $\overline{S}$ and $\overline{T}$ in co-slithers.

Return to our running example, the length of the fundamental slither $\overline{D}\,\overline{2}$ is $\overline{\beta}_1 = \beta/\operatorname{codeg}(p_1^*)$, the number of co-ouroboroi, and the length of the fundamental co-slither $\overline{S}$ is $\overline{\alpha}_1 = \alpha/\deg(p_1^*)$, the number of ouroboroi. As guaranteed by Lemma 1, the (co-)slithers of $\mathcal{S}$ and $\mathcal{T}$ are related by

$$(D2)^{\operatorname{codeg}(p_1^*)} = (D2)^3 \qquad \text{and} \qquad S^{\deg(p_1^*)} = S^2.$$

The 2-fold orbit table of our running example, shown in Figure 4, has two ouroboroi with slither $\overline{D}\,\overline{2}$, and two co-ouroboroi with co-slither $\overline{S}^2$. The ouroboros degree is thus $\deg(p_2^*) = 2/2 = 1$, and the co-ouroboros degree is $\operatorname{codeg}(p_2^*) = 6/2 = 3$. As predicted by Lemma 1, we have

$$(D2)^{\operatorname{codeg}(p_2^*)} = (D2)^3 \qquad \text{and} \qquad (S^2)^{\deg(p_2^*)} = S^2.$$

4 Combinatorial properties of the dynamics

Recall from the introduction that there are several notions of "period": the period of a scroll is the minimal number of rows before it repeats, and the period of a ticker tape is the length of a minimal invariant shift.

Definition 5. The *fundamental period* of a scroll $\mathcal{S} = (X_{i,j})$, denoted $T(\mathcal{S})$, is the smallest $k > 0$ such that $X_{i+k,j} = X_{i,j}$ for all i, j. The *fundamental period* of a ticker tape $\mathcal{X} = (X_k)$, denoted $T(\mathcal{X})$, is the smallest $\ell > 0$ such that $X_{k+\ell} = X_k$ for all k.

It is clear that $T(\mathcal{X})$ divides $nT(\mathcal{S})$, which is the length of the fundamental orbit vector. In our running example, the fundamental orbit vector has length $nT(\mathcal{S}) = 11 \cdot 7 = 77$, but the period of the ticker tape is just $T(\mathcal{X}) = 7$ because the ticker tape is generated by the subsequence 1010000. In this section, we will derive a formula relating these two periods.

Consider the equivalence relation on $\mathsf{Live}(\mathcal{S})$ defined by the intersection of snakes and co-snakes. In other words, the equivalence classes, which we will call *fibers*,

$$\langle s \rangle \cap \langle c \rangle = \langle s^{\beta} \rangle = \langle c^{\alpha} \rangle \leq G(\mathcal{S}).$$

In both notations, we will denote the fiber containing a live entry by

$$\mathsf{Fiber}(i,j) = \mathsf{Snake}(i,j) \cap \mathsf{CoSnake}(i,j), \qquad \mathsf{Fiber}(k) = \mathsf{Snake}(k) \cap \mathsf{CoSnake}(k).$$

Since the fibers are the orbits under the action of a cyclic group, there is some integer σ such that two live entries in $\mathcal{X}$ are in the same fiber if and only if they differ by a multiple of σ in the ticker tape.

Definition 6. The *scale* of a ticker tape $\mathcal{X}$ (or scroll $\mathcal{S}$) is the minimal distance $\sigma > 0$ between any two live entries in the same fiber. That is, in ticker tape notation, for any $k \in \mathbb{Z}$,

$$\sigma = \mathsf{Scale}(\mathcal{S}) = \mathsf{Scale}(\mathcal{X}) = s^{\beta}(k) - k = c^{\alpha}(k) - k.$$

From some fixed $k \in \mathsf{Live}(\mathcal{X})$, we can compute the scale in two ways: (i) by applying the successor function β times, or (ii) applying the shadow function α times. We can compute the scale by summing the number of positions we increase at each step, and this is useful for deriving elementary properties, some of which appear below.

Henceforth, given a scroll $\mathcal{S}$ with α snakes and β co-snakes, let β_T and β_D denote the number of instances of 2 and D in $\mathsf{Slither}(\mathcal{S})$, respectively. Let α_S and α_L denote the number of instances of S and L in $\mathsf{CoSlither}(\mathcal{S})$, respectively. Clearly, the lengths of the slither and co-slither are

$$\big|\mathsf{Slither}(\mathcal{S})\big| = \beta = \beta_T + \beta_D \qquad \text{and} \qquad \big|\mathsf{CoSlither}(\mathcal{S})\big| = \alpha = \alpha_S + \alpha_L.$$

Given an orbit table $\mathcal{T}_\omega$, let $\overline{\beta}_{\omega,T}$ and $\overline{\beta}_{\omega,D}$ denote the number of instances of 2 and D in $\mathsf{Slither}(\mathcal{T}_\omega)$, respectively. Similarly, let $\overline{\alpha}_{\omega,S}$ and $\overline{\alpha}_{\omega,L}$ denote the number of instances of S and L, respectively. The length $\overline{\beta}_\omega$ (resp. $\overline{\alpha}_\omega$) of the ω-slither (resp. ω-coslither) is

$$\big|\mathsf{Slither}(\mathcal{T}_\omega)\big| = \overline{\beta}_{\omega,T} + \overline{\beta}_{\omega,D} = \frac{\beta}{\operatorname{codeg}(p_\omega^*)}, \quad \big|\mathsf{CoSlither}(\mathcal{T}_\omega)\big| = \overline{\alpha}_{\omega,S} + \overline{\alpha}_{\omega,L} = \frac{\alpha}{\deg(p_\omega^*)}.$$

Lemma 2. *The scale of a ticker tape is* $\sigma = 2\beta_T + (n+1)\beta_D = (2n-1)\alpha_S + (2n-2)\alpha_L$.

Lemma 2 has several seemingly unrelated consequences. The first one involves a property that was pointed out in the introduction: that the column sums, as a cyclic vector, has odd period.

Proposition 1. *If n is even, then $\mathsf{Scale}(\mathcal{S})$ is odd. It follows that the sum vector of a scroll, and hence of any orbit table, has odd cyclic period.*

Lemma 2 also implies that the fundamental ouroboros and co-ouroboros degrees are relatively prime. Using this, along with the facts that $\deg(p_\omega^*)$ divides $\deg(p_1^*)$ and $\operatorname{codeg}(p_\omega^*)$ divides $\operatorname{codeg}(p_1^*)$, we can make a stronger statement.

Proposition 2. *For any $\omega \geq 1$, we have* $\gcd\big(\deg(p_\omega^*), \operatorname{codeg}(p_\omega^*)\big) = 1$.

From here, we can characterize the number of ouroboroi and co-ouroboroi in the ω-fold orbit table in terms of the number of them in the fundamental orbit table.

Proposition 3. *Suppose a scroll $\mathcal{S}$ has α snakes and β co-snakes and that its fundamental orbit table has $\overline{\alpha}$ ouroboroi and $\overline{\beta}$ co-ouroboroi. For $\omega > 1$, the numbers $\overline{\alpha}_\omega$ and $\overline{\beta}_\omega$ of ouroboroi and co-ouroboroi in its ω-fold orbit table satisfy*

$$\overline{\alpha}_\omega = \overline{\alpha} \cdot \gcd(\deg(p_1^*), \omega) \qquad \textit{and} \qquad \overline{\beta}_\omega = \overline{\beta} \cdot \gcd(\operatorname{codeg}(p_1^*), \omega).$$

Putting together the prior results yields a relationship between the periods of the scroll and ticker tape, and their fundamental degree, co-degree, and scale.

Theorem 3. *The periods of the ticker tape $\mathcal{X}$ and scroll $\mathcal{S}$ are*

$$T(\mathcal{X}) = \frac{\mathsf{Scale}(\mathcal{X})}{\deg(p_1^*)\operatorname{codeg}(p_1^*)}, \qquad T(\mathcal{S}) = \frac{\mathsf{Scale}(\mathcal{X})}{\deg(p_1^*)\operatorname{codeg}(p_1^*)\gcd(T(\mathcal{X}), n)},$$

and thus are related by $T(\mathcal{X}) = \gcd(T(\mathcal{X}), n) \cdot T(\mathcal{S})$.

A natural next question to ask is what scrolls and/or ticker tapes are possible for a given n, and how many are there. Suppose the slither has exactly a instances of 2, and its co-slither has exactly b instances of S and exactly c instances of L. This forces the slither to have exactly $2(b+c)-1$ instances of D, and these quantities must satisfy

$$2a + 3b + 4c = n + 1. \tag{4.1}$$

This is a straightforward necessary condition governing the slithers and co-slithers that can exist in a scroll for a given n. However, the next theorem ensures that it is also sufficient. In particular, *any* solution to Equation (4.1) with $a, b, c \geq 0$ and $b + c > 0$ gives a set of potential slithers and co-slithers that only differ by rearrangement,[2] and every one of these slither and co-slither combinations corresponds to a valid ticker tape.

[2]Note that if $b = c = 0$, our co-slither would be empty, which is impossible.

Theorem 4. *For a fixed n, we can construct all orbit tables that begin with a live entry through the following procedure:*

1. *Take a solution to the equation $2a + 3b + 4c = n + 1$, with $a, b, c \in \mathbb{Z}_{\geq 0}$, $b + c > 0$.*

2. *Construct the slither by choosing any sequence of $2(b + c) - 1$ instances of D and a instances of 2.*

3. *Construct the co-slither by choosing any sequence of b instances of S and c instances of L.*

Different solutions correspond to distinct ticker tapes, up to cyclic shift of the (co-)slither.

It is important to note that though every scroll determines a unique ticker tape, and vice-versa, two ticker tapes that differ by a shift can lead to different scrolls. However, this is an artifact of that representation. Intuitively, Theorem 4 is a characterization of the different possible dynamics that can arise by toggling independent sets in $\mathcal{I}_n$.

5 Concluding remarks

The original problem posed in this paper arose as a natural next step of the second author's work on toggling independent sets over a path graph [2]. The torsor structure came as a surprise, and it led us down a much more interesting mathematical road than we had expected. It all works because of the commuting (co-)successor bijections that happen to act simply transitively on the live entries, and it opens up a slew of questions. This can all be framed in terms of asynchronous cellular automata [1], which leads to new interesting questions in that field, such as exploring these ideas using other elementary cellular automata rules. In another direction, in ongoing work, we have found other similar torsor structures from toggling, such as over different graphs and using different toggle actions. This suggests that some of the results in this paper are special cases of an unexplored more general framework, relating algebra and topology to combinatorial dynamics in new ways.

References

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