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Bruhat interval polytopes, 1-skeleton lattices, and smooth torus orbit closures

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Abstract. Introduced by Kodama and Williams, *Bruhat interval polytopes* are generalized permutohedra closely connected to the study of torus orbit closures and total positivity in Schubert varieties. We show that the 1-skeleton posets of these polytopes are lattices and classify when the polytopes are simple, thereby resolving open problems and conjectures of Fraser, of Lee–Masuda, and of Lee–Masuda–Park. In particular, we classify when generic torus orbit closures in Schubert varieties are smooth.

Keywords: Bruhat interval polytope, weak order, Bruhat order, lattice, Schubert variety, torus orbit, smooth

1 Introduction

1.1 Bruhat interval polytopes

For a permutation w in S_n, write $\boldsymbol{w}$ for the vector $(w^{-1}(1), \dots, w^{-1}(n)) \in \mathbb{R}^n$. The *Bruhat interval polytope* Q_w is defined as the convex hull:

$$Q_w := \mathrm{Conv}(\{\boldsymbol{u} \mid u \preceq w\}) \subset \mathbb{R}^n,$$

where $\preceq$ denotes Bruhat order on S_n (see Section 2). Bruhat interval polytopes were introduced by Kodama and Williams in [18], where it is shown that they are the images under the *moment map* of the *Schubert variety* X_w in the flag variety, and also of the *totally positive part* $X_w^{\geq 0}$ of the Schubert variety. Therefore, the combinatorics of Q_w encodes information about the actions of the torus and positive torus on X_w and $X_w^{\geq 0}$ respectively.

The combinatorics of Q_w was studied further by Tsukerman and Williams [31], who showed that Q_w is a *generalized permutohedron* in the sense of Postnikov [27] and the matroid polytope of a flag *positroid*. Additional connections to the geometry of matroids were made in [6], and Bruhat interval polytopes have also appeared [32] in the context of BCFW-bridge decompositions [1] from physics, and in the study [20, 21, 22, 24] of *generic torus orbit closures* Y_w in X_w.

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1.2 The 1-skeleton of Q_w as a lattice

Throughout this work, we study the *1-skeleton poset* P_w of Q_w, a partial order on the lower Bruhat interval $[e, w] = \{u \mid u \preceq w\}$.

Definition 1.1. The poset $(P_w, \leq_w)$ has underlying set the Bruhat interval $[e, w]$ and cover relations $u \lessdot_w v$ whenever Q_w has an edge between vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ and $\ell(v) > \ell(u)$, where ℓ denotes Coxeter length.

When $w = w_0$ is the *longest permutation*, the polytope Q_w is the *permutohedron*, a fundamental object in algebraic combinatorics, and the poset P_w is the very well-studied *right weak order* (see Section 2). For general w, since edges of Q_w must be Bruhat covers by [31], the order $\leq_w$ is intermediate in strength between right weak order and Bruhat order on $[e, w]$. Since the work of Björner [3] it has been known that the weak order P_{w_0} on S_n is a *lattice*; in our first main theorem, we generalize this to all of the posets P_w.

Theorem A (Proven as Theorem 4.5)**.** *Let $w \in S_n$, then P_w is a lattice.*

As explained below, special cases of this lattice structure confirm a conjecture of Fraser [12], recover several previous results of various authors, imply new properties of Q_w, and suggest interesting directions for future work.

1.2.1 BCFW-bridge decompositions

In the last decade, there has been an explosion of work (see [1]) relating the physical theory of *scattering amplitudes* to the combinatorics and geometry of the *totally nonnegative Grassmannian $Gr(k, n)_{\geq 0}$* by way of the *amplituhedron*. In this setting, *on-shell diagrams* from physics correspond to *reduced plabic graphs*, which give parametrizations of an important cell decomposition of $Gr(k, n)_{\geq 0}$ [26].

In [1] it is shown that reduced plabic graphs for a given cell may be built up recursively using *BCFW-bridge decompositions*. In [32], Williams showed that these decompositions of plabic graphs correspond to the maximal chains in P_v when v is a Grassmannian permutation, analogous to the fact that reduced words for the longest permutation w_0 correspond to maximal chains in P_{w_0} (weak order). Since weak order is a lattice, Theorem A extends this analogy and implies new structure within the set of BCFW-bridge decompositions. That P_v is a lattice for Grassmannian permutations was conjectured by Fraser [12]. Fraser also conjectured that a larger class of posets, which are not necessarily the 1-skeleton posets of any polytope, are lattices; this problem remains open.

1.2.2 Quotients of weak order

Theorem A is proven by realizing P_w as a quotient of weak order P_{w_0} by an equivalence relation Θ_w which respects the weak order join operation (but does *not* respect the meet operation!) Thus P_w is a *semilattice quotient* of P_{w_0} but not a *lattice quotient*. The lattice quotients and lattice homomorphisms of weak order have been classified [28, 29]. This work thus suggests that semilattice quotients and homomorphisms of weak order are an intriguing topic for further study.

1.2.3 The parabolic map and the mixed meet

Let $S_n(I)$ denote the subgroup of S_n generated by a subset I of the simple reflections. Billey, Fan, and Losonczy proved [2] that for any $w \in S_n$ the set $S_n(I) \cap [e, w]$ has a unique maximal element $m(w, I)$ under Bruhat order. Richmond and Slofstra [30] showed that this element $m(w, I)$ determines whether the projection of the Schubert variety $X_w \subset G/B$ to a partial flag variety G/P is a fiber bundle, and is thus important for understanding the singularities of X_w. We show in Theorem 4.6 that the element $m(w, I)$ is just the join in P_w of the simple reflections from I, demonstrating the richness of the lattice structure on P_w.

A related operation of *mixed meet* was studied by Bump and Chetard in [9] in relation to certain intertwining operators of representations of reductive groups over nonarchimedean local fields. The mixed meet of $u, v \in S_n$ is the unique Bruhat maximal permutation in $[e, u]_R \cap [e, v]$. In the language of Section 4, this element is $\mathrm{bot}_v(u)$, the unique minimal element under $\leq_R$ in the equivalence class of u under the equivalence relation Θ_v induced on S_n by the normal fan of Q_v. This element is a translate of $\mu_v(u)$, where μ_v is the *matroid map* obtained by viewing $[e, v]$ as a Coxeter matroid [7].

1.2.4 The non-revisiting path property

A polytope Q has the *non-revisiting path property* if no shortest path in its 1-skeleton between two vertices returns to a face after having left it. This property has long been of interest in the field of combinatorial optimization. In [14], Hersh proves that any simple polytope whose 1-skeleton poset is a lattice has the non-revisiting path property. Thus, combining Theorem A and the classification of simple Bruhat interval polytopes in Theorem B below, we obtain a rich new family of examples to which Hersh's theorem applies. Additionally, in Section 5 we observe that all polytopes Q_w are *directionally* simple. We therefore ask: does Hersh's theorem extend to directionally simple polytopes?

1.3 Bruhat interval polytopes and generic torus orbit closures

1.3.1 Simple Bruhat interval polytopes and smooth torus orbit closures

Let $G = GL_n(\mathbb{C})$, let B denote the Borel subgroup of upper triangular matrices, and let T denote the maximal torus of diagonal matrices. The *flag variety* $Fl_n = G/B$ and its Schubert subvarieties $X_w := \overline{BwB/B}$ are of fundamental importance in many areas of algebraic combinatorics, algebraic geometry, and representation theory. The torus T acts naturally on G/B via left multiplication, and the fixed points $(G/B)^T$ are the points wB for $w \in S_n$, where we identify w with its permutation matrix. The fixed points of the Schubert variety X_w are $\{uB \mid u \preceq w\}$.

Torus orbits in G/B and their closures are a rich family of varieties, studied since Klyachko [17] and Gelfand–Serganova [13] with close connections to matroids and Coxeter matroids [7]. One class of torus orbit closures has received considerable interest [20, 21, 22, 24] of late: *generic* torus orbit closures in Schubert varieties. A torus orbit closure $Y \subset X_w$ is called generic if $Y^T = X_w^T$; we write Y_w for a generic torus orbit closure in X_w.

One of the main properties of interest for torus orbits in the flag variety has historically been their singularities [10, 11], and in particular determining when they are smooth. For Schubert varieties themselves, smoothness was famously characterized by Lakshmibai–Sandhya [19] in terms of pattern avoidance. In our next main theorem, we resolve a conjecture of Lee and Masuda [20] by classifying when Y_w is smooth.

Theorem B (Conjectured by Lee–Masuda [20]; Proven below as Corollary 6.3)**.** *Let $w \in S_n$, then Q_w is a simple polytope if and only if it is simple at the vertex $\mathbf{w}$; equivalently, Y_w is a smooth variety if and only if it is smooth at the point wB.*

Theorem B is proven by showing (see Theorem 6.1) that the degree of a vertex of Q_w is an ordering preserving function of the poset P_w.

By [20], the condition that Y_w is smooth at wB can be checked combinatorially by determining whether a certain graph $\Gamma_w(w)$ is a tree (see Section 3). This tree condition has in turn been characterized combinatorially in terms of pattern avoidance [8], and shown [33] to characterize when X_w is *locally factorial*. By work of Björner–Ekedahl [5] it is also equivalent to the vanishing of the coefficient of q in the associated *Kazhdan–Lusztig polynomial* [15] and thus [16] the vanishing of a certain middle intersection cohomology group of X_w. It would be fascinating to give a purely geometric explanation for the equivalence (by Theorem B) of the smoothness of Y_w with these other geometric conditions on X_w.

1.3.2 Directionally simple polytopes and h-vectors

In Section 5 we show that, even when Q_w is not a simple polytope, it is still *directionally simple* (see Definition 5.1). This fact was also shown in [24] by an involved calculation, but follows directly from our results realizing P_w as a quotient of weak order. This property of Q_w implies that its *h-vector* has positive entries which count certain permutations according to their number of ascents. In Proposition 5.6 we resolve an open problem of Lee–Masuda–Park [23] by showing that Y_w is smooth if and only if this h-vector is palindromic.

2 Background on the weak and strong Bruhat orders

We refer the reader to [4] for basic definitions and results on Coxeter groups.

We view the symmetric group S_n as a Coxeter group with generators $s_1, \ldots, s_{n-1}$, where $s_i := (i\, i+1)$ is an adjacent transposition. An expression $w = s_{i_1} \cdots s_{i_\ell}$ of minimal length is a *reduced word* for w and in this case the quantity $\ell = \ell(w)$ is the *length* of w. There are two important partial orders on S_n, each graded by length. The *right weak order* $\leq_R$ by definition has cover relations $w \lessdot_R ws$ whenever s is a simple generator and $\ell(ws) = \ell(w) + 1$; the *(strong) Bruhat order* $\preceq$ has cover relations $w \prec wt$ whenever $\ell(wt) = \ell(w) + 1$ and t lies in the set T of transpositions (ij). We write $[v, w]_R$ and $[v, w]$ for the closed interval between v, w in right weak and Bruhat order respectively.

The *left inversions* of an element $w \in S_n$ are the reflections $T_L(w) := \{t \in T \mid \ell(tw) < \ell(w)\}$. It is well-known that weak order is characterized by containment of inversions:

Proposition 2.1. *Let $v, w \in S_n$, then $v \leq_R w$ if and only if $T_L(v) \subseteq T_L(w)$.*

The symmetric group contains a unique element w_0 of maximum length, and w_0 is the unique maximal element of S_n under both $\leq_R$ and $\preceq$. In fact, weak order is a lattice.

Theorem 2.2 (Björner [3])**.** *The poset $(S_n, \leq_R)$ is a lattice; we write $\wedge_R$ and $\vee_R$ for the meet and join in right weak order.*

3 The graphs $\widetilde{\Gamma}_w$ and Γ_w

Lee and Masuda [20] provided a combinatorial model for determining the edges incident to a vertex $\boldsymbol{u}$ of Q_w.

Definition 3.1 (Lee and Masuda [20])**.** For $u \preceq w$, the directed graph $\widetilde{\Gamma}_w(u)$ has vertex set $[n]$ with directed edges $(u(i), u(j))$ whenever $i < j$, $u(ij) \preceq w$, and $|\ell(u(ij)) - \ell(u)| = 1$. We write $\widetilde{E}_w(u)$ for this set of edges. The directed graph $\Gamma_w(u)$ is defined to be the transitive reduction of $\widetilde{\Gamma}_w(u)$, with edge set $E_w(u)$.

Proposition 3.2 (Lee and Masuda [20], Proposition 7.7)**.** *Two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ of Q_w are connected by an edge of the polytope if and only if $v = u(ij)$ where $(u(i), u(j)) \in E_w(u)$.*

The proofs of the main theorems rely on a detailed development of the properties of directed paths in the graphs $\Gamma_w(u)$ and $\widetilde{\Gamma}_w(u)$, and how these change as u varies, some of which are sampled below.

3.1 Directed paths and local changes

When the permutation w is understood, we write $a \xrightarrow{u} b$ when $(a,b) \in \widetilde{E}_w(u)$ and $a \overset{u}{\dashrightarrow} b$ when there is a directed path from a to b in $\widetilde{\Gamma}_w(u)$ (equivalently, in $\Gamma_w(u)$); we write $a \overset{u}{\Rightarrow} b$ when $(a,b) \in E_w(u)$.

Proposition 3.3. *Let $u, w \in S_n$ with $u \preceq w$, and suppose $(ab)u \preceq w$, where $u^{-1}(a) < u^{-1}(b)$, then $a \overset{u}{\dashrightarrow} b$. In particular, if $(ab) \in T_L(u)$, then $a \overset{u}{\dashrightarrow} b$.*

The following proposition is a fundamental ingredient in many of proofs of the subsequent theorems. Its proof is rather involved and must be omitted for the sake of space.

Proposition 3.4. *Suppose that $u, v \in S_n$ satisfy $u \lessdot_w v = (ab)u$, with $a < b$:*

(i) *If $c < d$ and $c \overset{v}{\dashrightarrow} d$, then $c \overset{u}{\dashrightarrow} d$;*

(ii) *If $c > d, c \neq b, d \neq a$, and $c \overset{v}{\dashrightarrow} d$, then $c \overset{u}{\dashrightarrow} d$.*

4 The lattice property

4.1 Generalized permutohedra

The normal fan of the permutohedron $\mathrm{Perm}_n = Q_{w_0}$ is the fan determined by the *braid arrangement*, which has defining hyperplanes $x_i - x_j = 0$ for $\forall i \neq j$. The top-dimensional cones $C(y)$ in this fan are naturally labelled by permutations $y \in S_n$ giving the relative order of the coordinates of a point $(x_1, \ldots, x_n) \in C(y)$; in particular we have $\boldsymbol{y} \in C(y)$.

Following Postnikov [27], a polytope whose normal fan coarsens the braid arrangement is called a *generalized permutohedron*. Kodama–Williams [18] showed that Bruhat interval polytopes are generalized permutohedra. This implies that each top-dimensional cone $C_w(u)$ (where u now runs over the elements of $[e, w]$) in the normal fan of Q_w contains some of the $C(y)$. We write $[y]_w$ for the equivalence class of y under the equivalence relation Θ_w induced on S_n by the normal fan of Q_w.

We say $y \in S_n$ is a *linear extension* of $\Gamma_w(u)$ (equivalently, of $\widetilde{\Gamma}_w(u)$) if $y^{-1}(i) < y^{-1}(j)$ whenever $i \overset{u}{\dashrightarrow} j$. The following proposition is immediate from the construction of $\Gamma_w(u)$ in Section 5 of [20] and the discussion of normal fans of generalized permutohedra in Section 3 of [25].

Proposition 4.1 (See [20, 25])**.** *Let $w \in S_n$ and $u \preceq w$, then $[u]_w$ is exactly the set of linear extensions of $\Gamma_w(u)$.*

Somewhat surprisingly, the equivalence classes $[x]_w$ turn out to be intervals in right weak order. This result was established by other means in [20], but the particulars of our proof will be important later.

Proposition 4.2. *Let $x, w \in S_n$, then $[x]_w$ contains a unique minimal element $\mathrm{bot}_w(x)$ and unique maximal element $\mathrm{top}_w(x)$ under right weak order.*

Proof. Let u be the unique element of $[e, w] \cap [x]_w$. By Proposition 4.1, the elements y of $[x]_w$ are exactly the linear extensions of $\widetilde{\Gamma}_w(u)$. Suppose that $(ab) \in T_L(u)$ with $a < b$, then by Proposition 3.3 we have $b \overset{u}{\dashrightarrow} a$, so by Proposition 4.1 we have $(ab) \in T_L(y)$ for any $y \in [x]_w$. Thus by Proposition 2.1 we have $u \leq_R y$, so $\mathrm{bot}_w(x) = u$.

The reflections occurring as left inversions of some linear extension of $\widetilde{\Gamma}_w(u)$ are exactly those in

$$I := \{(ab) \mid a < b \text{ and } a \overset{u}{\not\dashrightarrow} b\}.$$

To see that $\mathrm{top}_w(x)$ exists, we will demonstrate that the corresponding set of roots, $R = \{e_a - e_b \mid (ab) \in I\}$, is biclosed, so that $\mathrm{top}_w(x)$ will be the unique permutation with left inversion set I.

First, note that if $a \overset{u}{\dashrightarrow} b$ and $b \overset{u}{\dashrightarrow} c$, then $a \overset{u}{\dashrightarrow} c$, so R is coclosed.

For closedness, let $a < b < c$ and assume that $a \overset{u}{\dashrightarrow} c$, which implies that $u^{-1}(a) < u^{-1}(c)$. If $u^{-1}(b) < u^{-1}(a)$, then $(ab) \in T_L(u)$, so by Proposition 3.3 we have $b \overset{u}{\dashrightarrow} a \overset{u}{\dashrightarrow} c$, so $b \overset{u}{\dashrightarrow} c$. If instead $u^{-1}(b) > u^{-1}(c)$, then $(bc) \in T_L(u)$, so by Proposition 3.3 we have $a \overset{u}{\dashrightarrow} c \overset{u}{\dashrightarrow} b$, so $a \overset{u}{\dashrightarrow} b$. Otherwise we have $u^{-1}(a) < u^{-1}(b) < u^{-1}(c)$. Consider a path $a \to a_1 \to \cdots \to a_r \to c_1 \to \cdots \to c_s \to c$, where $u^{-1}(a_i) \leq u^{-1}(b)$ and $u^{-1}(b) < u^{-1}(c_j)$ for all i, j. If any $a_i > b$, then $a \overset{u}{\dashrightarrow} a_i \overset{u}{\dashrightarrow} b$. If any $c_j < b$, then $b \overset{u}{\dashrightarrow} c_j \overset{u}{\dashrightarrow} c$. Otherwise, since $(a_r\, c_1)u$ covers u in Bruhat order, we must have $a_r = b$, so $a \overset{u}{\dashrightarrow} b \overset{u}{\dashrightarrow} c$. In all cases, we see $a \overset{u}{\dashrightarrow} b$ or $b \overset{u}{\dashrightarrow} c$, so R is closed. □

4.2 The poset structure

Theorem 4.3. *Given $w \in S_n$, the map $\mathrm{top}_w : S_n \to S_n$ is order preserving with respect to right weak order. That is, if $x \leq_R y$ then $\mathrm{top}_w(x) \leq_R \mathrm{top}_w(y)$. Furthermore, the quotient $\mathrm{Weak}_R(S_n)/\Theta_w$ is isomorphic to P_w via the map $[x]_w \mapsto \mathrm{bot}_w(x)$.*

Proof. By the proof of Proposition 4.2, we have for any $z \in S_n$ that

$$T_L(\mathrm{top}_w(z)) = \{(cd) \mid c < d \text{ and } c \overset{\mathrm{bot}_w(z)}{\not\dashrightarrow} d\}. \tag{4.1}$$

Thus if $u = \lessdot_w = v = (ab)u$ with $a < b$ then by Proposition 3.4(i) and (4.1) we have that

$$T_L(\mathrm{top}_w(u)) \subset T_L(\mathrm{top}_w(v)).$$

By Proposition 2.1 we see $\mathrm{top}_w(u) <_R \mathrm{top}_w(v)$.

Now suppose that $x \lessdot_R y = xs = tx$ with $[x]_w \neq [y]_w$, so $C_w(u)$ and $C_w(v)$ share a facet along the hyperplane fixed by the reflection t, where $u := \mathrm{bot}_w(x)$ and $v :=$

$\mathrm{bot}_w(y)$. This implies that there is an edge of Q_w with vertices $\boldsymbol{u}$ and $\boldsymbol{v}$. Since $u \leq_R x$ by Proposition 4.2, we have that $t \notin T_L(u)$ by Proposition 2.1 and the fact that $t \notin T_L(x)$. Since the convex cones $C_w(u)$ and $C_w(v)$ share at most one facet, and since $v = ut'$ for some $t' \in T$, we must have in fact that $t = t'$, and thus $\ell(v) > \ell(u)$ and $u \lessdot_w v$. This establishes that $P_w \cong \mathrm{Weak}_n / \Theta_w$, and that top_w is order preserving, after applying the first paragraph and that fact that $\mathrm{top}_w(u) = \mathrm{top}_w(x)$ and $\mathrm{top}_w(v) = \mathrm{top}_w(y)$. □

Corollary 4.4. *Let $w \in S_n$, then the map $\mathrm{top}_w : [e, w] \to \mathrm{top}_w([e, w])$ is an isomorphism between the posets P_w and $(\mathrm{top}_w([e, w]), \leq_R)$.*

The fact that P_w is a lattice also follows easily from Theorem 4.3.

Theorem 4.5. *For any $w \in S_n$, the poset P_w is a lattice, with join operation given by*

$$u \vee_w v = \mathrm{bot}_w(\mathrm{top}_w(u) \vee_R \mathrm{top}_w(v)).$$

Proof. Let $z = \mathrm{bot}_w(\mathrm{top}_w(u) \vee_R \mathrm{top}_w(v))$. Then

$$u \leq_R \mathrm{top}_w(u) \leq_R \mathrm{top}_w(u) \vee_R \mathrm{top}_w(v),$$

so by Theorem 4.3 we have $u \leq_w z$, and similarly $v \leq_w z$. On the other hand, if $y \geq_w u, v$, then by Theorem 4.3 we have $\mathrm{top}_w(y) \geq \mathrm{top}_w(u), \mathrm{top}_w(v)$ so $\mathrm{top}_w(y) \geq \mathrm{top}_w(u) \vee_R \mathrm{top}_w(v)$. Thus $y \geq_w z$ and we see that z is the join of u, v in P_w. Since P_w is a finite poset with a join and a unique minimal element, it also has a meet and is thus a lattice. □

Unlike weak order, P_w is not in general a semidistributive lattice.

4.3 The Billey–Fan–Losonczy parabolic map

Let $S_n(I)$ denote the subgroup of S_n generated by a subset I of the simple reflections. For $w \in S_n$ let $m(w, I)$ denote the unique maximal element of $S_n(I) \cap [e, w]$ under Bruhat order [2] (see Richmond and Slofstra [30] for the importance of these elements in determining the fiber bundle structure of Schubert varieties).

Theorem 4.6. *Let $w \in S_n$, and let I be a set of simple generators, then:*

$$m(w, I) = \bigvee_w \{s_i \in I \mid s_i \preceq w\}.$$

5 Directionally simple polytopes

Given a polytope $Q \subset \mathbb{R}^d$, say that a cost vector $c \in \mathbb{R}^d$ is *generic* if c is not orthogonal to any edge of Q. A generic cost vector induces an acyclic orientation on the 1-skeleton $G(Q)$ by taking edges to be oriented in the direction of greater inner product with c; we write $G_c(Q)$ for the resulting acyclic directed graph. It is clear that every face F of Q contains a unique source $\min_c(F)$ and sink $\max_c(F)$ with respect to this orientation.

Definition 5.1. We say that a polytope $Q \subset \mathbb{R}^d$ is *directionally simple* with respect to the generic cost vector c if for every vertex v of Q and every set E of edges of $G_c(Q)$ with source v there exists a face F of Q containing v whose set of edges incident to v is E.

Proposition 5.2. *A simple polytope $Q \subset \mathbb{R}^d$ is directionally simple with respect to any generic cost vector.*

Theorem 5.3 was proven in [24] by an involved direct computation; here we give a new proof using the results of Section 4.

Theorem 5.3. *Let $w \in S_n$, then Q_w is a directionally simple polytope with respect to the cost vector $c = (n, n-1, \ldots, 1)$.*

Theorem 5.3 shows that Q_w is always directionally simple, in Section 6 we will determine when Q_w is in fact simple.

5.1 h-vectors of directionally simple polytopes

The *f-vector* of a polytope $Q \subset \mathbb{R}^d$ is the tuple $f(Q) = (f_0, \ldots, f_d)$ where f_i is the number of i-dimensional faces of Q. The *h-vector* $h(Q)$ is defined by the equality of polynomials

$$\sum_{i=0}^{d} f_i(x-1)^i = \sum_{k=0}^{d} h_k x^k. \tag{5.1}$$

Proposition 5.4. *Let $Q \subset \mathbb{R}^d$ be directionally simple with respect to the generic cost vector c, with h-vector $h(Q) = (h_0, \ldots, h_d)$. Then for all $k = 0, \ldots, d$ the entry h_k is the number of vertices of Q with out-degree exactly k in $G_c(Q)$.*

Remark. One implication of Proposition 5.4 is that $h_i \geq 0, \forall i$. This by itself is already a very special property of directionally simple polytopes; indeed h-vectors of non-simple polytopes are rarely considered, because they are rarely positive or otherwise interesting.

For $u \in S_n$, write $\operatorname{asc}(u)$ for the number $n - 1 - |D_R(u)|$ of right ascents of u. Corollary 5.5 below is an extension to Q_w of the kind of interpretation for h-vectors of *simple* generalized permutohedra given by Postnikov–Reiner–Williams [25].

Corollary 5.5. *Let $(h_0, h_1, \ldots)$ be the h-vector of Q_w, then for all k we have:*

$$h_k = \left|\{z \in \operatorname{top}_w([e, w]) \mid \operatorname{asc}(z) = k\}\right|.$$

As explained in [24], the h-vector of Q_w also gives the Poincaré polynomial of the toric variety Y_w, so Corollary 5.5 gives a new formula for that invariant. We can also resolve an open problem raised in [23]:

Proposition 5.6 (Resolves Problem 6.1 of [23])**.** *The variety Y_w is smooth if and only if its Poincaré polynomial is palindromic.*

6 Vertex-degree monotonicity

In Section 4 we applied properties of the relation $c \overset{u}{\dashrightarrow} d$ to prove that P_w is a lattice. It is possible to develop and use more refined information about the relation $c \overset{u}{\Rightarrow} d$ to prove that vertex-degrees of Q_w are monotonic with respect to the partial order $\leq_w$, but we lack the space to do this here. As an application, we resolve a conjecture of Lee–Masuda–Park [20] characterizing smooth generic torus orbit closures in Schubert varieties.

Write $\deg_w(\boldsymbol{u})$ for the number of edges of Q_w incident to the vertex $\boldsymbol{u}$.

Theorem 6.1. *Let $w \in S_n$. If $u \leq_w v$ then $\deg_w(\boldsymbol{u}) \leq \deg_w(\boldsymbol{v})$.*

Theorem 6.1 will follow from the stronger Theorem 6.4 below.

Corollary 6.2. *Let $w \in S_n$, then the polytope Q_w is simple if and only if it is simple at the vertex $\boldsymbol{w}$.*

Proof. It is clear from Proposition 3.2 and the definition of $E_w(e)$ that Q_w is always simple at the vertex $\boldsymbol{e}$. Thus if Q_w is also simple at $\boldsymbol{w}$, Theorem 6.1 implies that it is simple at every vertex. □

Corollary 6.2 resolves Conjecture 7.17 of Lee–Masuda [20]. As described there, Corollary 6.2 has the following geometric interpretation.

Corollary 6.3. *Let Y_w be a generic torus orbit closure in the Schubert variety $X_w := \overline{BwB/B}$, then Y_w is smooth if and only if it is smooth at the torus fixed point wB.*

Write $c \overset{u}{\Leftrightarrow} d$ if $c \overset{u}{\Leftarrow} d$ or $c \overset{u}{\Rightarrow} d$ (note that we never have both $c \overset{u}{\Leftarrow} d$ and $c \overset{u}{\Rightarrow} d$).

Theorem 6.4. *Let $w \in S_n$ and suppose $u \lessdot_w v = tu$ with $c \overset{u}{\Leftrightarrow} d$, then there is a unique edge of $E_w(v)$ described by $c \overset{v}{\Leftrightarrow} d$ or $t(c) \overset{v}{\Leftrightarrow} t(d)$. Moreover, the map $\varphi : E_w(u) \to E_w(v)$ sending the edge $c \overset{u}{\Leftrightarrow} d$ to the corresponding edge of $E_w(v)$ is an injection.*

Theorem 6.1 follows from Theorem 6.4 since, by Proposition 3.2 we have $\deg_w(\boldsymbol{u}) = |E_w(u)|$ for all $u \preceq w$.

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