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Automorphisms of undirected Bruhat graphs

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Abstract. The *undirected Bruhat graph* $\Gamma(u, v)$ has the elements of the Bruhat interval $[u, v]$ as vertices, with edges given by multiplication by a reflection. Famously, $\Gamma(e, v)$ is regular if and only if the Schubert variety X_v is smooth, and this condition on v is characterized by pattern avoidance. In this work, we classify when $\Gamma(e, v)$ is *vertex-transitive*; surprisingly this class of permutations is also characterized by pattern avoidance and sits nicely between the classes of smooth permutations and self-dual permutations. This leads us to a general investigation of automorphisms of $\Gamma(u, v)$ in the course of which we show that *special matchings*, which originally appeared in the theory of Kazhdan–Lusztig polynomials, can be characterized, for the symmetric and right-angled groups, as certain $\Gamma(u, v)$-automorphisms which are conjecturally sufficient to generate the orbit of e under $\mathrm{Aut}(\Gamma(e, v))$.

Keywords: Bruhat order, Bruhat graph, vertex transitive, Schubert variety, smooth

1 Introduction

The *(directed) Bruhat graph* $\widehat{\Gamma}$ of a Coxeter group W is the directed graph with vertex set W and directed edges $w \to wt$ whenever $\ell(wt) > \ell(w)$ and t is a reflection. Write $\widehat{\Gamma}(u, v)$ for its restriction to a Bruhat interval $[u, v] \subset W$, and simply $\widehat{\Gamma}(v)$ for its restriction to $[e, v]$. These graphs appear ubiquitously in the combinatorics of Coxeter groups and Bruhat order [14], the topology of flag, Schubert, and Richardson varieties as the GKM-graph for the natural torus action [17, 18], and in the geometry of these varieties and related algebra, for example in the context of Kazhdan–Lusztig polynomials [4, 5, 12, 13].

In all of these contexts, the directions of the edges, and sometimes additional edge labels, are centrally important. In this work, however, we study the associated *undirected* graphs $\Gamma(u, v)$ and $\Gamma(v) := \Gamma(e, v)$. In particular, from the perspective of the undirected graph, it is very natural to study graph automorphisms (in contrast, the directed Bruhat graph $\widehat{\Gamma}$ has very few automorphisms [23]), and these automorphisms end up having close connections to previous work on smooth Schubert varieties [20, 9], self-dual Bruhat intervals [15], Billey–Postnikov decompositions [2, 22], and special matchings [8].

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1.1 Regular, vertex-transitive, and self-dual Bruhat graphs

The following well-known theorem, combining results of Lakshmibai–Sandhya [20] and Carrell–Peterson [9], helped establish the fundamental nature of both the Bruhat graph and pattern avoidance conditions in the combinatorial and geometric study of Schubert varieties.

Theorem 1 (Lakshmibai–Sandhya [20], Carrell–Peterson [9])**.** *The following are equivalent for a permutation w in the symmetric group $\mathfrak{S}_n$:*

(S1) *the undirected Bruhat graph $\Gamma(w)$ is a regular graph,*

(S2) *the permutation w avoids the patterns* 3412 *and* 4231*,*

(S3) *the poset $[e, w]$ is rank-symmetric, and*

(S4) *the Schubert variety X_w is smooth.*

In light of (S3), it is natural to ask whether $[e, w]$ is in fact self-dual as a poset when X_w is smooth. This turns out to not always be the case, but the smaller class of self-dual intervals also admits a nice characterization by pattern avoidance:

Theorem 2 (G.–G. [15])**.** *The following are equivalent for a permutation $w \in \mathfrak{S}_n$:*

(SD1) *the Bruhat interval $[e, w]$ is self-dual as a poset, and*

(SD2) *the permutation w avoids the patterns* 3412 *and* 4231 *as well as* 34521, 54123, 45321, *and* 54312*.*

In our first main theorem here, we characterize by pattern avoidance those permutations w such that $\Gamma(w)$ is *vertex-transitive*; this characterization implies that this class of permutations sits nicely between the classes of self-dual permutations (Theorem 2) and smooth permutations (Theorem 1).

Theorem 3. *The following are equivalent for a permutation $w \in \mathfrak{S}_n$:*

(VT1) *the undirected Bruhat graph $\Gamma(w)$ is a vertex-transitive graph,*

(VT2) *the permutation w avoids the patterns* 3412 *and* 4231 *as well as* 34521 *and* 54123*.*

The proof of Theorem 3 is quite technical and is omitted in this extended abstract.

Since vertex-transitive graphs are necessarily regular, it is clear that the permutations from Theorem 3 are a subset of those from Theorem 1, and this is borne out by comparing conditions (S2) and (VT2). It is not at all conceptually clear, however, why the self-dual permutations of Theorem 2 should in turn be a subset of those from Theorem 3, even though this fact is easily seen by comparing conditions (VT2) and (SD2). A conceptual bridge between these two classes of permutations is provided by Conjecture 1.

Conjecture 1. *Let $w \in \mathfrak{S}_n$ and let $\mathcal{O} = \{\varphi(e) \mid \varphi \in \mathrm{Aut}(\Gamma(w))\}$ be the orbit of the identity under graph automorphisms of $\Gamma(w)$, then $\mathcal{O} = [e, v]$ for some $v \leq w$.*

Indeed, if $[e, w]$ is self-dual, then $w \in \mathcal{O}$, and so if Conjecture 1 holds we must have $\mathcal{O} = [e, w]$. That is, $\Gamma(w)$ must be vertex-transitive.

In the course of the proof of Theorem 3 and the refinement (see Section 3) of Conjecture 1, we are led to consider certain automorphisms of $\Gamma(u, v)$ arising from perfect matchings on the Hasse diagram of $[u, v]$. That these automorphisms are the same thing as the previously well-studied *special matchings* on $[u, v]$ is the subject of our second main theorem.

1.2 Special matchings and Bruhat automorphisms

Special matchings (see the definition in Section 2.3) on Bruhat intervals were introduced [6, 8] because they can be used to define a recurrence for *Kazhdan–Lusztig R-polynomials* [19] which allows for the resolution of the *Combinatorial Invariance Conjecture* in the case of lower intervals $[e, w]$. These matchings are intended to generalize many of the combinatorial properties of the matching on W induced by multiplication by a simple reflection s. Special matchings on Bruhat intervals and related posets have since found several other combinatorial and topological applications and been generalized in several ways [1, 7, 21], and are completely classified on lower Bruhat intervals. [11].

In Theorem 4 and Conjecture 2 below we give a new characterization of special matchings of Bruhat intervals $[u, v]$ in terms of automorphisms of $\Gamma(u, v)$. This characterization is notable because it expresses the special matching condition, originally formulated as a condition only on Bruhat covers, as a condition on the global structure of the undirected Bruhat graph.

A Coxeter group W is called *right-angled* if every pair of simple generators either commutes or generates an infinite dihedral group.

Theorem 4. *Let W be a right-angled Coxeter group or the symmetric group and let $u \leq v$ be elements of W. Then a perfect matching of the Hasse diagram of $[u, v]$ is a special matching if and only if it is an automorphism of $\Gamma(u, v)$.*

Conjecture 2. *Theorem 4 holds for arbitrary Coxeter groups W.*

1.3 Outline

In Section 2, we cover background and definitions relating to Coxeter groups, Bruhat order and Bruhat graphs, Billey–Postnikov decompositions, and special matchings. In Section 3 we give a more precise version of Conjecture 1 in terms of *almost reducible decompositions* and some partial results towards resolving the conjecture. Section 4 outlines the proof of Theorem 4. The proof of Theorem 4 relies on a structural property of Bruhat order, the existence of upper bounds of *butterflies*, which may be of independent interest. Most of the proofs are omitted for this extended abstract, notably including those of Theorem 3, Theorem 7, Proposition 4, Lemma 1 and Lemma 3. Readers may refer to [16] for additional details.

2 Background and definitions

2.1 Bruhat graphs and Bruhat order

The *directed Bruhat graph* $\widehat{\Gamma}$ of a Coxeter group W is the directed graph with vertex set W and directed edges $w \to wt$ whenever t is a reflection with $\ell(wt) > \ell(w)$. Note that, since T is closed under conjugation, the “left" and “right" versions of $\widehat{\Gamma}$ in fact coincide. The *(undirected) Bruhat graph* Γ is the associated simple undirected graph. The directed graph $\widehat{\Gamma}$ is much more commonly considered in the literature, and often called “the Bruhat graph" but, since our focus in this work is on the undirected graph Γ, when directedness is not specified we mean the undirected graph.

The *(strong) Bruhat order* $(W, \leq)$ is the partial order on W obtained by taking the transitive closure of the relation determined by $\widehat{\Gamma}$. We write $[u, v]$ for the interval $\{w \in W \mid u \leq w \leq v\}$ in Bruhat order. For $u \leq v$, we write $\widehat{\Gamma}(u, v)$ and $\Gamma(u, v)$ for the restrictions of $\widehat{\Gamma}, \Gamma$ to the vertex set $[u, v]$; when u is the identity element e, we sometimes write simply $\widehat{\Gamma}(v)$ and $\Gamma(v)$.

Fundamental properties of Bruhat order can be found in [3].

For $w \in W$, we write $\mathrm{Supp}(w)$ for the *support* of w: the set of simple reflections appearing in some (equivalently, every) reduced word for w. We say the element $w \in W$ has a *disjoint support decomposition* if it may be expressed as a nontrivial product $w = w'w''$ with $\mathrm{Supp}(w') \cap \mathrm{Supp}(w'') = \emptyset$ (note that, in this case, we have $w' = w^J$ and $w'' = w_J$ with $J = \mathrm{Supp}(w'')$).

Proposition 1. *Let $w = w'w''$ be a disjoint support decomposition, then:*

$$\widehat{\Gamma}(w) \cong \widehat{\Gamma}(w') \times \widehat{\Gamma}(w''),$$
$$\Gamma(w) \cong \Gamma(w') \times \Gamma(w''),$$
$$[e, w] \cong [e, w'] \times [e, w''].$$

In each case, the isomorphism is given by group multiplication.

Proposition 1 implies that when $w = w'w''$ is a disjoint support decomposition, $\Gamma(w), \widehat{\Gamma}(w)$, and $[e, w]$ have automorphisms induced by automorphisms for w', w''.

2.2 Billey–Postnikov decompositions

Definition 1 (Billey–Postnikov [2], Richmond–Slofstra [22])**.** *Let W be a Coxeter group and $J \subseteq S$, the parabolic decomposition $w = w^J w_J$ of w is a* Billey–Postnikov decomposition *or* BP-decomposition *if*

$$\mathrm{Supp}(w^J) \cap J \subseteq D_L(w_J).$$

BP-decompositions were introduced by Billey and Postnikov in [2] in the course of their study of pattern avoidance criteria for smoothness of Schubert varieties in all finite types. The following characterizations of BP-decompositions will be useful to us.

Proposition 2 (Richmond–Slofstra [22])**.** *For $w \in W$ and $J \subseteq S$, the following are equivalent:*

1. *$w = w^J w_J$ is a BP-decomposition,*
2. *the multiplication map $\left([e, w^J] \cap W^J\right) \times [e, w_J] \to [e, w]$ is a bijection,*
3. *w_J is the maximal element of $W_J \cap [e, w]$.*

2.3 Special matchings

The *Hasse diagram*, denoted $H(P)$, of a poset P is the undirected graph with vertex set P and edges (x, y) whenever $x \lessdot_P y$ is a cover relation in P. Note that the Hasse diagram $H(W)$ of Bruhat order on W is a (non-induced) subgraph of Γ. A *perfect matching* of a graph G is a fixed-point-free involution $M : G \to G$ such that $(x, M(x))$ is an edge of G for all $x \in G$.

Definition 2 (Brenti [6], Brenti–Caselli–Marietti [8])**.** *A perfect matching M on the Hasse diagram of a poset P is a* special matching *if, for every cover relation $x \lessdot_P y$, either $M(x) = y$ or $M(x) <_P M(y)$.*

For $u \leq v \in W$ it is not hard to check, using the Lifting Property, that for $s \in D_L(v) \setminus D_L(u)$ (resp. $s \in D_R(v) \setminus D_R(u)$) left (resp. right) multiplication by s determines a special matching of the Hasse diagram $H([u, v])$. In fact, this motivated the definition of special matching [6]. Proposition 3 below, a special case of the classification by Caselli and Marietti [10, 11] of special matchings of lower Bruhat interval, observes that *middle multiplication* is also a special matching for lower intervals.

Proposition 3 (Special case of Theorem 5.1 in [10])**.** *Suppose $w = w^J w_J$ is a BP-decomposition of w and in addition we have $\mathrm{Supp}(w^J) \cap \mathrm{Supp}(w_J) = \{s\}$, then the middle multiplication map*

$$\phi : x \mapsto x^J s x_J,$$

is a special matching of $[e, w]$.

2.4 From special matchings to Bruhat automorphisms

The following result of Waterhouse shows that $\widehat{\Gamma}$ has no nontrivial automorphisms as a directed graph.

Theorem 5 (Waterhouse [23])**.** *Let W be an irreducible Coxeter group which is not dihedral, then $\mathrm{Aut}((W, \leq))$ (equivalently, $\mathrm{Aut}(\widehat{\Gamma})$) is generated by the graph automorphisms of the Dynkin diagram of W and the group inversion map on W.*

In this paper we study the much richer sets of automorphisms of Γ and particularly of its subgraphs $\Gamma(u, v)$. Although it is stated only for lower intervals $[e, v]$, the proof of Theorem 10.3 in [8] also applies to general intervals $[u, v]$ and yields Theorem 6 below:

Theorem 6 (Theorem 10.3 of [8])**.** *Let $u \leq v$ be elements of a Coxeter group W. Any special matching M of the Hasse diagram $H([u, v])$ is an automorphism of $\Gamma(u, v)$.*

Corollary 1. *Let $w \in W$. If $s \in D_L(w)$ (resp. $D_R(w)$) then left (resp. right) multiplication by s is an automorphism of $\Gamma(w)$. If $w = w^J w_J$ is a BP-decomposition of w with $\mathrm{Supp}(w^J) \cap \mathrm{Supp}(w_J) = \{s\}$, then middle multiplication by s is an automorphism of $\Gamma(w)$.*

Theorem 6 provides one direction of Theorem 4 and Conjecture 2 for arbitrary Coxeter groups. This implies that special matchings on Bruhat intervals, although defined by a local condition (that is, a condition on cover relations), respect the global structure of Bruhat graphs. The reverse direction is the subject of Section 4.

3 Identity orbits in Bruhat graphs

We describe a more precise version of Conjecture 1, taking into account the automorphisms described in Section 2.4 and Theorem 3. In light of Proposition 1, it is sufficient to consider permutations $w \in \mathfrak{S}_n$ which have full support and do not admit a disjoint support decomposition; we call such permutations *Bruhat irreducible*.

Definition 3. *A Bruhat irreducible permutation $w \in \mathfrak{S}_n$ is* almost reducible *at (J, i) if $w = w^J w_J$ is a BP-decomposition with $\mathrm{Supp}(w^J) \cap J = \{s_i\}$ and $s_i \notin D_L(w), D_R(w)$.*

Definition-Proposition 1. *If a Bruhat irreducible $w \in \mathfrak{S}_n$ is almost reducible at (J, i), then $J = \{s_1, \ldots, s_i\}$ or $\{s_i, \ldots, s_{n-1}\}$. We say it is* left-almost-reducible *at i if $J = \{s_1, \ldots, s_i\}$, and* right-almost-reducible *at i if $J = \{s_i, \ldots, s_{n-1}\}$.*

Definition 4. *For a Bruhat irreducible permutation $w \in \mathfrak{S}_n$, let*

$$\{i_1 < \cdots < i_k\} = \{i \mid w \text{ is right-almost-reducible at } i\}$$

and define $A_R(w) := s_{i_1} \cdots s_{i_k}$. Similarly, let $\{j_1 < \cdots < j_t\}$ be the set of j at which w is left-almost-reducible and define $A_L(w) := s_{j_t} \cdots s_{j_1}$.

Theorem 7. *Let w be Bruhat irreducible. Then the following three elements commute pairwise:*

$$A_R(w), A_L(w), w_0(D_L(w) \cap D_R(w)).$$

The following is a strengthened version of Conjecture 1.

Conjecture 3. *Let $w \in \mathfrak{S}_n$ be Bruhat irreducible and let $\mathcal{O}$ denote the orbit of e under graph automorphisms of $\Gamma(w)$. Define*

$$v(w) := w_0(D_L(w)) \cdot A_R(w) \cdot w_0(D_L(w) \cap D_R(w)) \cdot A_L(w) \cdot w_0(D_R(w)),$$

then $\mathcal{O} = [e, v(w)]$.

Proposition 4. *Let $w \in \mathfrak{S}_n$ be Bruhat irreducible and such that $\Gamma(w)$ is vertex-transitive, then $v(w) = w$, so Conjecture 3 holds in this case.*

The following proposition shows that the element $v(w)$ is indeed in the identity orbit of $\Gamma(w)$. An automorphism of $\Gamma(w)$ sending e to $v(w)$ may be obtained by composing various left, right, and middle multiplication automorphisms (see Section 2.4).

Proposition 5. *Let $w \in \mathfrak{S}_n$ be Bruhat irreducible and let $\mathcal{O}$ be the orbit of e under graph automorphisms of $\Gamma(w)$, then $v(w) \in \mathcal{O}$.*

4 From Bruhat automorphisms to special matchings via butterflies

In Theorem 8 below we give a converse to Theorem 6 for certain Coxeter groups. Theorem 6 and Theorem 8 together imply Theorem 4.

Theorem 8. *Let $u \leq v$ be elements of a Coxeter group W which is right-angled or a symmetric group, then any perfect matching of $H([u, v])$ which is an automorphism of $\Gamma(u, v)$ is a special matching.*

The proof of Theorem 8 relies on Lemma 1, a structural property of Bruhat order involving *butterflies*.

Definition 5. *We say that elements x_1, x_2, y_1, y_2 of a Coxeter group W form a* butterfly *if $x_1 \lessdot y_1, y_2$ and $x_2 \lessdot y_1, y_2$.*

The butterfly structures are essential to the analysis of Bruhat automorphisms and special matchings, and are of interest on their own.

Lemma 1. *Let W be a Coxeter group which is right-angled or the symmetric group, and suppose that $x_1, x_2, y_1, y_2 \in [u, v]$ form a butterfly. Then there is an element $z \in [u, v]$ with $y_1, y_2 \lessdot z$.*

Proof of Theorem 8 given Lemma 1. Let $u \leq v$ be elements of a Coxeter group W which is right-angled or the symmetric group, and let M be a perfect matching of $H([u,v])$ which is an automorphism of $\Gamma(u,v)$. Suppose that M is not a special matching; since M is a $\Gamma(u,v)$-automorphism, the violation of the special matching property must consist of elements $x \lessdot y$ with $M(y) \lessdot M(x)$. Choose x, y so that y has maximal length among all such violations in $[u,v]$.

Now, note that $x, M(y), y, M(x)$ form a butterfly, so by Lemma 1 there exists an element $z \in [u,v]$ with $y, M(x) \lessdot z$. We must have $M(z) > z$, for otherwise each of $y, M(x)$, and $M(z)$ would each cover both x and $M(y)$, but this substructure cannot occur in Bruhat order of a Coxeter group (see Theorem 3.2 of [8]). Since height-two intervals in Bruhat order are diamonds (see Chapter 2 of [3]), there exists an element $w \neq z$ with $y \lessdot w \lessdot M(z)$.

Suppose that $M(w) < w$, then since M is an automorphism of the Bruhat graph we must have $M(w) \lessdot z$ and $M(y) \lessdot M(w)$. Now, since $y \lessdot z$, we know $M(y) \to M(z)$ in $\widehat{\Gamma}(u,v)$, but the height-three interval $[M(y), M(z)]$ contains at least three elements—$y, M(w)$, and $M(x)$ at height one, contradicting Proposition 3.3 of [14].

We conclude that $w \lessdot M(w)$. However this too is a contradiction, since $w \lessdot M(z)$ is a violation of the special matching condition with $\ell(M(z)) > \ell(y)$. Thus M must be a special matching. □

We conjecture that a slight weakening of Lemma 1 holds for arbitrary Coxeter groups.

Conjecture 4. *Let W be any Coxeter group, and suppose that the elements $x_1, x_2, y_1, y_2 \in [u,v]$ form a butterfly. Then there is an element $z \in [u,v]$ with $y_1, y_2 \lessdot z$ or with $z \lessdot x_1, x_2$.*

Remark 1. *The weakening of Lemma 1 conjectured for general Coxeter groups in Conjecture 4 is necessary even for finite Coxeter groups. For example, the finite Coxeter group of type F_4 has a butterfly:*

$$\begin{aligned}
x_1 &= s_2s_3s_4s_2s_3s_1s_2s_3s_4s_3s_2s_3s_1s_2s_3s_1s_2,\\
x_2 &= s_3s_2s_3s_4s_3s_2s_3s_1s_2s_3s_4s_2s_3s_1s_2s_3s_2,\\
y_1 &= s_2s_3s_1s_2s_3s_4s_3s_2s_3s_1s_2s_3s_4s_2s_3s_1s_2s_3,\\
y_2 &= s_3s_2s_3s_4s_3s_2s_3s_1s_2s_3s_4s_2s_3s_1s_2s_3s_1s_2,
\end{aligned}$$

which has a lower bound $z = s_2s_3s_4s_2s_3s_1s_2s_3s_4s_3s_2s_3s_1s_2s_3s_2 \lessdot x_1, x_2$ but no upper bound $y_1, y_2 \lessdot z'$. See Figure 1 for a Bruhat interval that contains $z \lessdot x_1, x_2 \lessdot y_1, y_2$.

We end with a few more structural results concerning butterflies, in the setting of finite Weyl groups.

Lemma 2. *In a finite Weyl group, $w \lessdot ws_\alpha$ if and only if $\alpha \notin \mathrm{Inv}_R(w)$ and there does not exist $\beta_1, \beta_2 \in \mathrm{Inv}_R(w)$ such that $\beta_2 = -s_\alpha\beta_1$. Moreover, if $w \lessdot ws_\alpha$ and $\beta \in \Phi^+$ satisfies $s_\alpha\beta \in \Phi^-$, then $\beta \in \mathrm{Inv}_R(w)$ if and only if $\beta \in \mathrm{Inv}_R(ws_\alpha)$.*

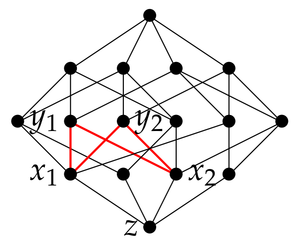

Figure 1: A butterfly in F_4 which does not admit an upper bound.

Lemma 3. *Let W be a finite Weyl group of simply-laced type, and let $x_1, x_2 \lessdot y_1, y_2$ form a butterfly. Then there exists $u \lessdot x_1, x_2$ and $z \gtrdot y_1, y_2$ in W.*

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References

[1] N. Abdallah, M. Hansson, and A. Hultman. "Topology of posets with special partial matchings". *Adv. Math.* **348** (2019), pp. 255–276. DOI.

[2] S. Billey and A. Postnikov. "Smoothness of Schubert varieties via patterns in root subsystems". *Adv. in Appl. Math.* **34**.3 (2005), pp. 447–466. DOI.

[3] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*. Vol. 231. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xiv+363.

[4] C. Blundell, L. Buesing, A. Davies, P. Veličković, and G. Williamson. "Towards combinatorial invariance for Kazhdan-Lusztig polynomials". 2021. arXiv:2111.15161.

[5] F. Brenti. "A combinatorial formula for Kazhdan-Lusztig polynomials". *Invent. Math.* **118**.2 (1994), pp. 371–394. DOI.

[6] F. Brenti. "The intersection cohomology of Schubert varieties is a combinatorial invariant". *European J. Combin.* **25**.8 (2004), pp. 1151–1167. DOI.

[7] F. Brenti, F. Caselli, and M. Marietti. "Diamonds and Hecke algebra representations". *Int. Math. Res. Not.* (2006), Art. ID 29407, 34. DOI.

[8] F. Brenti, F. Caselli, and M. Marietti. "Special matchings and Kazhdan-Lusztig polynomials". *Adv. Math.* **202**.2 (2006), pp. 555–601. DOI.

[9] J. B. Carrell. "The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties". *Algebraic groups and their generalizations: classical methods (University Park, PA, 1991)*. Vol. 56. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1994, pp. 53–61.

[10] F. Caselli and M. Marietti. "Special matchings in Coxeter groups". *European J. Combin.* **61** (2017), pp. 151–166. DOI.

[11] F. Caselli and M. Marietti. "A simple characterization of special matchings in lower Bruhat intervals". *Discrete Math.* **341**.3 (2018), pp. 851–862. DOI.

[12] A. Davies, P. Veličković, L. Buesing, S. Blackwell, D. Zheng, N. Tomašev, R. Tanburn, P. Battaglia, C. Blundell, A. Juhász, et al. "Advancing mathematics by guiding human intuition with AI". *Nature* **600**.7887 (2021), pp. 70–74.

[13] M. J. Dyer. "Hecke algebras and shellings of Bruhat intervals". *Compositio Math.* **89**.1 (1993), pp. 91–115. Link.

[14] M. Dyer. "On the "Bruhat graph" of a Coxeter system". *Compositio Math.* **78**.2 (1991), pp. 185–191. Link.

[15] C. Gaetz and Y. Gao. "Self-dual intervals in the Bruhat order". *Selecta Math. (N.S.)* **26**.5 (2020), Paper No. 77, 23. DOI.

[16] C. Gaetz and Y. Gao. "On automorphisms of undirected Bruhat graphs". *Math. Z.* **303**.2 (2023), Paper No. 31. DOI.

[17] M. Goresky, R. Kottwitz, and R. MacPherson. "Equivariant cohomology, Koszul duality, and the localization theorem". *Invent. Math.* **131**.1 (1998), pp. 25–83. DOI.

[18] V. Guillemin, T. Holm, and C. Zara. "A GKM description of the equivariant cohomology ring of a homogeneous space". *J. Algebraic Combin.* **23**.1 (2006), pp. 21–41. DOI.

[19] D. Kazhdan and G. Lusztig. "Representations of Coxeter groups and Hecke algebras". *Invent. Math.* **53**.2 (1979), pp. 165–184. DOI.

[20] V. Lakshmibai and B. Sandhya. "Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$". *Proc. Indian Acad. Sci. Math. Sci.* **100**.1 (1990), pp. 45–52. DOI.

[21] M. Marietti. "Algebraic and combinatorial properties of zircons". *J. Algebraic Combin.* **26**.3 (2007), pp. 363–382. DOI.

[22] E. Richmond and W. Slofstra. "Billey-Postnikov decompositions and the fibre bundle structure of Schubert varieties". *Math. Ann.* **366**.1-2 (2016), pp. 31–55. DOI.

[23] W. C. Waterhouse. "Automorphisms of the Bruhat order on Coxeter groups". *Bull. London Math. Soc.* **21**.3 (1989), pp. 243–248. DOI.