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Extremal tensor products of Demazure crystals are direct sums of Demazure crystals

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Abstract. We give a new necessary and sufficient condition for when tensor products of Demazure crystals decompose as direct sums of Demazure crystals. Our local criterion depends on the string property which Demazure crystals, and more generally, extremal crystals, exhibit. Our characterization implies that tensor products of Demazure crystals are direct sums of Demazure crystals if and only if they are extremal.

Keywords: Demazure crystals, extremal crystals, excellent filtrations, tensor products.

1 Introduction

In his study of the representations of quantum groups $U_q(\mathfrak{g})$ for $\mathfrak{g}$ a complex semisimple Lie algebra, Kashiwara [9], based on work of Lusztig [15], introduced *crystal bases* upon which, in the $q \to 0$ limit, the action of the Chevalley operators could be easily described. The crystal bases form the vertices of a *crystal graph*, a directed, colored graph with edges given by deformed Chevalley operators. The combinatorial structure of the crystal encodes the highest weight theory of the corresponding $U_q(\mathfrak{g})$-modules. Thus to any irreducible highest weight representation $V(\lambda)$, we associate the highest weight crystal $\mathcal{B}(\lambda)$ whose character agrees with the Weyl character of the module.

Given the monoidal structure of the category of $U_q(\mathfrak{g})$-modules, Kashiwara defined a crystal structure on the set $\mathcal{B}_1 \otimes \mathcal{B}_2$ which aligns with the tensor product of the corresponding modules. In particular, the fact that $V(\lambda) \otimes V(\mu)$ admits a *good filtration*, i.e. a filtration by Weyl modules, is reflected in the fact that $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ decomposes as a direct sum of highest weight crystals.

Demazure [4] considered a family of submodules generated by *extremal weight elements* under the Borel subalgebra, known eponymously as *Demazure modules*. The associated *Demazure crystals*, introduced by Littelmann [14] and generalized by Kashiwara [10], arise as truncations of the crystals for $U_q(\mathfrak{g})$-modules. As in the classical case, Demazure crystals encode the combinatorial structure of the corresponding Demazure

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modules. Hence, each Demazure module $V_w(\lambda)$ has an associated crystal $B_w(\lambda)$, indexed by a highest weight λ and an element w of the Weyl group W of $\mathfrak{g}$.

Filtrations by Demazure modules are known as *excellent filtrations*. Unlike with tensor products of Weyl modules, tensor products of Demazure modules do not always admit excellent filtrations [8]. Thus a natural question to consider is when can $V_w(\lambda) \otimes V_u(\nu)$ be filtered by Demazure modules.

In this paper we answer this question from a crystal theoretic perspective by considering a larger family of subcrystals, which we call *extremal subsets*. Extremal subsets are characterized by the string property which states that every i-string of the crystal which intersects the subset is either entirely contained in the subset or intersects in only the top element. Kashiwara [10] showed every Demazure crystal is extremal, though the converse does not hold. We show that tensor products of Demazure crystals $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\nu)$ decompose as sums of Demazure crystals if and only if $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\nu)$ is extremal. By studying tensor products of extremal subcrystals, we give a local criterion for when tensor products of Demazure crystals are extremal, thus giving a local characterization of precisely when $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\nu)$ decomposes as a sum of Demazure crystals.

Our results generalize work of Lakshimbai, Littelmann, and Magyar [13] and Joseph [6] in which they prove $\{u_\lambda\} \otimes \mathcal{B}_u(\nu)$ decomposes as a direct sum of Demazure crystals. Our local criterion also provides an alternative characterization to Kouno's global condition [12] for when $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\nu)$ remains Demazure. For full details, see [1].

2 Crystal graphs

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In this section, we review normal $\mathfrak{g}$-crystals. For a thorough treatment of crystals, see [11].

2.1 Highest weight crystals

Let P be the weight lattice of $\mathfrak{g}$ and let I be the vertex set of the Dynkin diagram. For every $i \in I$ we have a simple root $\alpha_i \in P$ and a simple coroot $\alpha_i^\vee \in P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$. Given $\lambda \in P$ and $\mu^\vee \in P^\vee$ we write $\langle \mu^\vee, \lambda \rangle$ for the integer obtained by the natural symmetric pairing on weights and coweights. Write W for the Weyl group generated by the set of of simple reflections s_i associated to $\alpha_i^\vee \in P^\vee$ and P^+ for the set of dominant weights $\{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha_i^\vee \in P^\vee\}$.

Definition 1. A (finite) normal $\mathfrak{g}$-*crystal* is a nonempty set $\mathcal{B}$, together with *crystal operators* $e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$, a *weight map* $\mathrm{wt} : \mathcal{B} \to P$, and *string operators* $\varepsilon_i(b) := \max\left\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k(b) \in \mathcal{B}\right\}$ and $\varphi_i(b) := \max\left\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k(b) \in \mathcal{B}\right\}$, such that for every $i \in I$ and for every $b, b' \in \mathcal{B}$:

(1) $b' = e_i(b)$ if and only if $b = f_i(b')$ in which case $\mathrm{wt}(b') = \mathrm{wt}(b) + \alpha_i$;

(2) $\varphi_i(b) - \varepsilon_i(b) = \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$.

The finite-dimensional, irreducible, integrable representations of $U_q(\mathfrak{g})$ are naturally indexed by the integral dominant weights. For each $\lambda \in P^+$, let $\mathcal{B}(\lambda)$ denote the crystal for the irreducible highest weight representation $V(\lambda)$.

Given a highest weight crystal $\mathcal{B}$, the associated *crystal graph* is the directed, I-colored graph with vertex set $\mathcal{B}$ and with an i-edge from b to $f_i(b)$ provided the latter is nonzero.

A crystal is *connected* if its underlying (undirected) graph is connected. Henceforth, we refer to (elements of) crystals and (vertices of) their graphs interchangeably.

Example A. The *standard crystal* $\mathcal{B}(1, 0^{n-1})$ for $\mathfrak{sl}_n(\mathbb{C})$ has basis $\{i \mid i = 1, \dots, n\}$, weight map $\mathrm{wt}(\,i\,) = (0^{i-1}, 1, 0^{n-i-1})$, and lowering operators $f_j(i) = i+1$ if $j = i+1$ and $f_j(i) = 0$ otherwise. We draw the crystal graph for $\mathcal{B}(1, 0^{n-1})$ as shown in Figure 1.

$$1 \xrightarrow{\;1\;} 2 \xrightarrow{\;2\;} 3 \xrightarrow{\;3\;} \cdots \xrightarrow{\;n-1\;} n$$

Figure 1: The $\mathfrak{sl}_n(\mathbb{C})$-crystal $\mathcal{B}(1, 0^{n-1})$.

For any $i \in I$ and $X \subseteq \mathcal{B}$, let $\mathcal{F}_i(X) = \left\{ f_i^m(x) \mid x \in X \text{ and } m \in \mathbb{Z}_{\geq 0} \right\} \setminus \{0\}$. For $s_{i_1} \cdots s_{i_\ell}$ a reduced expression for $w \in W$, let $\mathcal{F}_w(X) = \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_\ell}(X)$. When $w = w_0$ is the longest element we write omit the subscript and write $\mathcal{F}(X)$.

Joseph [7] proves that the set $\mathcal{F}_w(X)$ is independent of the choice of reduced expression for w and so is well-defined. The sets $\mathcal{E}_i(X)$, $\mathcal{E}_w(X)$ and $\mathcal{E}(X)$ are similarly defined using raising operators.

An element $b \in \mathcal{B}$ is a *highest weight element* if $\mathcal{E}_i(\{b\}) = \{b\}$ for all i. Let b_λ denote the highest weight element of the irreducible highest weight crystal $\mathcal{B}(\lambda)$.

2.2 Demazure crystals

The Weyl group W is equipped with a partial order $\prec$ called *Bruhat order* defined on any $u, v \in W$ by $u \prec v$ if and only if there exists a reduced word for v which contain a reduced word for u as a subword. See [3] for a reference on Bruhat order.

Demazure crystals are subsets $\mathcal{B}_w(\lambda) \subseteq \mathcal{B}(\lambda)$ depending on a choice of $w \in W$. They were introduced by Littelmann who showed for classical $\mathfrak{g}$ that their characters are the characters of Demazure modules $V_w(\lambda)$ [5, 6].

Definition 2 ([10])**.** For $\lambda \in P^+$ and $w \in W$, the *Demazure crystal* $\mathcal{B}_w(\lambda)$ is

$$\mathcal{B}_w(\lambda) = \mathcal{F}_w(\{b_\lambda\}). \tag{2.1}$$

Kashiwara [10] generalized Littelmann construction to arbitrary $\mathfrak{g}$ and showed $\mathcal{B}_w(\lambda)$ satisfies the following properties.

(1) $\mathcal{E}\left(\mathcal{B}_w(\lambda)\right) \subset \mathcal{B}_w(\lambda)$;

(2) if $s_i w \prec w$, then $\mathcal{B}_w(\lambda) = \{f_i^m(b) \mid m \geq 0, b \in \mathcal{B}_{s_i w}(\lambda), e_i(b) = 0\} \setminus \{0\}$;

(3) for any i-string S, $S \cap \mathcal{B}_w(\lambda)$ is either $\varnothing$ or S or $\{b\}$, where $b \in S$ and $e_i(b) = 0$.

For any $i \in I$, an *i-string* is any connected subset of a crystal closed under both $\mathcal{E}_i$ and $\mathcal{F}_i$. Equivalently, an i-string is a subset of the form $\mathcal{F}_i(\{b\})$ where $e_i(b) = 0$.

Demazure crystals are nested according to Bruhat order [10], i.e. $\mathcal{B}_v(\lambda) \subseteq \mathcal{B}_w(\lambda)$ whenever $v \prec w$. We tighten this result as follows.

Given $\lambda \in P^+$, let W_λ be the stabilizer subgroup of λ in W. The minimal (resp. maximal) length coset representatives of wW_λ are denoted by $\lfloor w \rfloor^\lambda$ (resp. $\lceil w \rceil^\lambda$).

Proposition 3. *Let $\lambda \in P^+$ and $v, w \in W$. Then $v \preceq \lceil w \rceil^\lambda$ if and only if $\mathcal{B}_v(\lambda) \subseteq \mathcal{B}_w(\lambda)$. Moreover, $\mathcal{B}_v(\lambda) = \mathcal{B}_w(\lambda)$ only when $v \in wW_\lambda$.*

Example B. Consider $\mathcal{B}_{s_2}(2,2,0) \subset \mathcal{B}(2,2,0)$ in Figure 2. Here $w = s_1$ and $\lambda = (2,2,0)$. Since $s_1 \in W_\lambda$, we have $\lceil w \rceil^\lambda = s_2 s_1 \succ s_1$, and so $\mathcal{B}_{s_2 s_1}(2,2,0) = \mathcal{B}_{s_2}(2,2,0)$. Likewise, since $\lceil w \rceil^\lambda \prec s_1 s_2 s_1$, we have $\mathcal{B}_{s_2 s_1}(2,2,0) \subsetneqq \mathcal{B}_{s_1 s_2 s_1}(2,2,0) = \mathcal{B}_{s_1 s_2}(2,2,0) = \mathcal{B}(2,2,0)$.

2.3 Extremal crystals

Following work of the extremal authors [2], we consider subsets satisfying property (3).

Definition 4. A subset $X \subseteq \mathcal{B}(\lambda)$ is *extremal* if X is nonempty and for any i-string S of $\mathcal{B}(\lambda)$, $S \cap X$ is either $\varnothing$ or S or $\{b\}$ where $b \in S$ and $e_i(b) = 0$.

Notice any subset of $\mathcal{B}(\lambda)$ satisfying Kashiwara's property (3) necessarily satisfies property (1) as well. In particular, if $X \subset \mathcal{B}(\lambda)$ is extremal, then $\mathcal{E}X \subset X$, and so $b_\lambda \in X$.

As Kashiwara proves [10], all Demazure crystals are extremal subsets. The converse, however, is false. Not all extremal subsets are Demazure crystals.

Example C. Let $\mathfrak{g} = \mathfrak{sl}_3$ and $\lambda = (2,2,0)$. Then $X = \{b_\lambda, f_2(b_\lambda), f_2^2(b_\lambda), f_1 f_2(b_\lambda)\}$ (seen in the middle of Figure 2) is extremal, but not Demazure. In particular, $B_{s_2}(2,2,0) \subsetneqq X \subsetneqq B(2,2,0)$. Similarly, $Y = \{b_\lambda, f_2(b_\lambda), f_2^2(b_\lambda), f_1 f_2^2(b_\lambda), f_1^2 f_2^2(b_\lambda)\}$ is also an extremal subset of $B(2,2,0)$ containing $B_{s_1}(2,2,0)$ that is not Demazure.

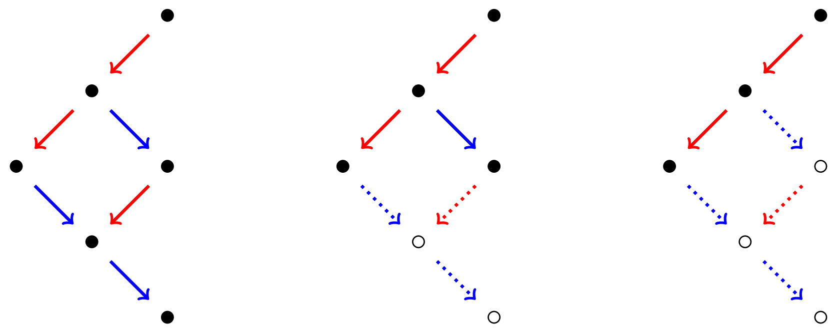

Figure 2: The $\mathfrak{sl}_3$-crystal $B(2,2,0)$ (left), an extremal subset (middle), and the Demazure crystal $B_{s_2}(2,2,0)$ (right) with f_1 and f_2 depicted by blue and red arrows, respectively.

3 Tensor products of crystals

3.1 Kashiwara's tensor product rule

Given $\mathfrak{g}$-crystals $\mathcal{B}_1$ and $\mathcal{B}_2$, the *direct sum* $\mathcal{B}_1 \oplus \mathcal{B}_2$ is their disjoint union with corresponding operators. Since any graph decomposes into the disjoint union of its connected components, every $\mathfrak{g}$-crystal decomposes as a direct sum of highest weight crystals.

Definition 5. The *tensor product* $\mathcal{B}_1 \otimes \mathcal{B}_2$ has vertex set $\{b_1 \otimes b_2 \mid b_1 \in \mathcal{B}_1 \text{ and } b_2 \in \mathcal{B}_2\}$, crystal operator f_i defined by

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \\ b_1 \otimes f_i(b_2) & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \end{cases}$$

e_i defined analogously, $\mathrm{wt}_i(b) = \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$, $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$, and $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}_i(b_1))$ and $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \mathrm{wt}_i(b_2))$.

Kashiwara [9] proves this tensor product is associative and noncommutative and proves $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is a crystal for $V(\lambda) \otimes V(\mu)$.

Example D. Consider the tensor product $\mathcal{B}(1,1,0) \otimes \mathcal{B}(1,0,0)$, where

$$\mathcal{B}(1,1,0) = a_1 \to a_2 \to a_3 \quad \text{and} \quad \mathcal{B}(1,0,0) = b_1 \to b_2 \to b_3\,.$$

Then, $\varphi_2(a_1) = \varphi_1(a_2) = \varepsilon_1(b_2) = \varepsilon_2(b_3) = 1$ and $\varphi_1(a_1) = \varphi_2(a_2) = \varepsilon_2(b_2) = \varepsilon_1(b_3) = 0$. Thus, as seen in Figure 3, $\mathcal{B}(1,1,0) \otimes \mathcal{B}(1,0,0)$ will decompose into two connected components with highest weights $(2,1,0)$ and $(1,1,1)$, respectively. Thus $\mathcal{B}(1,1,0) \otimes \mathcal{B}(1,0,0) \cong \mathcal{B}(2,1,0) \oplus \mathcal{B}(1,1,1)$, as expected from the decomposition of the tensor product of the corresponding modules.

3.2 Tensor products of Demazure crystals

The tensor product $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is not always a direct sum of Demazure crystals.

Example E. Consider the $\mathfrak{sl}_3$-crystals $\mathcal{B}_{s_2}(1,1,0)$ and $\mathcal{B}_{s_1}(1,1,0)$. Their tensor product, show in the middle diagram of Figure 3, is not a direct sum of Demazure crystals. In fact, it is not even extremal.

Kouno [12] characterized w, u, λ, μ such that $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is a direct sum of Demazure crystals.

Recall that for any $\lambda \in P^+$, we denote by W_λ be the stabilizer subgroup of λ in W and by $\lfloor w \rfloor^\lambda$ and $\lceil w \rceil^\lambda$ the minimal and maximal length coset representatives of wW_λ, respectively. For any $\sigma \in W$, let $W_\sigma \subseteq W$ denote the parabolic subgroup

$$W_\sigma = \langle\, s_i \in W \mid s_i\sigma \prec \sigma \,\rangle.$$

Theorem 6 (Kouno [12])**.** *Let $\lambda, \mu \in P^+$ and $u, w \in W$. Then $B_w(\lambda) \otimes B_u(\mu)$ is a direct sum of Demazure crystals if and only if $\lfloor w \rfloor^\lambda \in W_{\lceil u \rceil^\mu}$.*

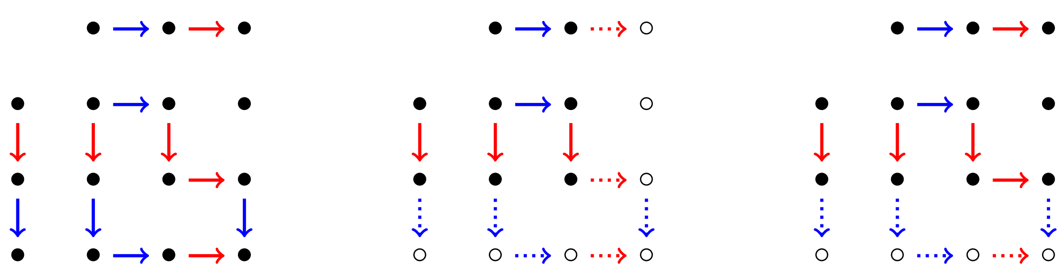

Figure 3: The tensor products $B(1,1,0) \otimes B(1,0,0)$ (left), $B_{s_2}(1,1,0) \otimes B_{s_1}(1,0,0)$ (middle), and $B_{s_2}(1,1,0) \otimes B(1,0,0) = B_{s_2 s_1}(2,1,0) \oplus B_e(1,1,1)$ (right) with f_1 and f_2 depicted by blue and red arrows, respectively.

Example F. Consider $\lambda, \mu \in P^+$, and suppose W has a longest element, which we denote by w_0. Then $\lfloor w \rfloor^\lambda \in W = W_{w_0}$ for any $w \in W$, and so $B_w(\lambda) \otimes B(\mu)$ always decomposes into Demazure crystals; see Figure 3.

Example G. Let $\mathfrak{g} = \mathfrak{sl}_3$, and consider $B_{s_2}(1,1,0) \otimes B_{s_1}(1,0,0)$. Then $W_{(1,0,0)} = \{s_2, e\}$, thus $\lceil s_1 \rceil^{(1,0,0)} = s_1 s_2$ and so $W_{s_1 s_2} = \{s_1\}$. However, $W_{(1,1,0)} = \{s_1, e\}$ so that $\lfloor s_2 \rfloor^{(1,1,0)} = s_2 \notin W_{s_1 s_2}$. As seen in Figure 3, $B_{s_2}(1,1,0) \otimes B_{s_1}(1,1,0)$ is indeed not Demazure.

Recall the tensor product of crystals is not commutative, though Kashiwara [10] showed $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is isomorphic to $\mathcal{B}(\mu) \otimes \mathcal{B}(\lambda)$. We remark this does *not* hold for Demazure crystals; that is, $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is *not* isomorphic to $\mathcal{B}_u(\mu) \otimes \mathcal{B}_w(\lambda)$ in general.

Indeed, by Kouno's characterization $B_w(\lambda) \otimes \{u_\mu\}$ is a direct sum of Demazure crystals only when $w \in W_\mu$. However, Joseph [7] proved $\{u_\mu\} \otimes B_w(\lambda)$ always decomposes as a direct sum of Demazure crystals.

Example H. Take $\mathfrak{g} = \mathfrak{sl}_3$, then $B_e(1,1,0) \otimes B_{s_2}(1,1,0) \cong B_e(2,2,0) \oplus B_e(2,1,1)$, as seen in Figure 5 (middle) , is a direct sum of Demazure crystals. However, $B_{s_2}(1,1,0) \otimes B_e(1,1,0)$ in Figure 5 (right) is not even extremal, let alone Demazure.

3.3 Tensor products of extremal crystals

Just as tensor products of Demazure crystals are not always Demazure, tensor products of extremal subsets are not always extremal. For instance, in the rightmost diagram of Figure 5, we see that $B_{s_1}(1,1,0) \otimes B_e(1,1,0)$ is not extremal even though both factors are.

Example I. Consider $X = \{b_\lambda, f_2(b_\lambda), f_2^2(b_\lambda), f_1 f_2(b_\lambda)\} \subset B_{(2,2,0)}$, an extremal though not Demazure subset. As seen in Figure 4, $X \otimes X$ decomposes into connected components $Y_1 \oplus Y_2 \oplus Y_3 \subset B_{(4,4,0)} \oplus B_{(4,3,1)} \oplus B_{(4,2,2)}$ where neither Y_1 nor Y_2 are extremal subsets.

However, if the resulting tensor product of two subsets of crystals is itself extremal, this imposes some structure on the underlying subsets themselves.

Proposition 7. *If $X \otimes Y \subset \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is an extremal subset, then $\mathcal{E}(X) \subset X \sqcup \{0\}$. Furthermore, if $\mathcal{E}(Y) \subset Y \sqcup \{0\}$, then $X \subset \mathcal{B}(\lambda)$ is an extremal subset.*

Example J. Let $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = (4,4,0)$, and consider $X = \{b_\lambda\}$ and $Y = \{b_\lambda, f_2(b_\lambda), f_2^2(b_\lambda)\}$ subsets of $B(4,4,0)$. Then $X \otimes Y \cong B_e(8,8,0) \oplus B_e(8,7,1) \oplus B_e(8,6,2)$ is a sum of Demazure subsets, though Y is not extremal (but it is closed under $\mathcal{E}$).

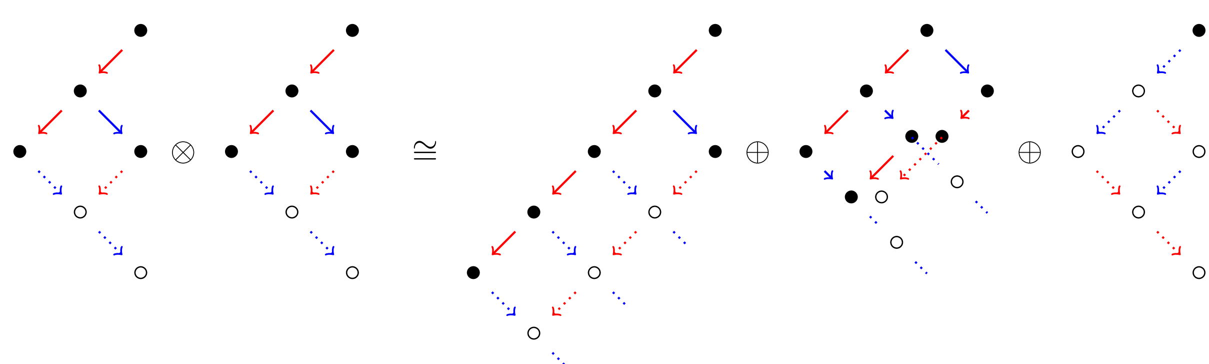

Figure 4: The summands of the extremal but not Demazure subgraph of $B_{(2,2,0)} \otimes B_{(2,2,0)} \cong B_{(4,4,0)} \oplus B_{(4,3,1)} \oplus B_{(4,2,2)}$ with f_1 and f_2 depicted by blue and red arrows.

4 Characterization of extremal tensor products

Determining when the tensor product of extremal subsets remains extremal depends solely on the following elements.

Definition 8. For $i \in I$, an element $x \otimes y \in \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is called an *i-hinge* if $e_i(x \otimes y)$ and $f_i(x \otimes y)$ are both nonzero with $e_i(x \otimes y) = e_i(x) \otimes y$ and $f_i(x \otimes y) = x \otimes f_i(y)$.

We say $x \otimes y \in X \otimes Y \subset \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is a *broken i-hinge* if $f_i(y) \notin Y$.

Example K. Consider $B(1,1,0) = a_1 \rightarrow a_2 \rightarrow a_3$. Then the element $a_2 \otimes a_1 \in B(1,1,0) \otimes B(1,1,0)$ (seen in the leftmost diagram of Figure 5) is a 2-hinge since $\varepsilon_2(a_2) = 1$ and $\varphi_2(a_2) = 0$ but $\varepsilon_2(a_1) = 0$ with $\varphi_2(a_1) = 1$. In particular, the subset $B_{s_2}(1,1,0) \otimes B_e(1,1,0)$ (rightmost in Figure 5) contains a broken 2-hinge since $f_2(a_1) \notin B_e(1,1,0)$.

Theorem 9. *Let $X \subset \mathcal{B}(\lambda)$ and $Y \subset \mathcal{B}(\mu)$ be extremal subsets. Then $X \otimes Y$ is an extremal subset of $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ if and only if $X \otimes Y$ contains no broken i-hinge for any $i \in I$.*

In particular, if $X = \{b_\lambda\} \subset \mathcal{B}(\lambda)$ has only the highest weight element or if $Y = \mathcal{B}(\mu)$ contains all possible elements, then $X \otimes Y$ contains no i-hinges for any i. Thus both $\{b_\lambda\} \otimes \mathcal{B}_u(\mu)$ and $\mathcal{B}_w(\lambda) \otimes \mathcal{B}(\mu)$ are extremal subsets of $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$.

Recall every Demazure subset is extremal, though the converse is false.

Any subset $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu) \subset \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ which is a direct sum of Demazure crystals is also an extremal subset. Amazingly, the converse of this statement is also true.

Theorem 10. *For $\lambda, \mu \in P^+$ and $w, u \in W$, we have $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is an extremal subset of $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ if and only if $\lfloor w \rfloor^\lambda \in W_{\lceil u \rceil^\mu}$.*

Combining this with Theorem 6, we derive the following result.

Corollary 11. *For $\lambda, \mu \in P^+$ and $w, u \in W$, the tensor product $\mathcal{B}_v(\lambda) \otimes \mathcal{B}_w(\mu)$ is a direct sum of extremal subsets if and only if $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is a direct sum of Demazure crystals.*

Thus, $\mathcal{B}_v(\lambda) \otimes \mathcal{B}_w(\mu)$ is a sum of Demazure crystals precisely when it doesn't contain a broken i-hinge for any $i \in I$. Hence Corollary 11 gives a local characterization of tensor products of Demazure crystals that does not rely on the values of λ, μ, w, u.

Example L. Take $\lambda = (1,1,0)$ and $w = e$ and $v = s_2$ as in Figure 5. Then $\lfloor w \rfloor^\lambda = e$ and $\lceil v \rceil^\lambda = s_2 s_1$, thus $\lfloor w \rfloor^\lambda \in W_{\lceil v \rceil^\lambda}$ so $B_e(1,1,0) \otimes B_{s_2}(1,1,0)$ is extremal. Conversely, $\lfloor v \rfloor^\lambda = s_2$ and $\lceil w \rceil^\lambda = s_1$ so $\lfloor v \rfloor^\lambda \notin W_{\lceil w \rceil^\lambda}$ and thus $B_{s_2}(1,1,0) \otimes B_e(1,1,0)$ is not extremal.

It is important to note that Corollary 11 is false if we replace $\mathcal{B}_v(\lambda)$ and $\mathcal{B}_w(\mu)$ with arbitrary extremal subsets. This can seen in Figure 4, where $X = Y$ is extremal and non-Demazure but the tensor product is neither.

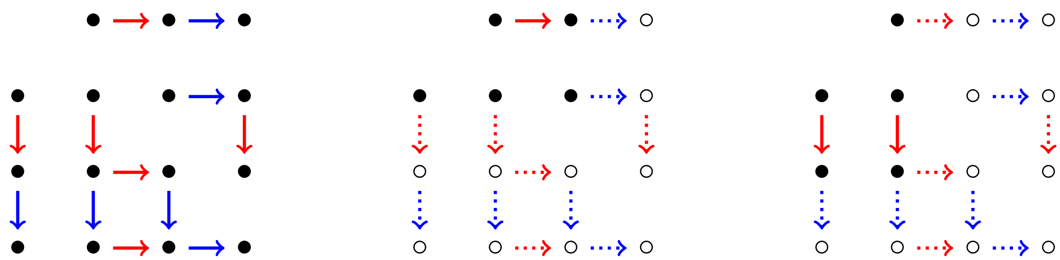

Figure 5: The tensor products $B(1,1,0) \otimes B(1,1,0)$ (left), $B_e(1,1,0) \otimes B_{s_2}(1,1,0)$ (middle), and $B_{s_2}(1,1,0) \otimes B_e(1,1,0)$ (right) with f_1 and f_2 depicted by blue and red arrows.

5 Application to tensor squares

Even when $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is not a direct sum of Demazure crystals, some connected components of it may be. For instance, in Example M and Figure 6, $B_{s_2 s_1}(2,1,0)^{\otimes 2}$ decomposes into four connected components, two of which are Demazure and two of which are not even extremal. In particular, the component of weight $(4,2,0)$ is a Demazure crystal. Using Corollary 11, we show that the highest weight component is always Demazure.

Theorem 12. *For $\lambda \in P^+$ and $w \in W$, the m-fold tensor product*

$$\mathcal{F}\left(\{b_\lambda \otimes \cdots \otimes b_\lambda\}\right) \cap \mathcal{B}_w(\lambda) \otimes \cdots \otimes \mathcal{B}_w(\lambda) \subset \mathcal{B}(\lambda) \otimes \cdots \otimes \mathcal{B}(\lambda)$$

is isomorphic to $\mathcal{B}_w(m\lambda)$. In particular, it is a Demazure crystal.

Example M. Let $\mathfrak{g} = \mathfrak{sl}_3$ and consider $B(2,1,0) \otimes B(2,1,0) \cong B(4,2,0) \oplus B(3,3,0) \oplus B(4,1,1) \oplus B(3,2,1)^{\oplus 2} \oplus B(2,2,2)$. The subset $B_{s_2 s_1}(2,1,0) \otimes B_{s_2 s_1}(2,1,0)$ decomposes into four connected components; see Figure 6. Only the components with highest weights $(4,2,0)$ and $(3,3,0)$ are Demazure. The components with highest weights $(3,2,1)$ and $(4,1,1)$ are not even extremal. The remaining two highest weights do not appear.

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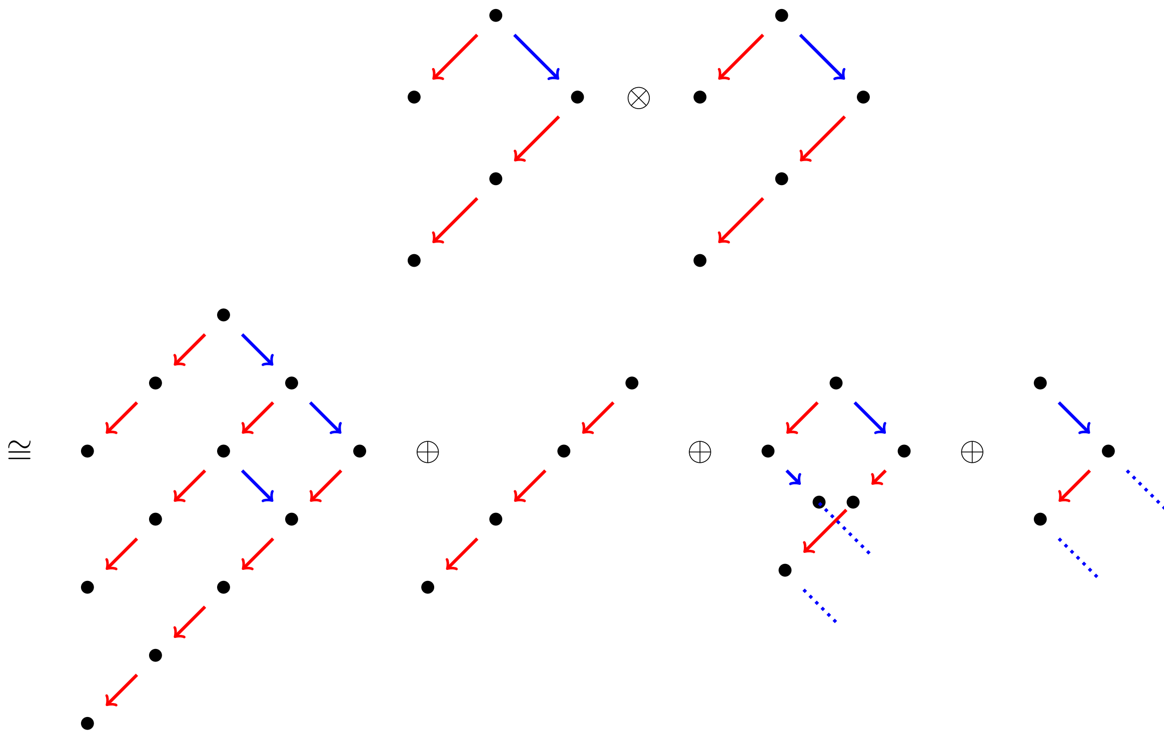

Figure 6: The tensor product of $B_{s_2 s_1}(2,1,0) \otimes B_{s_2 s_1}(2,1,0)$ decomposed into connected components, some of which are Demazure and some of which are not extremal.

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