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Invariant theory for the free left-regular band and a q-analogue

Sarah Brauner*[1], Patricia Commins†[1], and Victor Reiner‡[1]

[1] *School of Mathematics, University of Minnesota, Minneapolis, MN*

Abstract. We examine from an invariant theory viewpoint the monoid algebras for two monoids having large symmetry groups. The first monoid is the free left-regular band on n letters, defined on the set of all injective words, that is, the words with at most one occurrence of each letter. This monoid carries the action of the symmetric group. The second monoid is one of its q-analogues, considered by K. Brown, carrying an action of the finite general linear group. In both cases, we show that the invariant subalgebras are semisimple commutative algebras, and characterize them using *Stirling* and *q-Stirling numbers*.

We then use results from the theory of random walks and random-to-top shuffling to decompose the entire monoid algebra into irreducibles, simultaneously as a module over the invariant ring and as a group representation. Our irreducible decompositions are described in terms of *derangement symmetric functions* introduced by Désarménien and Wachs.

Keywords: left-regular band, random-to-top, Stirling number, symmetric group, general linear group, unipotent character.

To the memory of Georgia Benkart.

1 Introduction

Motivated by results on mixing times for shuffling algorithms on permutations, Bidigare [2] and Bidigare, Hanlon and Rockmore [1] developed a complete spectral analysis for a class of random walks on the chambers of a hyperplane arrangement. Their work relied heavily on the *Tits semigroup* structure on the cones of the arrangement. Later, Brown [4] generalized their analysis to random walks coming from semigroups $\mathcal{L}$ which form a *left-regular band (LRB)*, meaning that $x^2 = x$ and $xyx = xy$ for all x, y in $\mathcal{L}$. Left-regular bands have since been studied by many others; see Margolis, Saliola, and Steinberg [10] for extensive work along with a historical discussion in their Chapter 1.

*braun622@umn.edu. Supported by NSF Graduate Research Fellowship
†commi010@umn.edu. Supported by NSF Graduate Research Fellowship
‡reiner@umn.edu. Supported by NSF grant DMS-2053288

Some left-regular bands come equipped with natural symmetry groups. In his PhD thesis [2], Bidigare studied the Tits semigroup algebra of a reflection arrangement under the action of its corresponding reflection group. He discovered that the invariant subalgebra is anti-isomorphic to a well-studied algebra: Solomon's descent algebra. The shuffling operators contained in the invariant subalgebra are random walks on the reflection group (see [4, Theorem 8]) and include interesting examples such as random-to-top shuffling and inverse riffle shuffling. The close relationship between the Tits semigroup algebra and the descent algebra has proved to be useful beyond shuffling; for example, Saliola used this viewpoint of the descent algebra in his computation of its quiver [16].

This abstract is based on [3], where we study two examples of left-regular bands M, related to those discussed by Brown, with large groups of monoid automorphisms G:

- the *free LRB on n letters* [4, §1.3], denoted $\mathcal{F}_n$, with G the symmetric group $\mathfrak{S}_n$, and
- a q-analogue $\mathcal{F}_n^{(q)}$ related to monoids in [4], and G the general linear group $GL_n(\mathbb{F}_q)$.

Inspired by Bidigare's work, we study these left-regular bands under the action of their symmetry groups. In particular, for both monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, we examine the *monoid algebra* $R := \mathbf{k}M$ with coefficients in a commutative ring $\mathbf{k}$ with 1, and answer the two *main questions of invariant theory* for G acting on R:

Question 1.1. *What is the structure of the invariant subalgebra R^G?*

Question 1.2. *What is the structure of R, simultaneously as an R^G-module and a G-representation?*

Notably, both questions are answered for the two monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$ in parallel.

We answer Question 1.1 by showing that (when $q \in \mathbf{k}^\times$) the invariant subalgebra R^G for both monoids is semisimple, commutative, and generated by a single element. This generator acts semisimply on R with eigenvalues in $0, 1, \cdots, n$ for $M = \mathcal{F}_n$ and $[0]_q, [1]_q, \cdots, [n]_q$ for $M = \mathcal{F}_n^{(q)}$. Our analysis uses the combinatorics of Stirling and q-Stirling numbers.

Our answer to Question 1.2 involves decomposing the eigenspaces of the generator of R^G on R as G-representations (for $\mathbf{k}$ a field in which $|G|$ is invertible). We do so by (i) introducing and studying filtrations on R and (ii) inductively constructing eigenvectors. We describe these eigenspaces in terms of *derangement symmetric functions* first introduced by Désarménien and Wachs [5]. Derangement symmetric functions have connections to many well-studied objects in combinatorics such as the complex of injective words [13], random-to-top and random-to-random shuffling [21], higher Lie characters [21], and configuration spaces [9]; see [3, §3.3] for historical details. We add to this list by showing they form crucial building blocks for the invariant theory of $\mathbf{k}\mathcal{F}_n$ and $\mathbf{k}\mathcal{F}_n^{(q)}$.

1.1 Outline

In Section 2 we define the two monoids of interest and their respective symmetry groups. In Section 3 we determine the structure of the invariant subalgebras for both monoid algebras, answering Question 1.1 with Theorem 3.4. In order to present our answer to Question 1.2 for both monoids in parallel, we use the language of symmetric functions. Accordingly, Section 4 reviews the famous Frobenius characteristic map and its lesser-known q-analogue. We also give background on the *derangement symmetric function* here. Finally, in Section 5 we answer Question 1.2 with Theorem 5.1.

2 The left-regular bands and their symmetry groups

Definition 2.1. The *free left-regular band (or LRB)* on n letters $\mathcal{F}_n$ (see [4, §1.3], [18, §14.3.1]) consists, as a set, of all words $\mathbf{a} = (a_1, a_2, \ldots, a_\ell)$ with letters a_i from $\{1, 2, \ldots, n\}$ and no repeated letters, that is, $a_i \neq a_j$ for $1 \leq i < j \leq n$. The set $\mathcal{F}_n$ becomes a semigroup under the following operation: if $\mathbf{b} = (b_1, \ldots, b_m)$ is another word in $\mathcal{F}_n$, then their product is

$$\mathbf{a} \cdot \mathbf{b} := (a_1, \ldots, a_\ell, b_1, \ldots, b_m)^\wedge,$$

where we adopt the notation from Brown [4] that for a sequence $\mathbf{c} = (c_1, \ldots, c_p)$, the subsequence $\mathbf{c}^\wedge = (c_1, \ldots, c_p)^\wedge$ is obtained by removing any letter c_i that appears already in the prefix $(c_1, c_2, \ldots, c_{i-1})$.

Example 2.2. *In $\mathcal{F}_5$, one has $(2, 5, 1) \cdot (4, 2, 3, 1) = (2, 5, 1, 4, 3)$.*

One can check that the empty word $()$ is an identity element for this operation, and hence $\mathcal{F}_n$ is not only a semigroup, but a *monoid*. For any word $\mathbf{a} \in \mathcal{F}$, the *length* $\ell(\mathbf{a}) := \ell$ lies anywhere in the range $0 \leq \ell \leq n$. There is a left action on $\mathcal{F}_n$ by $\mathfrak{S}_n$, the symmetric group on n letters, where $w \in \mathfrak{S}_n$ acts by $w(a_1, \ldots, a_\ell) = (w(a_1), \ldots, w(a_\ell))$.

The q-analogue of $\mathcal{F}_n$ that we will consider will be denoted $\mathcal{F}_n^{(q)}$, defined as follows.

Definition 2.3. As a set, $\mathcal{F}_n^{(q)}$ consists of all partial flags of subspaces $\mathbf{A} = (A_1, \ldots, A_\ell)$ where A_i is an i-dimensional $\mathbb{F}_q$-linear subspace of $(\mathbb{F}_q)^n$, and $A_1 \subset A_2 \subset \cdots \subset A_\ell$. Again the *length* $\ell(\mathbf{A}) := \ell$ lies in the range $0 \leq \ell \leq n$. We multiply $\mathbf{A}$ with another flag $\mathbf{B} = (B_1, \ldots, B_m)$ in $\mathcal{F}_n^{(q)}$ by:

$$\mathbf{A} \cdot \mathbf{B} := (A_1, \ldots, A_\ell, A_\ell + B_1, A_\ell + B_2, \ldots, A_\ell + B_m)^\wedge.$$

As before, for a sequence $\mathbf{C} = (C_1, \ldots, C_p)$ of nested subspaces $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_p$, the subsequence $\mathbf{C}^\wedge$ is obtained by removing any C_i appearing in the prefix $(C_1, \ldots, C_{i-1})$.

As in the $q = 1$ case, $\mathcal{F}_n^{(q)}$ is not only a semigroup but a *monoid* since the empty flag $()$ is an identity element[1]. Our symmetry group in this case is $GL_n := GL_n(\mathbb{F}_q)$, and $g \in GL_n$ acts on the left by $g(A_1, \ldots, A_\ell) = (g(A_1), \ldots, g(A_\ell))$.

Let $\mathbf{k}$ be a commutative ring with 1. For any finite monoid M (such as $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$), the *monoid algebra* $R = \mathbf{k}M$ is the free $\mathbf{k}$-module with basis elements given by the elements $\mathbf{a}$ of M, and multiplication extended $\mathbf{k}$-linearly. A group G of monoid automorphisms of M (such as $G = \mathfrak{S}_n, GL_n$) acts as ring automorphisms on $\mathbf{k}M$.

Following Brown [4], we use *chambers* to refer to words and flags of maximum length in $\mathcal{F}_n$ and $\mathcal{F}_n^{(q)}$. In $\mathbf{k}\mathcal{F}_n$, the chamber subspace is $\mathbf{k}\mathfrak{S}_n$. In $\mathbf{k}\mathcal{F}_n^{(q)}$, the chamber subspace is spanned by complete flags of $(\mathbb{F}_q)^n$, equivalent to the coset space $\mathbf{k}[GL_n/B]$.

3 Answer to Question 1.1

Our first goal is to answer Question 1.1 by describing the invariant rings R^G. Importantly, since the groups G permute the monoid elements M, the monoid algebra $R = \mathbf{k}M$ becomes a permutation representation of G. Therefore the invariant subalgebra R^G has as a $\mathbf{k}$-basis the orbit sums $\{\sum_{\mathbf{a} \in \mathcal{O}} \mathbf{a}\}$ as one runs through all G-orbits $\mathcal{O}$ on M.

For both monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, one can easily identify the G-orbits as elements of a fixed length. In particular, the G-invariant subalgebras R^G have $\mathbf{k}$-bases $\{x_\ell\}_{\ell=0,1,\ldots,n}$ and $\{x_\ell^{(q)}\}_{\ell=0,1,\ldots,n}$, defined by

$$x_\ell := \sum_{\substack{\mathbf{a} \in \mathcal{F}_n \\ \ell(\mathbf{a}) = \ell}} \mathbf{a}, \qquad x_\ell^{(q)} := \sum_{\substack{\mathbf{A} \in \mathcal{F}_n^{(q)} \\ \ell(\mathbf{A}) = \ell}} \mathbf{A}.$$

Example 3.1. *Let $q = 2$, $n = 3$, $\ell = 1$, and let e_1, e_2, e_3 be standard basis vectors for $V = (\mathbb{F}_2)^3$. Using the notation $\langle v_1, v_2, \ldots, v_m \rangle$ for the $\mathbb{F}_q$-span of the vectors $\{v_1, v_2, \ldots, v_m\}$ in V, one has*

$$x_1^{(2)} = (\langle e_1 \rangle) + (\langle e_2 \rangle) + (\langle e_3 \rangle) + (\langle e_1 + e_2 \rangle) + (\langle e_1 + e_3 \rangle) + (\langle e_2 + e_3 \rangle) + (\langle e_1 + e_2 + e_3 \rangle).$$

We will show that both invariant subalgebras R^G have another natural basis consisting of *powers* of $x_1, x_1^{(q)}$ when $q \in \mathbf{k}^\times$. Henceforth, set $x := x_1$ and $x^{(q)} := x_1^{(q)}$.

Our proof that the powers of x and $x_1^{(q)}$ form a basis of R^G involves Stirling numbers and one of their q-analogues[2]. Recall the standard q-analogue of $n \in \mathbb{Z}_{\geq 0}$:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

[1] The authors thank an anonymous referee for pointing out that both $\mathcal{F}_n, \mathcal{F}_n^{(q)}$ are instances of the (right-) *free Rhodes expansion* of the lattice semigroups for the Boolean algebra and finite vector space lattices, using the rank one elements as generators; see [12, §4].

[2] See [15, §1.1, 1.2] for a history of two standard q-analogues, one being the $S_q(n,k)$ in Definition 3.2.

Definition 3.2. *Define the classical Stirling numbers $S(n,k)$ and a q-analogue $S_q(n,k)$ recursively as follows. When $n=k=0$, let $S(n,k)=S_q(n,k)=1$. When $n+k\geq 1$ with $n=0$ or $k=0$, set $S(n,k)=S_q(n,k)=0$. For n and k both at least 1, define*

$$\begin{array}{rcccl} S(n,k) & = & S(n-1,k-1) & + & k\cdot S(n-1,k), \\ S_q(n,k) & = & q^{k-1}\cdot S_q(n-1,k-1) & + & [k]_q\cdot S_q(n-1,k). \end{array} \tag{3.1}$$

The recurrences in Equation (3.1) lead to a change of basis expression in R^G.

Lemma 3.3. *Let $\mathbf{k}$ be a commutative ring with 1, and let $R=\mathbf{k}M$ with $M=\mathcal{F}_n$ or $\mathcal{F}_n^{(q)}$. Then the (q-)Stirling numbers $S(m,k), S_q(m,k)$ are the expansion coefficients for the powers $\{x^m\}_{m=0,1,\ldots,n}$ and $\{(x^{(q)})^m\}_{m=0,1,\ldots,n}$ in the $\mathbf{k}$-bases $\{x_k\}_{k=0,1,\ldots,n}$, $\{x_k^{(q)}\}_{k=0,1,\ldots,n}$ of R^G:*

$$x^m = \sum_k S(m,k)\, x_k,$$
$$(x^{(q)})^m = \sum_k S_q(m,k)\, x_k^{(q)}.$$

Thus unitriangularity of $\{S(m,k)\}$ shows $\{x^k\}_{k=0,1,\ldots,n}$ always gives a $\mathbf{k}$-basis for R^G, while triangularity of $\{S_q(m,k)\}$ shows $\{(x^{(q)})^k\}_{k=0,1,\ldots,n}$ are a $\mathbf{k}$-basis for R^G if and only if $q\in\mathbf{k}^\times$.

Proof Idea. The lemma follows from induction on the Stirling recurrences (3.1) after proving the following multiplication rules in R^G: for $\ell=0,1,\ldots,n$,

$$x\cdot x_\ell = \ell x_\ell + x_{\ell+1},$$
$$x^{(q)}\cdot x_\ell^{(q)} = [\ell]_q\, x_\ell^{(q)} + q^\ell\, x_{\ell+1}^{(q)}.$$

Note that these rules allow one to solve for each x_ℓ or $x_\ell^{(q)}$ in terms of powers of x or $x^{(q)}$ inductively. The Stirling numbers just happen to be the change-of-basis coefficients. $\square$

We use Lemma 3.3, as well as properties of minimal polynomials, to describe the structure of the invariant subalgebras, answering Question 1.1.

Theorem 3.4. *Let $\mathbf{k}$ be any commutative ring with 1, and $R=\mathbf{k}M$ for either of the monoids $M=\mathcal{F}_n,\mathcal{F}_n^{(q)}$, with symmetry groups $G=\mathfrak{S}_n, GL_n$. If $M=\mathcal{F}_n^{(q)}$, assume that q is in $\mathbf{k}^\times$.*

(i) The unique $\mathbf{k}$-algebra map $\mathbf{k}[X]\longrightarrow R$ defined by

$$X\longmapsto\begin{cases} x & \text{if } M=\mathcal{F}_n,\\ x^{(q)} & \text{if } M=\mathcal{F}_n^{(q)},\end{cases}$$

induces an algebra isomorphism $\mathbf{k}[X]/(f(X))\cong R^G$ where

$$f(X):=\begin{cases} X(X-1)(X-2)\cdots(X-n) & \text{if } M=\mathcal{F}_n,\\ X(X-[1]_q)(X-[2]_q)\cdots(X-[n]_q) & \text{if } M=\mathcal{F}_n^{(q)}.\end{cases}$$

Hence R^G is commutative and generated by x or $x^{(q)}$.

(ii) If $\mathbf{k}$ is a field where $|G|$ is invertible, then x or $x^{(q)}$ acts semisimply on any finite-dimensional R^G-module, with eigenvalues contained in the lists

$$\begin{cases} 0, 1, 2, \ldots, n & \text{if } M = \mathcal{F}_n, \\ [0]_q, [1]_q, [2]_q, \ldots, [n]_q & \text{if } M = \mathcal{F}_n^{(q)}. \end{cases}$$

An old, but interesting, observation is that multiplication by x acts on the chamber subspace of $\mathbf{k}\mathcal{F}_n$ as a (rescaled) version of the random-to-top operator, see for instance B. Steinberg [18, Prop. 14.5]. This observation will be used in the proof of Theorem 5.1.

Example 3.5. *If $n = 4$ and $w = (3, 1, 4, 2)$, then*

$$\begin{aligned} x \cdot w &= ((1) + (2) + (3) + (4)) \cdot (3, 1, 4, 2) \\ &= (1, 3, 4, 2) + (2, 3, 1, 4) + (3, 1, 4, 2) + (4, 3, 1, 2) \end{aligned}$$

which (after scaling by $\frac{1}{4}$) is the result of random-to-top shuffling on w as an element of $\mathbf{k}\mathfrak{S}_4$.

Remark 3.6. In unpublished notes, Garsia [7] (see also Tian [20]), studies the *top-to-random* shuffling operator, which is adjoint or transpose to the random-to-top operator. There he sketches a proof that its minimal polynomial is $X(X-1)(X-2)\cdots(X-n)$. In light of the fact that an operator and its transpose have the same minimal polynomial, Garsia's sketch is closely related to the part of our proof of Theorem 3.4 dealing with $M = \mathbf{k}\mathcal{F}_n$.

4 Symmetric function background

4.1 Symmetric functions, $\mathfrak{S}_n$- and unipotent GL_n-representations

We review here the relation between the *ring of symmetric functions* Λ and representations of $\mathfrak{S}_n$; see Sagan [14], Stanley [17] as references, and for undefined terminology. We then review the parallel story for R. Steinberg's *unipotent representations* of GL_n; see [8, §4.2, 4.6, 4.7] as a reference.

The *ring of symmetric functions* Λ (of bounded degree, in infinitely many variables) may be viewed as a polynomial algebra $\mathbb{Z}[h_1, h_2, \ldots] = \mathbb{Z}[e_1, e_2, \ldots]$ where h_n, e_n are the *complete homogeneous* and *elementary* symmetric functions of degree n. One may view Λ as a graded $\mathbb{Z}$-algebra $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda^n$, which we wish to relate to the direct sum

$$\mathcal{C}(\mathfrak{S}) := \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathfrak{S}_n),$$

where $\mathcal{C}(\mathfrak{S}_n)$ denotes the $\mathbb{Z}$-module of *virtual characters* of $\mathfrak{S}_n$. That is, $\mathcal{C}(\mathfrak{S}_n)$ is the free $\mathbb{Z}$-module on the basis of irreducible characters $\{\chi^\lambda\}$ indexed by the partitions λ of n, or alternatively, the $\mathbb{Z}$-submodule of class functions on $\mathfrak{S}_n$ of the form $\chi - \chi'$ for genuine characters χ, χ'. One makes $\mathcal{C}(\mathfrak{S})$ a graded algebra via the *induction product* defined by

$$\begin{array}{rcl} \mathcal{C}(\mathfrak{S}_{n_1}) \times \mathcal{C}(\mathfrak{S}_{n_2}) & \longrightarrow & \mathcal{C}(\mathfrak{S}_{n_1+n_2}) \\ (f_1, f_2) & \longmapsto & f_1 * f_2 := (f_1 \otimes f_2) \uparrow_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}}^{\mathfrak{S}_{n_1+n_2}} \end{array}$$

where $(-) \uparrow_H^G$ is the usual *induction* of class functions on a subgroup H to class functions on G. One then has the *Frobenius characteristic isomorphism* of $\mathbb{Z}$-algebras $\mathcal{C}(\mathfrak{S}) \xrightarrow{\mathrm{ch}} \Lambda$, mapping

$$\begin{array}{rcl} \mathcal{C}(\mathfrak{S}) & \xrightarrow{\mathrm{ch}} & \Lambda \\ \mathbf{1}_{\mathfrak{S}_n} & \longmapsto & h_n, \\ \chi^\lambda & \longmapsto & s_\lambda. \end{array}$$

Here s_λ is the *Schur function*. For a composition $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_\ell$, we use the standard shorthand $h_\alpha := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_\ell}$.

There is a parallel story for a certain subset of GL_n-representations. Specifically, there is a collection of irreducible GL_n-representations $\{\chi_q^\lambda\}$, indexed by partitions λ of n, which are the irreducible constituents occurring within the GL_n-permutation action on the set GL_n/B of *complete flags of subspaces* in $V = (\mathbb{F}_q)^n$. They were studied by R. Steinberg [19], and are now called the *unipotent characters* of GL_n. Denoting by $\mathcal{C}(GL_n)$ the free $\mathbb{Z}$-submodule of the class functions on GL_n with unipotent characters $\{\chi_q^\lambda\}$ as a basis, one can define the *parabolic* or *Harish-Chandra induction* product on the direct sum $\mathcal{C}(GL) := \bigoplus_{n=0}^\infty \mathcal{C}(GL_n)$ as follows:

$$\begin{array}{rcl} \mathcal{C}(GL_{n_1}) \times \mathcal{C}(GL_{n_2}) & \longrightarrow & \mathcal{C}(GL_{n_1+n_2}) \\ (f_1, f_2) & \longmapsto & f_1 * f_2 := \left((f_1 \otimes f_2) \Uparrow_{GLn_1 \times GL_{n_2}}^{P_{n_1,n_2}} \right) \uparrow_{P_{n_1,n_2}}^{GLn_1+n_2}. \end{array}$$

Here P_{n_1,n_2} is the maximal parabolic subgroup of $GL_{n_1+n_2}$ setwise stabilizing the $\mathbb{F}_q$-span of the first n_1 standard basis vectors, and $(-) \Uparrow_{GLn_1 \times GL_{n_2}}^{P_{n_1,n_2}}$ is the *inflation* operation that creates a $GLn_1 \times GL_{n_2}$-representation from a P_{n_1,n_2}-representation, by precomposing with the surjective homomorphism $P_{n_1,n_2} \twoheadrightarrow GL_{n_1} \times GL_{n_2}$ sending $\left[\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right] \longmapsto \left[\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix}\right]$.

This parabolic induction turns out to make $\mathcal{C}(GL)$ into an associative, commutative $\mathbb{Z}$-algebra. One then has a q-analogue of the Frobenius isomorphism $\mathcal{C}(GL) \xrightarrow{\mathrm{ch}_q} \Lambda$ that sends $\mathbf{1}_{GL_n} \longmapsto h_n$ and $\chi_q^\lambda \longmapsto s_\lambda$. For example, the chamber subspaces $\mathbf{k}\mathfrak{S}_n$ and $\mathbf{k}[GL_n/B]$ of $\mathbf{k}\mathcal{F}_n$ and $\mathbf{k}\mathcal{F}_n^{(q)}$ carry the same (q)-Frobenius characteristic image h_{1^n}.

4.2 (q-)derangement numbers and representations

A central role in this story is played by the classical *derangement numbers* d_n, and the *q-derangement numbers* $d_n(q)$ of Wachs [22]:

$$\begin{aligned} d_n &:= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = n! \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^n}{n!} \right), \\ d_n(q) &:= [n]!_q \sum_{k=0}^{n} \frac{(-1)^k}{[k]!_q}. \end{aligned}$$

There are two well-known combinatorial models for d_n counting permutations in $\mathfrak{S}_n$:

- *derangements*, which are the fixed-point free permutations, or
- *desarrangements*, which are permutations $w = (w_1, w_2, \ldots, w_n)$ whose first ascent position i with $w_i < w_{i+1}$ (using $w_{n+1} = n+1$) occurs for an even position i.

Wachs [22] and later Désarménien and Wachs [6] gave various interpretations for $d_n(q)$. In particular, $d_n(q)$ is still closely related to derangements and desarrangements. Letting D_n, E_n denote the derangements and desarrangements in S_n, and defining the *major index* statistic of a permutation $w = (w_1, \ldots, w_n)$ as $\mathrm{maj}(\sigma) = \sum_{i : w_i > w_{i+1}} i$, one has

$$d_n(q) = \sum_{\sigma \in D_n} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in E_n} q^{\mathrm{maj}(\sigma^{-1})}.$$

These $d_n, d_n(q)$ are the dimensions for a pair of representations of $\mathfrak{S}_n$ and GL_n, which we call the *derangement representation* $\mathcal{D}_n$ and its (unipotent) q-analogue $\mathcal{D}_n^{(q)}$. Both have the same symmetric function image $\mathfrak{d}_n$ under the Frobenius maps ch and ch_q. The symmetric function $\mathfrak{d}_n$ was originally introduced by Désarménien and Wachs [5] and has many equivalent descriptions (see [3, Proposition 3.1]). Here, we define $\mathfrak{d}_n$ by its decomposition into Schur functions; this description is due to Reiner and Webb [13]. To do so, we must define desarrangement tableaux.

A standard Young tableau Q with n cells written in English notation, has *descent set*

$$\mathrm{Des}(w) := \{ i \in \{1, 2, \ldots, n-1\} : i+1 \text{ appears south and weakly west of } i \text{ in } Q \}.$$

For example, $Q = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$ has $\mathrm{Des}(Q) = \{1, 3, 4\}$. A *desarrangement tableau* is a standard Young tableau Q with n cells for which the smallest element of $\{1, 2, \ldots, n\} \setminus \mathrm{Des}(Q)$ is even. Thus the example tableau Q given above is a desarrangement tableau.

Definition 4.1. $\mathfrak{d}_n = \sum_Q s_{\lambda(Q)}$ *where Q runs through the desarrangement tableaux of size n.*

Example 4.2. *We compute $\mathfrak{d}_n$ for $0 \leq n \leq 4$.*

n	*desarrangement tableaux* Q	$\mathfrak{d}_n$
0	$\varnothing$	1
1	*(none)*	0
2	$\begin{ytableau} 1 \\ 2 \end{ytableau}$	$s_{(1,1)}$
3	$\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$	$s_{(2,1)}$
4	$\begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau}$ $\begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}$ $\begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau}$ $\begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}$	$s_{(1,1,1,1)} + s_{(2,1,1)} + s_{(2,2)} + s_{(3,1)}$

5 Answer to Question 1.2

We are now prepared to answer Question 1.2 by examining the monoid algebras $R = \mathbf{k}M$ as modules over both the symmetry groups $\mathbf{k}G$ and their invariant subalgebras R^G. Since R carries commuting actions of R^G and of $\mathbf{k}G$, we will describe R simultaneously as a module over both. Henceforth, assume that $\mathbf{k}$ is a field in which $|G|$ is invertible.

We will utilize two important features of our setting. First, since $\mathbf{k}M$ is finite-dimensional over $\mathbf{k}$, it is semisimple both as an R^G-module from Theorem 3.4(ii), and as a $\mathbf{k}G$-module by Maschke's Theorem. Second, by Theorem 3.4(ii), we have that R^G is generated by the single element x or $x^{(q)}$, which acts diagonalizably with certain eigenvalues λ all lying in $\mathbf{k}$. It follows that in order to understand the simultaneous R^G- and $\mathbf{k}G$-module structure of any module V, it suffices to decompose the eigenspaces $(\ker((x-\lambda)|_V)$ or $\ker((x^{(q)}-\lambda)|_V))$ as G-representations.

Hence, we are able to answer Question 1.2 for $j = 0, 1, \ldots n$

- when $M = \mathcal{F}_n$ by describing $\ker\left((x-j)|_{\mathbf{k}\mathcal{F}_n}\right)$, the j-eigenspace of x acting on $\mathbf{k}\mathcal{F}_n$, as an $\mathfrak{S}_n$-representation, and
- when $M = \mathcal{F}_n^{(q)}$ by describing $\ker\left((x^{(q)}-[j]_q)|_{\mathbf{k}\mathcal{F}_n^{(q)}}\right)$, the $[j]_q$-eigenspace of $x^{(q)}$ acting on $\mathbf{k}\mathcal{F}_n^{(q)}$, as a GL_n- representation.

Theorem 5.1. *Let $\mathbf{k}$ be a field in which $|G|$ is invertible. Then $x, x^{(q)}$ act diagonalizably on $\mathbf{k}\mathcal{F}_n, \mathbf{k}\mathcal{F}_n^{(q)}$, and for each $j = 0, 1, 2, \ldots, n$, their eigenspaces carry G-representations with the same Frobenius map images*

$$\operatorname{ch}\ker\left((x-j)|_{\mathbf{k}\mathcal{F}_n}\right) = \sum_{\ell=j}^{n} h_{(n-\ell,j)} \cdot \mathfrak{d}_{\ell-j} = \operatorname{ch}_q \ker\left((x^{(q)}-[j]_q)|_{\mathbf{k}\mathcal{F}_n^{(q)}}\right).$$

In other words, one has G-representation isomorphisms

$$\ker\left((x-j)|_{\mathbf{k}\mathcal{F}_n}\right) \cong \bigoplus_{\ell=j}^{n} \mathbf{1}_{\mathfrak{S}_{n-\ell}} * \mathbf{1}_{\mathfrak{S}_j} * \mathcal{D}_{\ell-j},$$
$$\ker\left(\left(x^{(q)}-[j]_q\right)|_{\mathbf{k}\mathcal{F}_n^{(q)}}\right) \cong \bigoplus_{\ell=j}^{n} \mathbf{1}_{GL_{n-\ell}} * \mathbf{1}_{GL_j} * \mathcal{D}^{(q)}_{\ell-j}.$$

Proof Idea. The crucial idea is to introduce a filtration on $\mathbf{k}M$ by length,

$$0=\mathbf{k}M_{\geq n+1} \subset \mathbf{k}M_{\geq n} \subset \mathbf{k}M_{\geq n-1} \subset \cdots \subset \mathbf{k}M_{\geq 1} \subset \mathbf{k}M_{\geq 0}=\mathbf{k}M,$$

where $M_{\geq \ell}$ is the $\mathbf{k}$-span of words of length at least ℓ for $M=\mathcal{F}_n$ and flags of length at least ℓ for $M=\mathcal{F}_n^{(q)}$. By semisimplicity, there is an isomorphism of both R^G-modules and G-representations:

$$\mathbf{k}M \cong \bigoplus_{\ell=0}^{n} \mathbf{k}M_{\geq \ell}/\mathbf{k}M_{\geq \ell+1}. \tag{5.1}$$

Our approach is to study the eigenspaces of $x, x^{(q)}$ on each summand in Equation (5.1).

The bottom of the filtration (when $\ell=n$) is the chamber subspace of $M=\mathbf{k}\mathcal{F}_n, \mathbf{k}\mathcal{F}_n^{(q)}$, which we write as $\mathfrak{C}_n$, $\mathfrak{C}_n^{(q)}$, respectively. We prove that

$$\operatorname{ch}\ker\left((x-j)|_{\mathfrak{C}_n}\right) = h_j\mathfrak{d}_{n-j} = \operatorname{ch}_q \ker\left(\left(x^{(q)}-[j]_q\right)|_{\mathfrak{C}_n^{(q)}}\right). \tag{5.2}$$

The key ingredients in proving Equation (5.2) are (i) inductive constructions of explicit j-, $[j]_q$-eigenvectors[3] of $x|_{\mathfrak{C}_n}$, $x^{(q)}|_{\mathfrak{C}_n^{(q)}}$ from nullvectors of $x|_{\mathfrak{C}_{n-j}}$, $x^{(q)}|_{\mathfrak{C}_{n-j}^{(q)}}$, (ii) the dimensions of $\ker\left(x|_{\mathfrak{C}_n}\right)$, $\ker\left(x^{(q)}|_{\mathfrak{C}_n^{(q)}}\right)$ following from work of Phatarfod [11], Brown [4], and (iii) a recursive description of $\mathfrak{d}_n$ from [5]: $h_{1^n}=\sum_{j=0}^{n}\mathfrak{d}_j h_{n-j}$.

Finally, we address the remaining summands of Equation (5.1) by reinterpreting the action of x, $x^{(q)}$ on $\mathbf{k}M_{\geq \ell}/\mathbf{k}M_{\geq \ell+1}$ in terms of the eigenspaces of $\mathfrak{C}_\ell, \mathfrak{C}_\ell^{(q)}$ and using properties of induced representations. □

Example 5.2. *We illustrate Theorem 5.1 computing the Frobenius map image for each j-eigenspace of x on $\mathbf{k}\mathcal{F}_3$, or equivalently the q-Frobenius map image for each $[j]_q$-eigenspace of $x^{(q)}$ on $\mathbf{k}\mathcal{F}_3^{(q)}$. The table below shows these symmetric functions in the j^{th} row, decomposed into columns labeled by ℓ, which index the filtration factors from Equation* (5.1) *that contribute a term.*

[3]The third author is grateful to Michelle Wachs for explaining to him the constructions in the case that $M=\mathbf{k}\mathcal{F}_n$ in 2002, in the context of random-to-top shuffling.

Frobenius map images for eigenspaces of $x, x^{(q)}$ on $\mathbf{k}\mathcal{F}_3, \mathbf{k}\mathcal{F}_3^{(q)}$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$
$j = 0$	$h_3 \cdot \mathfrak{d}_0 = h_3 \cdot s_{()} = s_{(3)}$	$h_2 \cdot \mathfrak{d}_1 = h_2 \cdot 0 = 0$	$h_1 \cdot \mathfrak{d}_2 = h_1 \cdot s_{(1,1)} = s_{(2,1)} + s_{(1,1,1)}$	$h_0 \cdot \mathfrak{d}_3 = h_0 \cdot s_{(2,1)} = s_{(2,1)}$
$j = 1$		$h_{(2,1)} \cdot \mathfrak{d}_0 = h_{(2,1)} \cdot s_{()} = s_{(3)} + s_{(2,1)}$	$h_{(1,1)} \cdot \mathfrak{d}_1 = h_{(1,1)} \cdot 0 = 0$	$h_1 \cdot \mathfrak{d}_2 = h_1 \cdot s_{(1,1)} = s_{(2,1)} + s_{(1,1,1)}$
$j = 2$			$h_{(2,1)} \cdot \mathfrak{d}_0 = h_{(2,1)} \cdot s_{()} = s_{(3)} + s_{(2,1)}$	$h_2 \cdot \mathfrak{d}_1 = h_2 \cdot 0 = 0$
$j = 3$				$h_3 \cdot \mathfrak{d}_0 = h_3 \cdot s_{()} = s_{(3)}$

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