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Rowmotion on 321-avoiding permutations

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Abstract. We give a natural definition of rowmotion for 321-avoiding permutations, by translating, through bijections involving Dyck paths and the Lalanne–Kreweras involution, the analogous notion for antichains of the positive root poset of type A. We prove that some permutation statistics, such as the number of fixed points, are homomesic under rowmotion, meaning that they have a constant average over its orbits. Finally, we show that the Armstrong–Stump–Thomas equivariant bijection between antichains in types A and B and non-crossing matchings can be described more naturally in terms of the Robinson–Schensted–Knuth correspondence on permutations.

Keywords: rowmotion, homomesy, pattern avoidance

1 Introduction

Let $\mathcal{S}_n$ denote the set of permutations of $\{1, 2, \ldots, n\}$. We say that $\pi \in \mathcal{S}_n$ is 321-avoiding if there do not exist $i < j < k$ such that $\pi(i) > \pi(j) > \pi(k)$. Let $\mathcal{S}_n(321)$ denote the set of 321-avoiding permutations in $\mathcal{S}_n$. We can represent $\pi \in \mathcal{S}_n(321)$ as an $n \times n$ array with crosses in squares $(i, \pi(i))$ for $1 \leq i \leq n$; we call this the *array* of π. Rows and columns are indexed using cartesian coordinates, so that (i, j) denotes the cell in the ith column from the left and jth row from the bottom. We say that $(i, \pi(i))$ is a *fixed point* (respectively *excedance*, *weak excedance*, *deficiency*, *weak deficiency*) if $\pi(i) = i$ (respectively $\pi(i) > i$, $\pi(i) \geq i$, $\pi(i) < i$, $\pi(i) \leq i$).

Let P be a finite poset, and let $\mathcal{A}(\mathsf{P})$ denote the set of antichains of P. *Antichain rowmotion* is the map $\rho_{\mathcal{A}} : \mathcal{A}(\mathsf{P}) \to \mathcal{A}(\mathsf{P})$ defined as follows: for $A \in \mathcal{A}(\mathsf{P})$, let $\rho_{\mathcal{A}}(A)$ be the minimal elements of the complement of the order ideal generated by A. See Section 2.3 for more details and definitions.

Historically, rowmotion was first described for general posets by Brouwer and Schrijver [2], and then again by Cameron and Fon-der-Flaas [3] as a composition of certain involutions called toggles. The name of rowmotion comes from the work of Striker and Williams [17] where, for certain posets, rowmotion is described as a composition of toggles along the rows, and related to another operation called promotion.

Restricting our attention to the poset of positive roots for the type A root system, we will show that antichains of this poset are in bijection with Dyck paths and with

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321-avoiding permutations. This will allow us to define a natural rowmotion operation on $\mathcal{S}_n(321)$.

When studying rowmotion, it is common to look for statistics that exhibit a property called *homomesy* [12]. Given a set S and a bijection $\tau : S \to S$ so that each orbit of the action of τ on S has finite order, we say that a statistic on S is *homomesic* under this action if its average on each orbit is constant. More specifically, the statistic is said to be *c-mesic* if its average over each orbit is c. We prove that several statistics on 321-avoiding permutations, including the number of fixed points, are homomesic under rowmotion.

Furthermore, we use the viewpoint of 321-avoiding permutations to shed new light into a celebrated bijection of Armstrong, Stump and Thomas [1] between antichains in root posets of finite Weyl groups (also known as nonnesting partitions) and noncrossing partitions. We will show that, in the case of types A and B, the Armstrong–Stump–Thomas (AST) bijection has a simple interpretation in terms of the Robinson–Schensted–Knuth (RSK) correspondence applied to 321-avoiding permutations.

2 Background

In this section we review some notions about Dyck paths, noncrossing matchings, and the RSK correspondence, in particular as it applies to 321-avoiding permutations. We also provide a basic overview of rowmotion.

2.1 Dyck paths

Let $\mathcal{D}_n$ be the set of words over $\{\mathtt{u},\mathtt{d}\}$ consisting of n $\mathtt{u}$s and n $\mathtt{d}$s, and satisfying that every prefix contains at least as many $\mathtt{u}$s as $\mathtt{d}$s. Elements of $\mathcal{D}_n$ are called Dyck paths, and they will be drawn in three different ways as lattice paths in $\mathbb{Z}^2$ starting at the origin. Replacing $\mathtt{u}$ and $\mathtt{d}$ with $(0,1)$ and $(1,0)$ (respectively, $(1,0)$ and $(0,1)$), we obtain paths that stay weakly above (respectively, below) the diagonal $y = x$. We denote these by $\mathcal{D}_n^{\triangledown}$ (respectively, $\mathcal{D}_n^{\triangleleft}$). The sets $\mathcal{D}_n^{\triangledown}$ and $\mathcal{D}_n^{\triangleleft}$ are in bijection with each other, by simply reflecting along the diagonal. The third way to draw Dyck paths that we will use is when $\mathtt{u}$ and $\mathtt{d}$ are replaced with $(1,1)$ and $(1,-1)$, respectively. In all cases, a pair of consecutive steps $\mathtt{ud}$ is called a *peak*, and a pair $\mathtt{du}$ is called a *valley*. Interpreting $\mathtt{u}$ and $\mathtt{d}$ steps of $D \in \mathcal{D}_n$ as opening and closing parentheses, respectively, and matching them in the usual way, a pair of matched steps will be called a *tunnel*, following [5].

Several bijections between 321-avoiding permutations and Dyck paths are known. For $\pi \in \mathcal{S}_n(321)$, let $E_p(\pi) \in \mathcal{D}_n^{\triangledown}$ be the path whose peaks occur at the weak excedances of π, let $E_v(\pi) \in \mathcal{D}_n^{\triangledown}$ be the path whose valleys occur at the excedances of π, and let $D_v(\pi) \in \mathcal{D}_n^{\triangleleft}$ be the path whose valleys occur at the weak deficiencies of π. The bijection that maps $E_p(\pi)$ to $D_v(\pi)$ is known as the *Lalanne–Kreweras involution* on Dyck paths [8, 9], which we denote by $\mathrm{LK} = D_v \circ E_p^{-1}$; see the examples in Figure 1.

Let $\mathcal{N}_n$ denote the set of *noncrossing matchings* of $\{1, 2, \ldots, 2n\}$, i.e., perfect matchings with the property that there do not exist $i < j < k < \ell$ such that i is matched with k and j is matched with ℓ. We will draw the points $1, 2, \ldots, 2n$ around a circle in clockwise order, with a line segment connecting each pair of matched points. There is a straightforward bijection between Dyck paths and noncrossing matchings.

Definition 2.1. Let $\mathrm{Match} : \mathcal{D}_n \to \mathcal{N}_n$ be the bijection defined as follows. Given $D \in \mathcal{D}_n$, the points i and j are matched in $\mathrm{Match}(D)$ if the steps of D in positions i and j form a tunnel.

Define promotion of Dyck paths to be the following map $\mathrm{Pro} : \mathcal{D}_n \to \mathcal{D}_n$. Given $D \in \mathcal{D}_n$, consider its first-return decomposition $D = \mathtt{u}A\mathtt{d}B$ where A, B are Dyck paths, and let $\mathrm{Pro}(D) = A\mathtt{u}B\mathtt{d}$. The reason for this name is that, as observed by White [14, Sec. 8], applying this operation to a path $D \in \mathcal{D}_n$ is equivalent to applying promotion on standard Young tableaux of shape (n, n).

2.2 RSK and 321-avoiding permutations

The RSK correspondence is a bijection between permutations and pairs of standard Young tableaux of the same shape. We refer the reader to [16, Sec. 7.11] for definitions. Given a permutation $\pi \in \mathcal{S}_n$, we denote its image by $\mathrm{RSK}(\pi) = (P, Q)$, where P and Q are standard Young tableaux of the same shape. In, particular, π is 321-avoiding if and only if P and Q have at most two rows. This property is used to define the following map from $\mathcal{S}_n(321)$ to $\mathcal{D}_n$.

Definition 2.2. Let $\pi \in \mathcal{S}_n(321)$, and suppose that $\mathrm{RSK}(\pi) = (P, Q)$. Define a Dyck path $\widehat{\mathrm{RSK}}(\pi)$ as follows. For $1 \le i \le n$, let the ith step be a $\mathtt{u}$ if i is in the top row of P, and a $\mathtt{d}$ otherwise; let the $(2n + 1 - i)$th step be a $\mathtt{d}$ if i is in the top row of Q, and a $\mathtt{u}$ otherwise.

2.3 Rowmotion

Let P be a finite poset. An *antichain* of P is a subset of pairwise incomparable elements. An *order ideal* of P is a subset I with the property that if $x \in I$ and $y \le x$, then $y \in I$. Similarly, an *order filter* of P is a subset F with the property that if $x \in F$ and $x \le y$, then $y \in F$. Let $\mathcal{A}(P)$, $\mathcal{I}(\mathsf{P})$ and $\mathcal{F}(\mathsf{P})$ denote the sets of antichains, order ideals and order filters of P, respectively. The complementation map Θ, defined on subsets S of P by $\Theta(S) = \mathsf{P} \setminus S$, restricts to a bijection between $\mathcal{I}(P)$ and $\mathcal{F}(P)$.

Following [7], define the up-transfer map $\Delta : \mathcal{I}(\mathsf{P}) \to \mathcal{A}(\mathsf{P})$ by letting $\Delta(I)$ be the set of maximal elements of $I \in \mathcal{I}(P)$, and the down-transfer map $\nabla : \mathcal{F}(\mathsf{P}) \to \mathcal{A}(\mathsf{P})$ by letting $\nabla(F)$ be the set of minimal elements of $F \in \mathcal{F}(P)$. The inverses of these maps are given by

$$\Delta^{-1}(A) = \{x \in \mathsf{P} : x \leq y \text{ for some } y \in A\},$$
$$\nabla^{-1}(A) = \{x \in \mathsf{P} : x \geq y \text{ for some } y \in A\},$$

for $A \in \mathcal{A}(P)$. Antichain rowmotion is the map $\rho_{\mathcal{A}} : \mathcal{A}(\mathsf{P}) \to \mathcal{A}(\mathsf{P})$ defined as the composition $\rho_{\mathcal{A}} = \nabla \circ \Theta \circ \Delta^{-1}$.

The poset of positive roots in type A, which we denote by $\mathbf{A}^{n-1}$, can be described as the set of intervals $\{[i,j] : 1 \leq i \leq j \leq n-1\}$ ordered by inclusion. It is a ranked poset, with rank function given by $\mathrm{rk}([i,j]) = j - i$. The set $\mathcal{A}(\mathbf{A}^{n-1})$ is in bijection with $\mathcal{D}_n$. One such bijection consists of mapping each path P to the antichain $\alpha(P)$ whose elements are at the valleys of P. This bijection allows us to define rowmotion on Dyck paths as $\rho_{\mathcal{D}} = \alpha^{-1} \circ \rho_{\mathcal{A}} \circ \alpha$. A second bijection between $\mathcal{A}(\mathbf{A}^{n-1})$ and $\mathcal{D}_n$ consists of mapping the antichain A to the path $\delta(A)$ whose peaks are at the elements of the antichain. Note that $\rho_{\mathcal{A}} = \alpha \circ \delta$ and $\rho_{\mathcal{D}} = \delta \circ \alpha$.

Panyushev [10, 11] considered the map on $\mathcal{A}(\mathbf{A}^n)$ defined by mapping an antichain given by $\{[i_1, j_1], \dots [i_k, j_k]\}$ to $\{[i'_1, j'_1], \dots [i'_{n-k}, j'_{n-k}]\}$ with $\{i'_1, \dots, i'_{n-k}\} = \{1, 2, \dots, n\} \setminus \{j_1, \dots, j_k\}$ and $\{j'_1, \dots j'_{n-k}\} = \{1, 2, \dots, n\} \setminus \{i_1, \dots, i_k\}$. As noted in [7], this map equals the composition $\alpha \circ \mathrm{LK} \circ \alpha^{-1}$, so we will denote it by $\mathrm{LK}_{\mathcal{A}}$. Hopkins and Joseph show in [7, Thm. 3.5] that a certain map known as *antichain rowvacuation*, denoted in general by $\mathrm{Rvac}_{\mathcal{A}}$, coincides with $\mathrm{LK}_{\mathcal{A}}$ in the case of the type A root poset.

3 A rowmotion operation on 321-avoiding permutations

To define rowmotion on 321-avoiding permutations, first consider the following bijection to antichains of $\mathbf{A}^{n-1}$.

Definition 3.1. Let $\mathrm{Exc} : \mathcal{S}_n(321) \to \mathcal{A}(\mathbf{A}^{n-1})$ be the bijection where, for $\pi \in \mathcal{S}_n(321)$, we define
$$\mathrm{Exc}(\pi) = \{[i, \pi(i) - 1] \,:\, (i, \pi(i)) \text{ is an excedance of } \pi\}.$$

See the left column of Figure 1 for an example of $\mathrm{Exc} : \mathcal{S}_9(321) \to \mathcal{A}(\mathbf{A}^8)$, which maps the permutation 241358967 to the antichain $\{[1,1], [2,3], [6,7], [7,8]\}$.

To see that Exc is a bijection, note that it can be written as $\mathrm{Exc} = \delta^{-1} \circ E_p = \alpha \circ E_v$. The map Exc provides the following natural way to translate rowmotion into an operation on $\mathcal{S}_n(321)$.

Definition 3.2. Rowmotion on 321-avoiding permutations is the map $\rho_{\mathcal{S}} : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$ defined by $\rho_{\mathcal{S}} = \mathrm{Exc}^{-1} \circ \rho_{\mathcal{A}} \circ \mathrm{Exc}$.

An equivalent description of $\rho_{\mathcal{S}}$ can be given in terms of the bijections to Dyck paths, as the composition

$$\rho_{\mathcal{S}} = E_v^{-1} \circ E_p. \tag{3.1}$$

Figure 1 gives examples of $\rho_{\mathcal{S}}$ computed in both ways. Note that if $\pi \in \mathcal{S}_n(321)$ has upper and lower paths given by $P = E_p(\pi)$ and $Q = D_v(\pi)$, then $\sigma = \rho_{\mathcal{S}}(\pi)$ has upper and lower paths given by $\rho_{\mathcal{D}}(P) = E_p(\sigma)$ and $\rho_{\mathcal{D}}^{-1}(Q) = D_v(\sigma)$. Equivalently, at the level of antichains, we have $\mathrm{Exc}(\sigma) = \rho_{\mathcal{A}}(\mathrm{Exc}(\pi))$ and $\mathrm{Exc}(\sigma^{-1}) = \rho_{\mathcal{A}}^{-1}(\mathrm{Exc}(\pi^{-1}))$, noting that $\mathrm{Exc}(\pi^{-1})$ is the antichain formed by the deficiencies of π. This follows from the fact that

$$\mathrm{LK}_{\mathcal{A}} \circ \rho_{\mathcal{A}} = \rho_{\mathcal{A}}^{-1} \circ \mathrm{LK}_{\mathcal{A}}, \tag{3.2}$$

which was proved by Panyushev [11, Thm. 3.5].

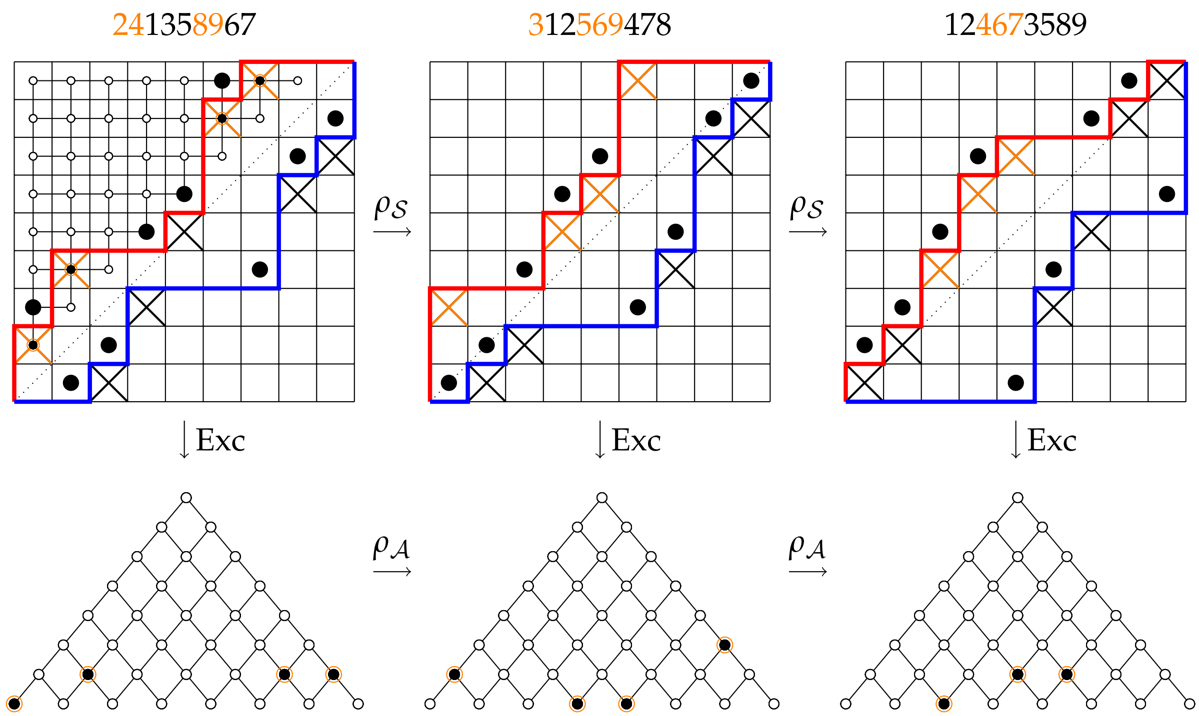

Figure 1: Two applications of rowmotion starting at $\pi = 241358967 \in \mathcal{S}_9(321)$, computed using Definition 3.2, or alternatively the composition (3.1). In the upper left diagram, the crosses represent π, the red path is $E_p(\pi)$, the blue path is $D_v(\pi) = \mathrm{LK}(E_p(\pi))$, and the dots represent $\rho_{\mathcal{S}}(\pi) = E_v^{-1}(E_p(\pi)) = 312569478$.

A diagram of our bijections for permutations, paths and antichains, as well as their interactions, appears in Figure 2. We can translate the Lalanne–Kreweras involution for Dyck paths into an involution on $\mathcal{S}_n(321)$ by defining

$$\mathrm{LK}_{\mathcal{S}} = E_p^{-1} \circ \mathrm{LK} \circ E_p. \tag{3.3}$$

Using that $\mathrm{LK}_{\mathcal{A}} = \alpha \circ \mathrm{LK} \circ \alpha^{-1}$, the maps $\mathrm{LK}_{\mathcal{S}}$ and $\mathrm{LK}_{\mathcal{A}}$ are related by

$$\mathrm{LK}_{\mathcal{S}} = E_p^{-1} \circ \alpha^{-1} \circ \mathrm{LK}_{\mathcal{A}} \circ \alpha \circ E_p. \tag{3.4}$$

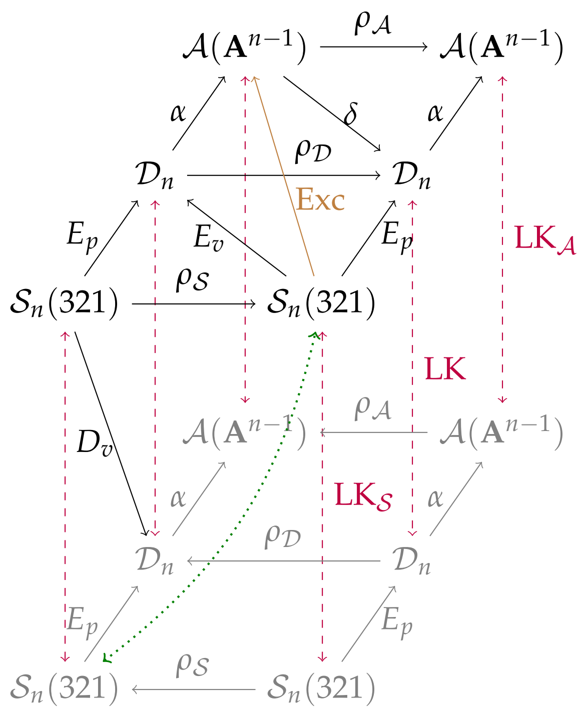

Figure 2: Diagram of the bijections $\rho_{\mathcal{S}}$, $\rho_{\mathcal{D}}$, $\rho_{\mathcal{A}}$, α, δ, E_p, E_v, D_v, LK_S, LK_A, and LK_D. The vertical dashed arrows are various versions of the Lalanne–Kreweras involution, and the dotted curved arrow is the map sending a permutation to its inverse.

The map $\mathrm{LK}_{\mathcal{S}}$ is closely related to the operation that sends each permutation to its inverse, as the next lemma shows.

Lemma 3.3. *If $\pi \in \mathcal{S}_n(321)$ then $\pi^{-1} = \rho_{\mathcal{S}}(\mathrm{LK}_{\mathcal{S}}(\pi)) = \mathrm{LK}_{\mathcal{S}}(\rho_{\mathcal{S}}^{-1}(\pi))$.*

Finally, let us mention that rowmotion on 321-avoiding permutations provides a more direct proof of a result of Hopkins and Joseph [7, Thm. 6.2] that the number of antichains of $\mathbf{A}^{n-1}$ that are fixed by the involution $\mathrm{LK}_{\mathcal{A}} \circ \rho_{\mathcal{A}}$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, by instead reducing it to a classical result of Simion and Schmidt [15] on the enumeration of 321-avoiding involutions.

4 Statistics and homomesies

In this section we show that certain statistics on 321-avoiding permutations exhibit homomesy under the action of $\rho_{\mathcal{S}}$.

4.1 The number of fixed points and the statistics h_i and ℓ_i

The first statistic that we consider is the number of fixed points of a permutation π, denoted by $\mathrm{fp}(\pi) = |\{i : \pi(i) = i\}|$. Let $\mathrm{exc}(\pi) = |\{i : \pi(i) > i\}|$ and $\mathrm{wexc}(\pi) = |\{i : \pi(i) \geq i\}|$ denote the number of excedances and weak excedances of π, respectively.

Theorem 4.1. *The statistic* fp *is 1-mesic under the action of* $\rho_{\mathcal{S}}$ *on* $\mathcal{S}_n(321)$.

Proof. Let $\pi \in \mathcal{S}_n(321)$, and let $\mathcal{O}$ be the orbit of π under $\rho_{\mathcal{S}}$. Then

$$\mathrm{fp}(\pi) = \mathrm{wexc}(\pi) - \mathrm{exc}(\pi) = n - \mathrm{exc}(\pi^{-1}) - \mathrm{exc}(\pi) = n - |\mathrm{Exc}(\pi)| - |\mathrm{Exc}(\pi^{-1})|.$$

Summing over the orbit,

$$\frac{1}{|\mathcal{O}|}\sum_{\pi\in\mathcal{O}} \mathrm{fp}(\pi) = n - \frac{1}{|\mathcal{O}|}\sum_{\pi\in\mathcal{O}} |\mathrm{Exc}(\pi)| - \frac{1}{|\mathcal{O}|}\sum_{\pi\in\mathcal{O}} |\mathrm{Exc}(\pi^{-1})|.$$

As noted above equation (3.2), applying $\rho_{\mathcal{S}}$ to π corresponds to applying $\rho_{\mathcal{A}}$ to $\mathrm{Exc}(\pi)$ and $\rho_{\mathcal{A}}^{-1}$ to $\mathrm{Exc}(\pi^{-1})$. It then follows by Lemma 3.3 that the sets $\{\mathrm{Exc}(\pi) : \pi \in \mathcal{O}\}$ and $\{\mathrm{Exc}(\pi^{-1}) : \pi \in \mathcal{O}\}$ are complete orbits under $\rho_{\mathcal{A}}$. It is known [1] that the antichain cardinality statistic is $\frac{n-1}{2}$-mesic under the action of $\rho_{\mathcal{A}}$ on $\mathcal{A}(\mathbf{A}^{n-1})$. Thus $\frac{1}{|\mathcal{O}|}\sum_{\pi\in\mathcal{O}} |\mathrm{Exc}(\pi)| = \frac{1}{|\mathcal{O}|}\sum_{\pi\in\mathcal{O}} |\mathrm{Exc}(\pi^{-1})| = \frac{n-1}{2}$, and we conclude that fp is 1-mesic. □

See Figure 3 for a computation of $\mathrm{fp}(\pi)$ over a rowmotion orbit for $n = 4$. As Theorem 4.1 asserts, the average of this statistic is 1 over each orbit.

Next we consider two families of statistics on 321-avoiding permutations, and show that they are also homomesic under rowmotion. The first family are the statistics h_i introduced by Hopkins and Joseph [7]. For $1 \leq i \leq n-1$, they define h_i on antichains $A \in \mathcal{A}(\mathbf{A}^{n-1})$ as

$$h_i(A) = \sum_{j=1}^{i} \mathbb{1}_{[j,i]}(A) + \sum_{j=i}^{n-1} \mathbb{1}_{[i,j]}(A),$$

where $\mathbb{1}_{[i,j]}(A)$ is the indicator function that equals 1 if $[i,j] \in A$ and 0 otherwise. For $\pi \in \mathcal{S}_n(321)$, we now define $h_i(\pi)$ by $h_i(\mathrm{Exc}(\pi))$. In terms of the array of $\pi \in \mathcal{S}_n(321)$, $h_i(\pi)$ counts the number of crosses of the form $(j, i+1)$ with $1 \leq j \leq i$, or (i,j) with $i+1 \leq j \leq n$.

Hopkins and Joseph prove in [7, Thm. 4.3] that the statistics h_i on antichains are 1-mesic under $\rho_{\mathcal{A}}$. This result can be translated in terms of 321-avoiding permutations as follows.

Theorem 4.2 ([7]). *For* $1 \leq i \leq n-1$, *the statistic* h_i *is* 1*-mesic under the action of* $\rho_{\mathcal{S}}$ *on* $\mathcal{S}_n(321)$.

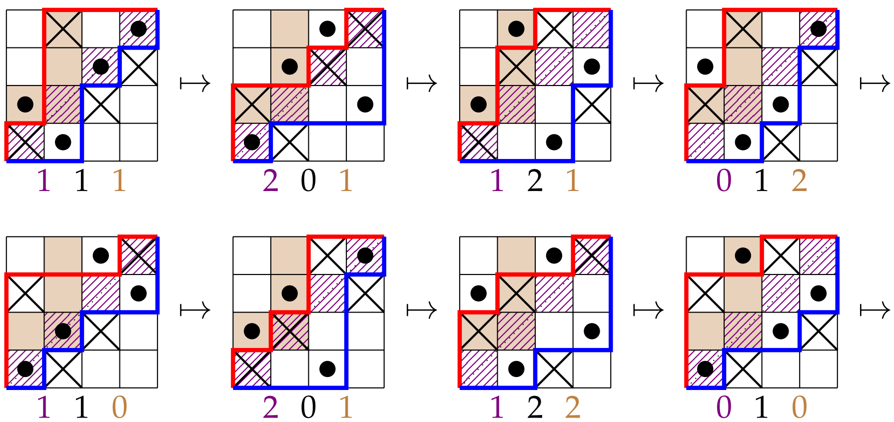

Figure 3: The rowmotion orbit containing 1423. The numbers below each diagram are the values $\mathrm{fp}(\pi)$, $h_2(\pi)$ and $\ell_2(\pi)$, from left to right.

Next we define a new family of permutation statistics, that we denote by ℓ_i for $1 \leq i \leq n$. For $\pi \in \mathcal{S}_n(321)$, let $\ell_i(\pi)$ be the number of crosses in the array of π of the form (j, i) with $1 \leq j \leq i$, plus the number of crosses of the form (i, j) with $i < j \leq n$. It turns out that the statistics ℓ_i are homomesic as well, see Figure 3 for an example.

Theorem 4.3. *For $1 \leq i \leq n$, the statistic ℓ_i is 1-mesic under the action of $\rho_{\mathcal{S}}$ on $\mathcal{S}_n(321)$.*

4.2 The sign statistic

In this subsection we describe how rowmotion interacts with the sign of a 321-avoiding permutation.

Theorem 4.4. *For all $\pi \in \mathcal{S}_n(321)$,*

$$\mathrm{sgn}(\rho_{\mathcal{S}}(\pi)) = \mathrm{sgn}(\mathrm{LK}_{\mathcal{S}}(\pi)) = \begin{cases} \mathrm{sgn}(\pi) & \text{if } n \text{ is odd,} \\ -\mathrm{sgn}(\pi) & \text{if } n \text{ is even.} \end{cases}$$

It is a classical result of Simion and Schmidt [15, Prop. 2] that, when n is even, the set $\mathcal{S}_n(321)$ contains the same number of odd and even permutations. A bijective proof of this fact was given by Reifegerste [13]. Theorem 4.4 gives two new bijections, $\rho_{\mathcal{S}}$ and $\mathrm{LK}_{\mathcal{S}}$, between the subsets of odd and even permutations in $\mathcal{S}_n(321)$. Furthermore, $\mathrm{LK}_{\mathcal{S}}$ has the additional property of being a sign-reversing involution on $\mathcal{S}_n(321)$.

Finally, we note the following two immediate consequences of Theorem 4.4.

Corollary 4.5. *For even n, the statistic* sgn *on $\mathcal{S}_n(321)$ is 0-mesic under the action of $\rho_{\mathcal{S}}$.*

Corollary 4.6. *For even n, the map* LK *on $\mathcal{D}_n$ has no fixed points.*

Corollary 4.6 is equivalent to [10, Thm. 4.6] for even n.

5 The Armstrong–Stump–Thomas bijection as a map on permutations

In [1], Armstrong, Stump and Thomas constructed a bijection between antichains in root posets of finite Weyl groups and noncrossing matchings, having the property that it translates rowmotion on antichains into rotation of noncrossing matchings. In type A, we can interpret antichains as 321-avoiding permutations via the bijection Exc. It turns out that, with this interpretation, the Armstrong–Stump–Thomas bijection is equivalent to the well-known Robinson–Schensted–Knuth correspondence restricted to 321-avoiding permutations. This fact extends to the root poset in type B.

5.1 AST in type A

The Armstrong–Stump–Thomas bijection is described in [1] in much more generality, but for our purposes we will be restricting exclusively to the cases of type A and B. We follow the description given by Defant and Hopkins [4]. In type A, we denote this bijection by AST.

Definition 5.1 (AST in type A [4])**.** Let $\mathrm{AST} : \mathcal{A}(\mathbf{A}^{n-1}) \to \mathcal{N}_n$ be the bijection where the image of $A \in \mathcal{A}(\mathbf{A}^{n-1})$ is obtained as follows. For each $1 \le i \le n$, the vertex $2n+1-i$ of the matching will be denoted by $\bar{i}$, so that the vertices are $1, 2, \ldots, n, \overline{n}, \overline{n-1}, \ldots, \overline{1}$ in clockwise direction. For each i from 1 to n, consider two options:

- if $[i, j-1] \in A$ for some j, match the vertex j with the nearest unmatched vertex in counterclockwise direction;
- otherwise, match the vertex $\bar{i}$ with the nearest unmatched vertex in clockwise direction.

We can now state our main result in this section.

Theorem 5.2. *Let* Match, $\widehat{\mathrm{RSK}}$, Exc *and* AST *be the bijections from Definitions 2.1, 2.2, 3.1 and 5.1, respectively. Then*

$$\mathrm{AST} = \mathrm{Match} \circ \widehat{\mathrm{RSK}} \circ \mathrm{Exc}^{-1}.$$

See Figure 4 for examples of these maps. It is shown in [1] that the bijection AST is equivariant in the sense that $\mathrm{AST} \circ \rho_{\mathcal{A}} = \mathrm{Rot} \circ \mathrm{AST}$.

5.2 AST in type B

Let $\mathbf{B}^m$ denote the positive root poset of the type B_m root system. The poset $\mathbf{B}^m$ is isomorphic to the quotient of $\mathbf{A}^{2m-1}$ by the relations of $[i, j] \sim [2m-j, 2m-i]$ for all

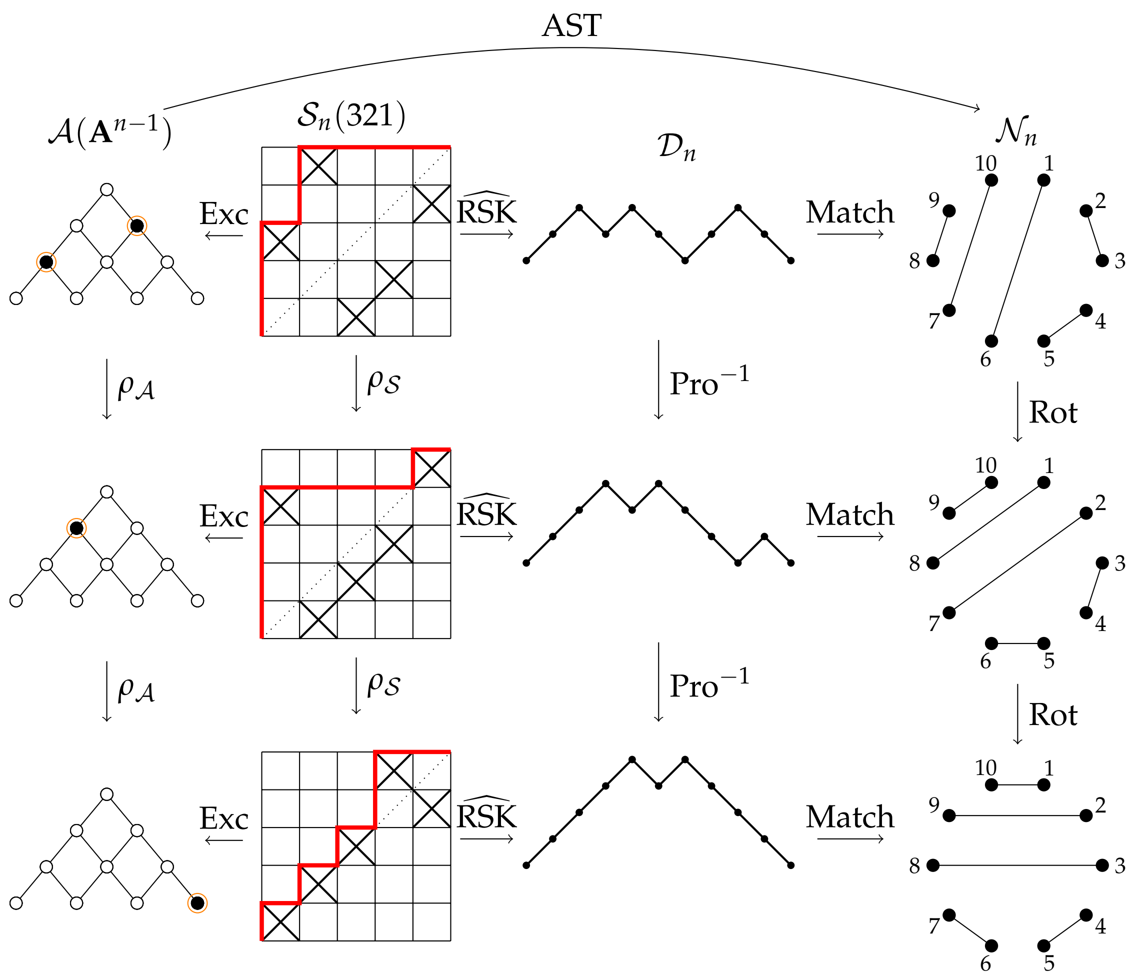

Figure 4: The three terms of the respective orbits of the elements of $\mathcal{A}(\mathbf{A}^4)$, $\mathcal{S}_5(321)$, $\mathcal{D}_5$, and $\mathcal{N}_5$ associated to and beginning with the permutation 35124 together with the respective actions of $\rho_{\mathcal{A}}$, $\rho_{\mathcal{S}}$, Pro^{-1}, and Rot.

$[i,j] \in \mathbf{A}^{2m-1}$. In type B, the Armstrong–Stump–Thomas map [1] is the map $\text{AST}_B : \mathcal{A}(\mathbf{B}^m) \to \mathcal{N}_{2m}$ defined as follows. For $A \in \mathcal{A}(\mathbf{B}^m)$, consider the antichain in $\mathbf{A}^{2m-1}$ given by

$$\hat{A} = \{[i,j] : [i,j] \in A \text{ or } [2m-j, 2m-i] \in A\}. \tag{5.1}$$

Now let $\text{AST}_B(A) = \text{AST}(\hat{A})$, where AST is the map from Definition 5.1.

It is shown in [1, Lemma 3.5] that the map $\rho_{\mathcal{A}}^n$ on $\mathcal{A}(\mathbf{A}^{n-1})$, obtained by applying rowmotion n times, sends an antichain $\{[i_1, j_1], \ldots, [i_k, j_k]\}$ to the antichain $\{[n-j_1, n-i_1], [n-j_2, n-i_2], \ldots, [n-j_k, n-i_k]\}$. The antichains that are invariant under this map are those that are symmetric under vertical reflection. When $n = 2m$, these are the antichains in the image of the embedding $A \mapsto \hat{A}$ from $\mathcal{A}(\mathbf{B}^m)$ into $\mathcal{A}(\mathbf{A}^{2m-1})$ given by (5.1). Thus the image of AST_B consists of the matchings in $\mathcal{N}_{2m}$ that are invariant under applying Rot^{2m}, i.e., under rotation by 180°. Denoting by $\mathcal{N}_n^S$ the set of centrally symmetric matchings in $\mathcal{N}_n$, one concludes that AST_B is a bijection between $\mathcal{A}(\mathbf{B}^m)$ and $\mathcal{N}_{2m}^S$.

Our alternative description of AST in terms of RSK allows us to give a new proof of this property of AST_B. Unlike the proof in [1], which relies on the properties of rowmotion, ours uses well-known properties of the RSK correspondence.

For $\pi \in \mathcal{S}_n$, its *reverse complement* is the permutation π^{rc} such that $\pi^{rc}(i) = n+1-\pi(n+1-i)$ for all i. The array of π^{rc} is obtained by rotating the array of π by 180°. Thus, the array of $(\pi^{rc})^{-1}$ is obtained by reflecting the array of π along the secondary diagonal, i.e., the one passing through the bottom-left and upper-right corners of the array. The map Exc from Definition 3.1 restricts to a bijection between permutations $\pi \in \mathcal{S}_n(321)$ such that $\pi = (\pi^{rc})^{-1}$, and antichains in $\mathcal{A}(\mathbf{A}^{2m-1})$ that are invariant under vertical reflection, which in turn are in bijection with $\mathcal{A}(\mathbf{B}^m)$. The behavior of RSK under these symmetries is well understood.

Theorem 5.3 ([16, Thm. A1.2.10])**.** *Suppose that* $\mathrm{RSK}(\pi) = (P, Q)$*. Then* $\mathrm{RSK}(\pi^{rc}) = (\mathrm{Evac}(P), \mathrm{Evac}(Q))$*, and* $\mathrm{RSK}((\pi^{rc})^{-1}) = (\mathrm{Evac}(Q), \mathrm{Evac}(P))$*.*

These properties allow us to show the following without use of AST.

Theorem 5.4. *For* $\pi \in \mathcal{S}_n(321)$*, the matchings* $\mathrm{Match}(\widehat{\mathrm{RSK}}((\pi^{rc})^{-1}))$ *and* $\mathrm{Match}(\widehat{\mathrm{RSK}}(\pi))$ *are* 180° *rotations of each other. Thus, the map* $\mathrm{Match} \circ \widehat{\mathrm{RSK}}$ *restricts to a bijection between* $\{\pi \in \mathcal{S}_n(321) : \pi = (\pi^{rc})^{-1}\}$ *and* $\mathcal{N}_n^S$*.*

Finally, another byproduct of our permutation perspective is that we can enumerate antichains of $\mathbf{B}^n$ which are fixed under the action of $\mathrm{LK}_{\mathcal{A}} \circ \rho_{\mathcal{A}}$, answering a question of Hopkins and Joseph [6, Remark 6.7].

Proposition 5.5.

$$\#\{A \in \mathcal{A}(\mathbf{B}^n) : \mathrm{LK}_{\mathcal{A}}(\rho_{\mathcal{A}}(A)) = A\} = 2^n.$$

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