Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #17, 12 pp.

Proceedings of the 35^{th} Conference on Formal Power Series and Algebraic Combinatorics (Davis)

Sums of Weighted Lattice Points of Polytopes

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Abstract. We study the problem of counting lattice points of a polytope that are weighted by an Ehrhart quasi-polynomial of a family of parametric polytopes. As applications one can compute integrals and maximum values of such quasi-polynomials. We not only recover some known identities in representation theory and semigroup theory but obtain new ones.

Keywords: quasi-polynomials, Ehrhart functions, combinatorial identities, polyhedra.

1 Introduction

We are given a rational convex polytope P in $\mathbb{R}^n$ and $w(x)$ a quasi-polynomial function in n variables. We call w a *weight function* (the precise definition of the quasi-polynomials we use here is below, but for now you can think of w as a polynomial). A computational problem arising throughout the mathematical sciences is to compute, or at least estimate,

$$L_P(w,t) = \sum_{x \in tP \cap \mathbb{Z}^n} w(x). \tag{1.1}$$

Note that, for a fixed w as the polytope P is dilated by an integer factor $t \in \mathbb{N}$ we obtain a function of t, which we call the *weighted Ehrhart quasi-polynomial* for the pair (P, w). The name is natural as when $w(x) = 1$ then $L_P(1,t)$ yields the classical *Ehrhart quasi-polynomial*. We recommend Chapter 4 of [25] or [8] and the references there for excellent introductions to Ehrhart functions and Ehrhart quasi-polynomials.

One can prove $L_P(w,t)$ is a quasi-polynomial in the sense that it is a function in the variable t which is a sum of monomials up to degree $d + M$, where $M = \deg w$, but whose coefficients E_m are periodic functions of $n \in \mathbb{N}$:

$$L_P(w,t) = \sum_{m=0}^{d+M} E_m t^m.$$

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The evaluation of Equation (1.1) at the dilation tP of P is obviously what we need in order to understand this Ehrhart quasi-polynomial. The leading coefficient of $L_P(w,t)$ is given by the integral of w over the polytope P. These integrals were studied in [6], [7] and more recently in [5].

We will illustrate soon many important examples of such weighted Ehrhart problems. For now note they appear in enumerative combinatorics [2], algebraic combinatorics [4, 10], statistics [16, 11], and in symbolic integration and optimization [5, 14], among others.

Our Contributions:

We now outline the main contributions whose details will appear in the forthcoming full version. The main theorem is a surprisingly simple way to evaluate the function $L_P(w,t)$ where P is a rational polytope and $w(x)$ is a very general weight function. The key idea is that we build a new polytope, the *weight lifting polytope* P^*, for which these functions become simply $L_{P^*}(1,t)$, in other words, just a "standard" lattice point counting function. This way (often) the weighted Ehrhart polynomial P is equivalent to the (usual) Ehrhart polynomial of P^*. Clearly, P^* will depend on both P and w:

Theorem 1.1 (The existence of weight lifting polytopes). *Let P be a rational convex polytope in the form $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, where $\mathbf{A} \in \mathbb{Z}^{s\times n}, \mathbf{b} \in \mathbb{Z}^s$. Let $Q(x_1,\ldots,x_n)$ be the parametric family of rational convex polytopes parameterized by $x_1,\ldots,x_n$, given by*

$$Q(x_1,\ldots,x_n) = \left\{ \mathbf{y} \mid \mathbf{C}\mathbf{y} = \sum_{i=1}^{n} x_i \mathbf{d_i} + \mathbf{e}, \mathbf{y} \geq \mathbf{0} \right\},$$

where $\mathbf{C} \in \mathbb{Z}^{r\times m}, \mathbf{d_i}, \mathbf{e} \in \mathbb{Z}^r$. Using Q define $w(\mathbf{x})$ to be the multivariate Ehrhart quasi-polynomial function in n variables that counts the number of lattice points in the parametric polytope $Q(x_1,\ldots,x_n)$ when x_i are chosen integers, i.e.,

$$w(x_1,\ldots,x_n) = |Q(x_1,\ldots,x_n) \cap \mathbb{Z}^m|.$$

1. *There is a* weight lifting polytope $P^* \subset \mathbb{R}^{n+m}$ *defined by*

$$P^* = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \mathbf{A}^* \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ -\mathbf{e} \end{pmatrix}, \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \right\}$$

where

$$\mathbf{A}^* = \begin{bmatrix} & & \mathbf{A} & & \mathbf{0} \\ \mathbf{d_1} & \mathbf{d_2} & \cdots & \mathbf{d_n} & -\mathbf{C} \end{bmatrix},$$

for which the summation of the lattice points of P weighted by w equals the number of lattice points of P^*.

2. *Moreover, when* $\mathbf{e} = 0$*, the construction is parametric in the sense that the weight* w *is a homogeneous function,* $(tP)^* = t(P^*)$*, and*

$$L_P(w,t) = |(tP)^* \cap \mathbb{Z}^{n+m}| = |t(P^*) \cap \mathbb{Z}^{n+m}| = L_{P^*}(1,t).$$

Remark 1.2. To the best of our knowledge the first version of Theorem 1.1 appeared in print in work by Ardila and Brugallé (see [4, Section 4]), but in [4] the weights $w(x)$ were special polynomials and in that case some of the consequences we show were not possible. In Section 2, we present a direct constructive/algorithmic proof of Theorem 1.1 and describe several interesting special cases depending on the type of Ehrhart quasi-polynomials (in particular, we recover the results of [4]).

Remark 1.3. The second half of Theorem 1.1 uses special weights that are by construction non-negative. But we note that most of the proof of the theorem works even when $w(x)$ takes negative or zero values over P. The function $L_P(w,t)$ still makes sense, but what we obtain is not a traditional Ehrhart polynomial, because, for example, the leading coefficient could be negative, and volumes are never negative.

Remark 1.4. Theorem 1.1 says the weight $w(x)$ can be any Ehrhart quasi-polynomial. In Section 2, we carefully discuss many ways to express polynomials in terms of these quasi-polynomial weights. A key point of our paper is that Theorem 1.1 is more versatile and expressive because it applies to more functions than just polynomial weights. In fact, Section 2 shows w can have many different representations (e.g., polynomials), some more efficient than others. To demonstrate the power in Section 3 we present applications to Combinatorial Representation Theory and Number Theory.

Corollary 1.5 below is a notable new consequence of Theorem 1.1 that can be applied to many problems of interest. For example, these ideas can be applied to integration and maximization of Kostka numbers, Littlewood–Richardson coefficients, and any other combinatorial invariant that is given by an Ehrhart quasi-polynomial.

Corollary 1.5. *Let* w *be weight obtained from an Ehrhart quasi-polynomial function of a parametric polyhedron* Q*, whose parameters are defined over the lattice points of a polytope* P*. Here* P, Q, w *are just as in Theorem 1.1. Using the weight lifting polytope construction of Theorem 1.1 one can integrate and maximize* w *over* P *as follows:*

- *One can compute the integral* $\int_P w(x)dx$ *reformulated as a volume computation of the weight lifting polytope* P^**.*
- *One can solve the maximization problem and determine* $\max_{\alpha\in P\cap\mathbb{Z}^n} w(\alpha)$*. It reduces to counting the lattice points of a finite sequence of weight lifting polytopes which contain each other and can be read from* P^* *efficiently.*

We sketch the proof of Corollary 1.5 in Section 2. In the forthcoming full version we will include computational experiments with LattE [13].

2 Proofs of Theorem 1.1 and sketch of other results

Here we present proofs of Theorem 1.1 and some variations of it.

Proof of Theorem 1.1. Note that there is a natural projection map $\pi : P^* \to P$ via $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}$. It suffices to show that for any fixed $\mathbf{x} \in P \cap \mathbb{Z}^n$, $w(\mathbf{x}) = |\pi^{-1}(\mathbf{x}) \cap \mathbb{Z}^{n+m}|$. Recall that $(\mathbf{x}, \mathbf{y}) \in \pi^{-1}(\mathbf{x})$ if and only if $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{C}\mathbf{y} = \sum_{i=1}^{n} x_i \mathbf{b_i} + \mathbf{e}$ where $\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}$. Given $\mathbf{x} \in P \cap \mathbb{Z}^n$, we see that $(\mathbf{x}, \mathbf{y}) \in \pi^{-1}(\mathbf{x}) \cap \mathbb{Z}^{n+m}$ if and only if $\mathbf{y} \in Q(x_1, \ldots, x_n) \cap \mathbb{Z}^m$. Hence, for a fixed $\mathbf{x} \in P \cap \mathbb{Z}^n$, $|\pi^{-1}(\mathbf{x}) \cap \mathbb{Z}^{n+m}| = |Q(\mathbf{x}) \cap \mathbb{Z}^m| = w(\mathbf{x})$.

We now consider second part of Theorem 1.1. We show that P^* is parametric with respect to $\mathbf{b}$ in the following sense. If $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, then

$$P^* = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \, \mathbf{A}^* \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ -\mathbf{e} \end{pmatrix}, \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \right\}.$$

Therefore, $tP = \{\mathbf{x} : \mathbf{A}\mathbf{x} = t\mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and

$$(tP)^* = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \, \mathbf{A}^* \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} t\mathbf{b} \\ -\mathbf{e} \end{pmatrix}, \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \right\}.$$

Given $\mathbf{e} = 0$, we can see that $(tP)^* = t(P^*)$. By the first part of the proof we conclude

$$L_P(w, t) = |(tP)^* \cap \mathbb{Z}^{n+m}| = |t(P^*) \cap \mathbb{Z}^{n+m}| = L_{P^*}(1, t). \qquad \square$$

Now we outline more results and corollaries of Theorem 1.1. From now on we deal with the most general *quasi-polynomial weighted* case, i.e., $w(x)$ is a non-constant quasi-polynomial as in the statement of Theorem 1.1.

Definition 2.1. Let $Q = \{\mathbf{y} \mid \mathbf{C}\mathbf{y} = \mathbf{d}, \mathbf{y} \geq \mathbf{0}\} \subseteq \mathbb{R}^m$ where $\mathbf{C} \in \mathbb{Z}^{r \times m}, \mathbf{d} \in \mathbb{Z}^r$ be a rational polytope. For every integer t, we define $tQ = \{\mathbf{y} \mid \mathbf{C}\mathbf{y} = t\mathbf{d}, \mathbf{y} \geq \mathbf{0}\}$.

Example 2.2. Consider the $(m-1)$-dimensional standard simplex

$$\Delta_{m-1} = \{\mathbf{y} \mid y_1 + \cdots + y_m = 1, y_i \geq 0\}.$$

Then $-2\Delta_{m-1} = \{\mathbf{y} \mid y_1 + \cdots + y_m = -2 \cdot 1, y_i \geq 0\}$.

Definition 2.3. A function $w(t)$ is a **late-dilated** Ehrhart quasi-polynomial if

$$w(t) = |(t-c)Q \cap \mathbb{Z}^m|,$$

where $c \in \mathbb{Z}$ and Q is a rational polytope of the form given in Definition 2.1.

Example 2.4. The function $\binom{t}{m-1}$ is a late-dilated Ehrhart polynomial in the variable t, because $\binom{t}{m-1} = |(t-m+1)\Delta_{m-1} \cap \mathbb{Z}^m|$.

Corollary 2.5. *Let $w_1, w_2, \cdots, w_n$ be n late-dilated Ehrhart quasi-polynomials, i.e., $w_i(t) = |(t-c_i)Q_i \cap \mathbb{Z}^{m_i}|$ where $Q_i = \{\mathbf{y_i} \mid \mathbf{C_i y_i} = \mathbf{d_i}, \mathbf{y_i} \geq \mathbf{0}\}$ and $\mathbf{C_i} \in \mathbb{Z}^{r_i \times m_i}, \mathbf{d_i} \in \mathbb{Z}^{r_i}, c_i \in \mathbb{Z}$. Consider a rational polytope of the form $P = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \subseteq \mathbb{R}^n$ where $\mathbf{A} \in \mathbb{Z}^{s \times n}, \mathbf{b} \in \mathbb{Z}^s$ and the multivariate function $w(\mathbf{x}) = \prod_{i=1}^n w_i(x_i)$. There exists a* weight lifting polytope *$P^* \subseteq \mathbb{R}^{n^*}$ of P, where $n^* = n + m_1 + \cdots + m_n$, such that*

$$\sum_{\mathbf{x} \in P \cap \mathbb{Z}^n} w(\mathbf{x}) = \left| P^* \cap \mathbb{Z}^{n^*} \right|.$$

Proof. We need only show that there is a rational polytope $Q(x_1, \ldots, x_n)$ of the form given in Theorem 1.1 for which $w(\mathbf{x}) = |Q(x_1, \ldots, x_n) \cap \mathbb{Z}^{m_1 + \cdots + m_n}|$ and then apply Theorem 1.1. Let $Q(x_1, \ldots, x_n) = \prod_{i=1}^n (x_i - c_i) Q_i$. Specifically, $Q(x_1, \ldots, x_n)$ has the form

$$\left\{ \begin{pmatrix} \mathbf{y_1} \\ \vdots \\ \mathbf{y_n} \end{pmatrix} \middle| \begin{pmatrix} \mathbf{C_1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{C_n} \end{pmatrix} \begin{pmatrix} \mathbf{y_1} \\ \vdots \\ \mathbf{y_n} \end{pmatrix} = x_1 \begin{pmatrix} \mathbf{d_1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} + \cdots + x_n \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{d_n} \end{pmatrix} + \mathbf{e}, \mathbf{y} \geq \mathbf{0} \right\}. \qquad \square$$

Corollary 2.6. *For every monomial $w(\mathbf{x}) = \mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, there exists a weight lifting polytope $P^* \subseteq \mathbb{R}^{n^*}$ where $n^* = n + 2|\alpha| = n + 2\sum_{i=1}^n \alpha_i$ such that*

$$\sum_{\mathbf{x} \in P \cap \mathbb{Z}^n} w(\mathbf{x}) = \left| P^* \cap \mathbb{Z}^{n^*} \right|.$$

Proof. By Corollary 2.5, we just need to show that $x_i^{\alpha_i}$ is a late-dilated Ehrhart polynomial. It is well known that $(k+1)^{\alpha_i}$ is the Ehrhart polynomial of the α_i-dimensional hypercube of length k. In particular, the hypercube has the form

$$tQ_i = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \\ z_1 \\ \vdots \\ z_{\alpha_i} \end{pmatrix} \middle| \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \\ z_1 \\ \vdots \\ z_{\alpha_i} \end{pmatrix} = t \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, y_i \geq 0, z_i \geq 0 \right\}. \qquad \square$$

Corollary 2.7. *For every polynomial $w(\mathbf{x}) = \sum_{\alpha \in I} c_\alpha \mathbf{x}^\alpha = \sum_{\alpha \in I} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, there exist $|I|$ weight lifting polytopes P^*_α indexed by the exponents of monomials such that*

$$\sum_{\mathbf{x} \in P \cap \mathbb{Z}^n} w(x) = \sum_{\alpha \in I} c_\alpha \left| P^*_\alpha \cap \mathbb{Z}^{n^*} \right|.$$

Proof. This follows directly from Corollary 2.6 $\square$

Remark 2.8. Corollary 2.7 implies that if $w(x)$ is a polynomial with $|I|$ nonzero monomials, then we can compute the sum of lattice points of P weighted by w by counting integral points in $|I|$ weight lifting polytopes.

We give another two corollaries of Theorem 1.1.

Corollary 2.9. *Consider the polynomial $w(\mathbf{x}) = \prod_{i=1}^{n} \binom{x_i+\alpha_i-1}{\alpha_i-1}$. There exists a weight lifting polytope $P^* \subseteq \mathbb{R}^{n^*}$ where $n^* = n + |\alpha|$ such that*

$$\sum_{\mathbf{x}\in P\cap\mathbb{Z}^n} w(\mathbf{x}) = \left|P^* \cap \mathbb{Z}^{n^*}\right|.$$

Proof. Recall that $\binom{x_i+\alpha_i-1}{\alpha_i-1}$ is the Ehrhart polynomial of the standard $(\alpha_i - 1)$-simplex $1 = y_1 + \cdots + y_{\alpha_i}$ with $y_i \geq 0$. In particular, the simplex has the form

$$tQ_i = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \end{pmatrix} \middle| \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \end{pmatrix} = t \cdot 1, y_i \geq 0 \right\}.$$

Applying Corollary 2.5 gives the weight lifting polytope from the statement. □

Corollary 2.10. *Consider the polynomial $w(\mathbf{x}) = \prod_{i=1}^{n} \binom{x_i}{\alpha_i-1}$. There exists a weight lifting polytope P^* of the dimension $n^* = n + |\alpha|$ such that $\sum_{\mathbf{x}\in P\cap\mathbb{Z}^n} w(\mathbf{x}) = \left|P^* \cap \mathbb{Z}^{n^*}\right|$.*

Proof. The function $\binom{x_i}{\alpha_i-1}$ is a late-dilated Ehrhart polynomial because $\binom{x_i+\alpha_i-1}{\alpha_i-1}$ is the Ehrhart polynomial of the standard $(\alpha_i - 1)$-simplex. Applying Corollary 2.5 gives the weight lifting polytope from the statement. □

Note that $\{\binom{x+k-1}{k-1} \mid k = 1, 2, \ldots\}$ and $\{\binom{x}{k-1} \mid k = 1, 2, \ldots\}$ are two well-known bases of the vector space of polynomials in x.

Corollary 2.11. *For every monomial $w(\mathbf{x}) = \mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, there exist at most $\alpha_1 \cdots \alpha_n$ weight lifting polytopes P^*_β indexed by the vector β and $P^*_\beta \subset \mathbb{R}^{n^*}$ where $n^* = n + |\beta|$ such that*

$$\sum_{\mathbf{x}\in P\cap\mathbb{Z}^n} w(\mathbf{x}) = \sum_{\beta\leq\alpha} c_\beta \left|P^*_\beta \cap \mathbb{Z}^{n^*}\right|.$$

Proof. Let $v_k(x)$ be one of the two binomial bases described above. We can transform the monomial basis $\{x^k \mid k = 0, 1, 2, \ldots\}$ into the binomial basis,

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = \sum_{\beta\leq\alpha} c(\alpha, \beta) \cdot v_{\beta_1}(x_1) v_{\beta_2}(x_2) \cdots v_{\beta_n}(x_n).$$

By Corollaries 2.9 and 2.10, for each β and each polynomial $v_{\beta_1}(x_1)v_{\beta_2}(x_2)\cdots v_{\beta_n}(x_n)$, there exists a corresponding weight lifting polytope $P^*_\beta \subset \mathbb{R}^{n+|\beta|}$. □

Remark 2.12. In Corollary 2.6 we express the weighted sum of lattice points of P using a single $P^* \subset \mathbb{R}^{n+2|\alpha|}$, but in Corollary 2.11 we express this sum using at most $\alpha_1\alpha_2\cdots\alpha_n$ polytopes of lower dimension $P^*_\beta \subset \mathbb{R}^{n+|\beta|}$.

Proof of Corollary 1.5. Applying Theorem 1.1 to P gives a weight lifting polytope P^* for which $L_P(w,t) = L_{P^*}(1,t)$. Applying a classical result relating the volume and lead coefficient of the Ehrhart quasi-polynomial of P^* completes the proof. Both $L_P(w,t)$ and $L_{P^*}(1,t)$ are quasi-polynomial functions of t, and concretely, this equality implies that their leading coefficients are the same.

We can then replace integration of $w(x)$ over P with computation of the leading coefficient of $L_{P^*}(1,t)$, which is equivalent to computing the volume of P^*. Note that this transformation can be carried out in a number of steps that is polynomial in the size of the inputs describing P^*.

For the second claim, we start by recalling an elementary fact. Let $S = \{s_1,\ldots,s_r\}$ be a set of non-negative real numbers. Then $\max\{s_i \mid s_i \in S\} = \lim_{k\to\infty} \sqrt[k]{\sum_{j=1}^r s_j^k}$. The arithmetic mean of S is at most its maximum value, which in turn is at most as big as $\sum_i s_i$. We apply these ideas to the set $S = \{w(\alpha) \mid \alpha \in P \cap \mathbb{Z}^n\}$. This gives upper and lower bounds for each positive integer k:

$$L_k = \sqrt[k]{\frac{\sum\limits_{\alpha\in P\cap\mathbb{Z}^n} w(\alpha)^k}{|P\cap\mathbb{Z}^d|}} \le \max\{w(\alpha) : \alpha \in P\cap\mathbb{Z}^n\} \le \sqrt[k]{\sum_{\alpha\in P\cap\mathbb{Z}^n} w(\alpha)^k} = U_k.$$

As $k\to\infty$, L_k and U_k approach this maximum value monotonically (from below and above, respectively). Trivially, if the difference between the (rounded) upper and lower bounds becomes strictly less than 1, we have determined $\max\{w(x) \mid x \in P\cap\mathbb{Z}^n\} = \lceil L_k \rceil$. Thus the process terminates with the correct value. Finally, the key value in the sequences L_k and U_k is the term $L_P(w^k,t) = \sum\limits_{\alpha\in tP\cap\mathbb{Z}^n} w(\alpha)^k$. Corollary 2.5 describes how to construct the weight lifting polytope P^* corresponding to the pair P and $w(\alpha)^k$. $\square$

3 Applications

Theorem 1.1 has applications beyond integration and maximization of Ehrhart quasi-polynomials. In this section we discuss how to use it to find new algebraic combinatorial identities by carefully choosing the polytope P and reinterpreting the weight function w in terms of Ehrhart quasi-polynomials of some polytopes Q_i.

3.1 Weighted Ehrhart in number theory

Simultaneous Core Partitions. We first describe an area in which weighted Ehrhart machinery has already been applied to prove a significant result. Let λ be a partition and $\mathcal{H}(\lambda)$ denote its multiset of hook lengths. The partition λ is called an *a-core partition* if no element of $\mathcal{H}(\lambda)$ is divisible by a. If λ is both an a-core partition and a b-core partition, then we say that it is an (a,b)-core partition. There is an extensive literature about statistical properties of sizes of simultaneous core partitions [12, 22]. Anderson proved that if a and b are relatively prime positive integers then the number of (a,b)-core partitions is $\frac{1}{a+b}\binom{a+b}{a}$ [1]. Johnson proved a conjecture of Armstrong, showing that the average size of an (a,b)-core partition is $(a+b+1)(a-1)(b-1)/24$ [18]. Johnson's proof fits into the framework of weighted Ehrhart theory.

Suppose that a and b are relatively prime positive integers. It is not hard to show that a-core partitions are in bijection with elements of $\Lambda_a = \{(c_0,\ldots,c_{a-1}) \in \mathbb{Z}^a \colon \sum_i c_i = 0\}$. Let $r_a(x)$ be the remainder when x is divided by a. We use cyclic indexing for elements $\mathbf{c} \in \Lambda_a$, that is, for $k \in \mathbb{Z}$ we set $c_k = c_{r_a(k)}$. Simultaneous (a,b)-core partitions are in bijection with the elements of Λ_a satisfying the inequalities $c_{i+b} - c_i \leq \left\lfloor \frac{b+i}{a} \right\rfloor$ for each $i \in \{0,1,\ldots,a-1\}$ [18, Lemma 23]. In this way, we see that (a,b)-core partitions are in bijection with integer points in a rational polytope $\mathrm{SC}_a(b)$. The size of the a-core partition corresponding to $\mathbf{c} = (c_0,\ldots,c_{a-1})$ is $h_a(\mathbf{c}) = \frac{a}{2}\sum_{i=0}^{a-1}\left(c_i^2 + ic_i\right)$ [18, Theorem 22]. Therefore, Anderson's theorem is equivalent to computing the number of integer points in $\mathrm{SC}_a(b)$, and Johnson's theorem is equivalent to computing $\sum_{\mathbf{c}\in \mathrm{SC}_a(b)} h_a(\mathbf{c})$.

Johnson computes this weighted sum of lattice points by relating it to a sum over the subset of integer points $(z_0,\ldots,z_{a-1})$ of the dilation of the standard simplex $b\Delta_{a-1}$ that satisfy $\sum iz_i \equiv 0 \pmod{a}$. Johnson then shows that the sum he needs to compute is equal to $1/a$ times the sum of a quadratic function w taken over all integer points of $b\Delta_{a-1}$. In order to conclude, he applies a result from Euler–Maclaurin theory, which is a version of the first part of Corollary 1.5, and also applies a version of weighted Ehrhart reciprocity that appears in [4].

By Corollary 2.7, there exists a family of weight lifting polytopes $P^*_\alpha \subset \mathbb{R}^{n^*}$ such that

$$\sum_{x\in b\triangle_{a-1}\cap\mathbb{Z}^a} w(x) = \sum_{\alpha\in I} c_\alpha \left| P^*_\alpha \cap \mathbb{Z}^{n^*} \right|.$$

It seems likely that further study of these kinds of weight lifting polytopes can lead to new techniques in the study of simultaneous core partitions.

Numerical Semigroups. A numerical semigroup S is an additive submonoid of $\mathbb{N}_0 = \{0,1,2,\ldots\}$ with finite complement. The elements of $\mathbb{N}_0 \setminus S$ are the *gaps* of S, denoted $G(S) = \{h_1,\ldots,h_g\}$. The *weight* of S is defined by $w(S) = (h_1+\cdots+h_g) - (1+2+\cdots+g)$. The motivation for studying $w(S)$ comes from the theory of Weierstrass semigroups of algebraic curves [3, Chapter 1, Appendix E].

Numerical semigroups containing m are in bijection with integer points $(x_1,\ldots,x_{m-1})$ in the *Kunz polyhedron* $P_m \subset \mathbb{R}^{m-1}$, which is defined via bounding inequalities

$$x_i + x_j \geq x_{i+j} \text{ if } i+j < m, \quad x_i + x_j + 1 \geq x_{i+j-m} \text{ if } i+j > m.$$

Let $NS(m,g)$ be the set of numerical semigroups containing m with genus g. These semigroups are in bijection with the integer points of $P_{m,g}$, the polytope we get from P_m by adding the additional constraint $\sum x_i = g$. For a more extensive discussion of the connection between numerical semigroups containing m and integer points in the Kunz polyhedron, see [19, Section 4]. If $(k_1, \ldots, k_{m-1})$ is the integer point corresponding to a semigroup S of genus g, then $w(S) = \frac{m}{2}\sum_{i=1}^{m-1} k_i(k_i-1) + \sum_{i=1}^{m-1} ik_i - g(g+1)/2$. There has been recent interest in statistical properties of weights of families of semigroups, see for example [21, Section 5] and [20].

By Corollary 2.7, there exists a family of weight lifting polytopes $P_\alpha^* \subset \mathbb{R}^{n^*}$ such that

$$\sum_{S \in NS(m,g)} w(S) = \sum_{S \in P_{m,g} \cap \mathbb{Z}^{m-1}} w(S) = \sum_{\alpha \in I} c_\alpha \left| P_\alpha^* \cap \mathbb{Z}^{n^*} \right|.$$

Studying this family of polytopes and applying a version of Corollary 1.5 suggests an approach to the following two questions:

1. What is the maximum of $w(S)$ for $S \in NS(m,g)$?
2. For fixed m, what is the main term in the expression for $\sum_{S \in NS(m,g)} w(S)$ as $g \to \infty$?

3.2 Weighted Ehrhart in combinatorial representation theory

There is a long tradition of using lattice points of polytopes in representation theory (see [15] and the references there). Here, as an application of Theorem 1.1, we provide new connections.

Maximizing Kostka numbers. Fix a partition $\lambda \vdash n$ and let $SSYT(\lambda)$ denote the set of semi-standard Young tableaux of shape λ. The Schur function s_λ is

$$s_\lambda(x) = \sum_{T \in SSYT(\lambda)} x^T = \sum_{\alpha \in \mathrm{comp}(n)} K_{\lambda\alpha} x^\alpha,$$

where $\mathrm{comp}(n)$ is the set of weak compositions n and $K_{\lambda\alpha}$ is the *Kostka number* that counts the number of tableaux in $SSYT(\lambda)$ with content α. Evaluating s_λ at $x_1 = 1, x_2 = 1, \ldots, x_N = 1, x_{N+1} = 0, x_{N+2} = 0, \ldots$ yields

$$|SSYT(\lambda, N)| = \sum_{\alpha \in N\text{-comp}(n)} K_{\lambda\alpha},$$

where $SSYT(\lambda, N)$ is the set of semi-standard Young tableaux of shape λ and entries bounded by N and N-comp(n) is the set of weak composition of n with N parts.

A weak composition of n with N parts is a lattice point in the scaled standard $(N-1)$-simplex $n\triangle_{N-1}$. The Kostka number $K_{\lambda\alpha}$ equals the number of lattice points in the Gelfand–Tsetlin polytope $GT(\lambda, \alpha)$ (see e.g., [15]), so $w(\alpha) = K_{\lambda\alpha}$ is a weight function. There have been contributions to understanding the behavior of $K_{\lambda\alpha}$ as (λ, α) vary and an example is [17] in which it is shown that they are log-concave. Applying the method in Corollary 1.5 one can use the weight lifting polytope given by Theorem 1.1 to compute $max_{\alpha \in N\text{-comp(n)}} K_{\lambda\alpha}$ and we will include generated data in the full version of this paper.

Robinson–Schensted–Knuth (RSK) identity. Fix partitions $\mu, \nu \vdash n$ and recall the famous *RSK identity* (for details see e.g., [23]):

$$\sum_{\lambda \vdash n} K_{\lambda\mu} K_{\lambda\nu} = N_{\mu,\nu}.$$

The left sum is over partitions of n and the summands are products of Kostka numbers. In fact, the left side of the identity is a weighted sum over the lattice points of

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 + \cdots + x_n = n, x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}. \tag{3.1}$$

This is because the weight function $w(\lambda) = K_{\lambda\mu} K_{\lambda\nu}$ is the number of lattice points in the Cartesian product $GT(\lambda, \mu) \times GT(\lambda, \nu)$ of two Gelfand–Tsetlin polytopes. The right-hand side of RSK, $N_{\mu\nu}$, is the number of lattice points in the transportation polytope

$$Mat_{n,n}(\mu, \nu) = \left\{ (z_{ij})_{1 \leq i,j \leq n} \mid \sum_j z_{ij} = \mu_i, \sum_i z_{ij} = \nu_j, z_{ij} \geq 0 \right\}.$$

While RSK provides more information (e.g., a bijection), Theorem 1.1 gives a new polytope whose number of lattice points is the sum $\sum_{\lambda \vdash n} K_{\lambda\mu} K_{\lambda\nu}$.

Corollary 3.1 (A new RSK-like identity)**.** *There exists a weight lifting polytope $P^*(\mu, \nu) \subseteq \mathbb{R}^{n^2+2n}$ which is combinatorially different from $Mat_{n,n}(\mu, \nu)$ such that*

$$\sum_{\lambda \vdash n} K_{\lambda\mu} K_{\lambda\nu} = |P^*(\mu, \nu) \cap \mathbb{Z}^{n^2+2n}|.$$

Littlewood–Richardson Coefficients. Schur functions are central objects in representation theory and combinatorics. The skew Schur function for partitions $\lambda, \mu \vdash n$ is

$$s_{\lambda/\mu}(x) = \sum_{\alpha \in \mathrm{comp}(n)} K_{\lambda/\nu,\alpha} x^\alpha,$$

where the sum is over all compositions of n and $K_{\lambda/\nu,\alpha}$ counts the number of skew semi-standard Young tableaux of shape λ/ν and weight α. The Littlewood–Richardson rule (see e.g., [24]) expresses the skew Schur functions in terms of Schur functions,

$$s_{\lambda/\mu}(x) = \sum_{\nu \vdash n} c^\lambda_{\mu\nu} s_\nu(x).$$

Comparing the expression of the coefficient of the monomial x^α yields

$$K_{\lambda/\nu,\alpha} = \sum_{\nu \vdash n} c^\lambda_{\mu\nu} K_{\nu\alpha}.$$

The Littlewood–Richardson coefficient $c^{\lambda}_{\mu\nu}$ counts the number of lattice points in the *hive polytope* $H^{\lambda}_{\mu\nu}$ (see e.g., [9]). Applying Theorem 1.1 to the simplex in (3.1) and $w(\nu) = c^{\lambda}_{\mu\nu}K_{\nu\alpha}$, which counts the number of lattice points in $H^{\lambda}_{\mu\nu} \times GT(\nu, \alpha)$, we obtain the following corollary.

Corollary 3.2. *There exists a weight lifting polytope* $P^*(\lambda/\mu, \alpha) \subseteq \mathbb{R}^{n^2+2n}$ *such that*

$$\sum_{\lambda \vdash n} c^{\lambda}_{\mu\nu}K_{\nu\alpha} = |P^*(\lambda/\mu, \alpha) \cap \mathbb{Z}^{n^2+2n}|.$$

Acknowledgements

We are truly grateful to Ardila who introduced us to his results with Brugallé that we have here extended to quasi-polynomial weights. Part of this work was done while most of the authors visited the American Institute of Mathematics in Palo Alto.

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