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Noncrossing partitions of an annulus

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Abstract. The noncrossing partition poset associated to a Coxeter group W and Coxeter element c is the interval $[1, c]_T$ in the absolute order on W. We construct a new model of noncrossing partitions for W of classical affine type, using planar diagrams. The model in type $\widetilde{\mathrm{A}}$ consists of noncrossing partitions of an annulus. In type $\widetilde{\mathrm{C}}$, the model consists of symmetric noncrossing partitions of an annulus or noncrossing partitions of a disk with two orbifold points. Following the lead of McCammond and Sulway, we complete $[1, c]_T$ to a lattice by factoring the translations in $[1, c]_T$, but the combinatorics of the planar diagrams leads us to make different choices about how to factor.

Résumé. Le poset de partitions noncroisées associé à un groupe de Coxeter W et un element c de Coxeter est l'intervalle $[1, c]_T$ dans l'order absolut sur W. Nous construisons un nouveau modèle de partitions noncroisées dans le type affine, en utilisant des diagrammes planaires. Le modèle de type $\widetilde{\mathrm{A}}$ se compose des partitions noncroisées d'un anneau. Le modèle de type $\widetilde{\mathrm{C}}$ se compose des partitions symmetriques noncroisées d'un anneau ou des partitions noncroisées d'un disque avec deux points d'orbifold. En suivant l'exemple de McCammond et Sulway, nous complétons $[1, c]_T$ à un treillis en factorisant les translations dans $[1, c]_T$, mais le combinatoire des diagrammes planaires nous conduit à faire dex choix différents sur la manière de factoriser.

Keywords: absolute order, affine Coxeter group, annulus, noncrossing partition

1 Introduction

Noncrossing partitions associated to a Coxeter group are algebraic/combinatorial objects that figure prominently in the analysis of the associated Artin group. In the classical finite types A, B, and D, noncrossing partitions are best understood in terms of certain planar diagrams. The full version [10] of this paper and its sequel [11] extend planar diagrams for noncrossing partitions to the classical affine types. Furthermore, in [20], which can be thought of as a "prequel" to these papers, the combinatorics of these planar diagrams is generalized to the setting of marked surfaces (in the sense of cluster algebras). Planar diagrams for noncrossing partitions of types A and $\widetilde{\mathrm{A}}$ are generalized

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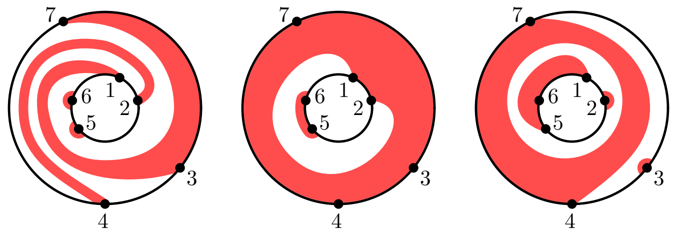


Figure 1: Some noncrossing partitions of an annulus

by noncrossing partitions of marked surfaces (without "punctures"), while in the other classical finite and affine types, the generalization is in terms of symmetric noncrossing partitions of a marked surface with (or without) double points.

Noncrossing partitions of a cycle were introduced by Kreweras [14]. Biane [3] connected noncrossing partitions of a cycle to finite Coxeter groups of type A (the symmetric groups) and showed that the lattice of noncrossing partitions is isomorphic to $[1, c]_T$, the interval between the identity and a Coxeter element c in the absolute order. The analogous interval in a finite Coxeter group of type B is modeled by centrally symmetric noncrossing partitions [1, 21], and there is an analogous planar model [1] for type D. These planar models can be understood uniformly in terms of the Coxeter plane, which was defined and studied in [12, 22]. Specifically, realizing group elements as permutations of an orbit, one projects to the Coxeter plane and interprets the cycle structure of the permutations as blocks in a partition [19].

The interval $[1, c]_T$ in a finite Coxeter group W serves as a Garside structure for the corresponding spherical Artin group $\mathrm{Art}(W)$, leading to a "dual presentation" of $\mathrm{Art}(W)$ [2, 7] and proving desirable properties of $\mathrm{Art}(W)$. The fact that $[1, c]_T$ is a lattice is crucial to the dual presentation.

Outside of finite type, the interval $[1, c]_T$ need not be a lattice. The case of affine type is treated in a series of papers that begins by extending crucial results on rigid motions [6] to affine type [5] and continues with an analysis of the failure of the lattice property in Coxeter groups of affine type [16]. The series culminates in [17], in which McCammond and Sulway extend the affine Coxeter group W to a larger group, thereby extending the interval $[1, c]_T$ to a lattice. The larger lattice serves as a Garside structure for a supergroup of the Euclidean Artin group $\mathrm{Art}(W)$, which inherits desirable (previously conjectured) properties from the supergroup.

In this extended abstract, we describe planar models, in affine types $\widetilde{\mathrm{A}}$ and $\widetilde{\mathrm{C}}$, for the intervals $[1, c]_T$ and the larger lattices constructed by McCammond and Sulway, **omitting all proofs**. In type $\widetilde{\mathrm{A}}$, the resulting model consists of ***noncrossing partitions of an annulus***, defined in Section 3 and exemplified in Figure 1. In type $\widetilde{\mathrm{C}}$, the model consists of ***symmetric noncrossing partitions of an annulus***, defined in Section 4. Some examples are shown in the top row of Figure 2. To see the symmetry, think of the pictures as

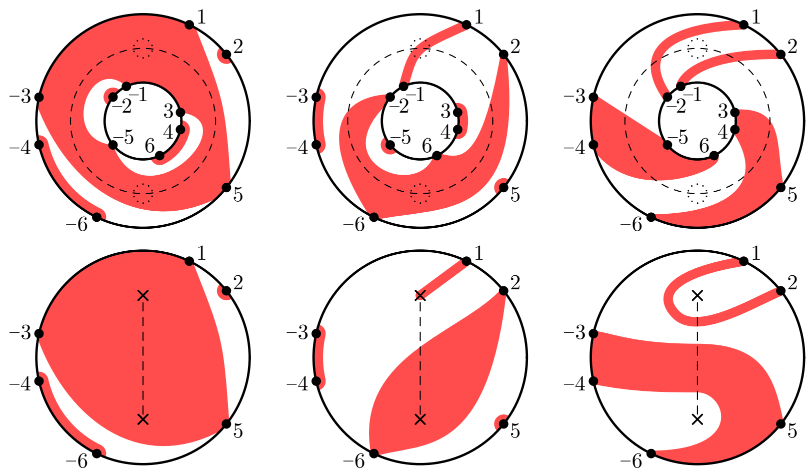


Figure 2: Some symmetric noncrossing partitions of an annulus and corresponding noncrossing partitions of the two-orbifold disk

perspective drawings of a circular cylinder; the symmetry is a rotation of the cylinder along a vertical axis. (The intersections of this axis with the cylinder are indicated in the pictures by small dotted circles). Symmetric noncrossing partitions of an annulus can equivalently be represented as ***noncrossing partitions of a two-orbifold disk***, as shown in the bottom row of Figure 2. The fixed points of the symmetry in the annulus become order-2 orbifold points in the disk, marked with an $\times$ in the pictures. The dotted circle cuts the annulus into two pieces (a fundamental domain of the symmetry and its one translate). The image of the circle under the quotient map is shown as a dotted line.

The key idea behind McCammond and Sulway's extension of $[1, c]_T$ to a lattice is the notion of "factoring" translations in an affine Coxeter group W. Certain elements of W act (in the affine representation) as translations, and the translations in $[1, c]_T$ were implicated (in [16]) in the failure of the lattice property. To overcome the failure, each translation in $[1, c]_T$ is factored into two or more translations not already contained in W. The larger group containing W is generated by W and these new ***factored translations***.

One of the motivating questions for the current project was how factored translations would fit into planar models. Would projecting the Coxeter plane lead to a useful model of $[1, c]_T$ only, or of the larger interval (incorporating factored translations) only, or of both? In type $\widetilde{\mathrm{A}}$, the best possible thing happened: The planar diagrams model the larger interval, and the elements of the smaller interval are distinguished by a simple criterion. The elements of $[1, c]_T$ are the noncrossing partitions of the annulus with no ***dangling annular blocks***. A dangling annular block is a block that is an annulus and has numbered points on only one component of its boundary. Thus the leftmost and middle picture in Figure 1 represent elements of $[1, c]_T$, while the rightmost picture represents an element in the larger interval, not in $[1, c]_T$.

We pause to mention two aspects of the project that are omitted from this extended abstract: First, we omit the construction of the Coxeter plane and the projection of an orbit to the Coxeter plane, which lead naturally to a model on the annulus. Second, we omit a detailed discussion connecting our constructions to the work of McCammond and Sulway. (We only mention here that the planar model on the annulus suggests a scheme for factoring translations that is different from that in [17].)

Remark 1.1. Most of the results of [10, 11] reported here have appeared in Laura Brestensky's thesis [9]. Some parts of [10, 11] (and thus of this extended abstract) are revisions of material from [9].

Remark 1.2. Some other constructions are superficially or more deeply similar to the constructions in this paper. Details are in [10], but here we mention [13, 18, 15].

2 Background

Given a Coxeter system (W,S), a ***Coxeter element*** is an element $c \in W$ that can be written as a product, in some chosen order, of the elements of S, each repeated exactly once in the product. The set of ***reflections*** of W is $T = \{wsw^{-1} : w \in W, s \in S\}$. The ***absolute length*** or ***reflection length*** $\ell_T(w)$ of an element $w \in W$ is the number of letters in a shortest expression for w as a product of elements of T. The ***absolute order*** $\leq_T$ is the partial order on W defined by $u \leq_T w$ if and only if $\ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$.

When W is finite, the interval $[1,c]_T$ is a lattice [8], often called the W-***noncrossing partition lattice***, but when W is infinite, $[1,c]_T$ can fail to be a lattice. Planar models of the W-noncrossing partition lattice are well known in types A, B, and D [1, 3, 14, 21]. In [10] and [11], we construct planar diagrams for $[1,c]_T$ in the classical affine types.

The Coxeter group W of type $\widetilde{\mathrm{A}}_{n-1}$ can be realized as the group $\widetilde{S}_n$ of ***affine permutations***. (See, e.g. [4, Section 8.3].) These are permutations π of $\mathbb{Z}$ such that $\pi(i+n) = \pi(i) + n$ for all $i \in \mathbb{Z}$ and $\sum_{i=1}^{n} \pi(i) = \binom{n+1}{2}$. We also consider the larger group $S_{\mathbb{Z}}(\mathrm{mod}\ n)$ (sometimes called the extended affine symmetric group) of ***periodic permutations***: permutations π of $\mathbb{Z}$ such that $\pi(i+n) = \pi(i) + n$ for all $i \in \mathbb{Z}$.

We will describe permutations $\pi \in S_{\mathbb{Z}}(\mathrm{mod}\ n)$ by their cycle structure. A finite cycle in π may not contain two entries that are equivalent modulo n, and for every finite cycle $(a_1\ \ a_2\ \cdots\ a_k)$ in π, the cycle $(a_1 + \ell n\ \ a_2 + \ell n\ \cdots\ a_k + \ell n)$ is also present in π for every $\ell \in \mathbb{Z}$. We write $(a_1\ \ a_2\ \cdots\ a_k)_n$ for the infinite product $\prod_{\ell \in \mathbb{Z}} (a_1 + \ell n\ \ a_2 + \ell n\ \cdots\ a_k + \ell n)$ of cycles. An infinite cycle in π necessarily has two entries that are equivalent modulo n, and, for any entry a_1 in the cycle, the cycle is completely determined by the sequence of entries from a_1 to the next entry a_{k+1} that is equivalent to a_1 modulo n. Thus we can represent infinite cycles in π by writing $(\cdots\ a_1\ \ a_2\ \cdots\ a_{k+1}\ \cdots)$.

A periodic permutation π is determined by the entries $\pi(1), \pi(2), \ldots, \pi(n)$, sometimes called the "window" of π. The affine permutations are the periodic permutations whose window sums to $\binom{n+1}{2}$.

Write $\ell_i = (\cdots\ i\ \ i+n\ \cdots)$ for $i = 1, \ldots, n$ and $L = \{\ell_1, \ldots, \ell_n\} \cup \left\{\ell_1^{-1}, \ldots, \ell_n^{-1}\right\}$. One can show that $S_\mathbb{Z}(\text{mod } n)$ is generated by $T \cup L$. We define a partial order $\le_{T\cup L}$ on $S_\mathbb{Z}(\text{mod } n)$ in the same way as the absolute order $\le_T$ on $\widetilde{S}_n$. We write $[1,c]_{T\cup L}$ for the interval between 1 and c in this order, where the subscript specifies not only the generating set, but indirectly also the group where the interval lives.

The Coxeter diagram of type $\widetilde{A}_{n-1}$ is a cycle on the simple reflections $\{s_1, \ldots, s_n\}$, with each s_i adjacent to s_{i-1} and s_{i+1} (taking indices modulo n). We record the choice of a Coxeter element c as follows: The numbers $1, \ldots, n$ are placed in clockwise order over one full turn about the center of the annulus. The number i is placed on the outer boundary if and only if s_{i-1} precedes s_i in c or on the inner boundary if and only if s_i precedes s_{i-1} in c. We identify the numbers $1, \ldots, n$ with their positions on the boundary of the annulus, and call them ***outer points*** or ***inner points***. This construction creates a bijection from the set of Coxeter elements to the set of all partitions of $\{1, \ldots, n\}$ into a nonempty set of outer points and a nonempty set of inner points.

Example 2.1. The pictures in Figure 1 shows the case where $W = \widetilde{S}_7$ and the Coxeter element is $c = s_6s_5s_2s_1s_3s_4s_7 = (\cdots\ 3\ \ 4\ \ 7\ \ 10\ \cdots)(\cdots\ 6\ \ 5\ \ 2\ \ 1\ \ {-1}\ \cdots)$.

Lemma 2.2. *Let c be a Coxeter element of $\widetilde{S}_n$, represented as partition of $\{1, \ldots, n\}$ into inner points and outer points. If $a_1, \ldots, a_k$ are the outer points in increasing (therefore clockwise) order and $b_1, \ldots, b_{n-k}$ are the inner points in decreasing (therefore counterclockwise) order, then c is*

$$(\cdots\ a_1\ \ a_2\ \cdots\ a_k\ \ a_1+n\ \cdots)(\cdots\ b_1\ \ b_2\ \cdots\ b_{n-k}\ \ b_1-n\ \cdots).$$

3 Noncrossing partitions of an annulus

Fix a Coxeter element c in $\widetilde{S}_n$, encoded by a choice of inner and outer points on an annulus as described in Section 2. We refer to this annulus as A. The inner and outer points are collectively called ***numbered points***. The pair consisting of A and the set of numbered points is an example of a marked surface in the sense of [20] (where the numbered points were called marked points). The results quoted here for the annulus were established in [9] and are quoted in [10] as special cases of results of [20].

A ***boundary segment*** of A is the portion of the boundary of A between two adjacent numbered points. A boundary segment may have both endpoints at the same numbered point if that is the only numbered point on one component of the boundary of A. Two subsets of A are related by ***ambient isotopy*** if they are related by a homeomorphism from A to itself, fixing the boundary ∂A pointwise and homotopic to the identity by a homotopy that fixes ∂A pointwise at every step.

An ***arc*** in A is a non-oriented curve in A, with endpoints at numbered points (possibly coinciding), that does not intersect itself except possibly at its endpoints and, except for its endpoints, does intersect the boundary of A. We exclude the possibility that an arc bounds a monogon in A and the possibility that an arc combines with a boundary segment to bound a digon in A. An ***embedded block*** in A is one of the following:

- a ***trivial block***, meaning a singleton consisting of a numbered point in A;
- an arc or boundary segment in A;
- a ***disk block***, meaning a closed disk in A whose boundary is a union of arcs and/or boundary segments of A;
- a ***dangling annular block***, meaning a closed annulus in A with one component of its boundary a union of arcs and/or boundary segments of A and the other component of its boundary a circle in the interior of the A that can't be contracted in A to a point; or
- a ***nondangling annular block***, meaning a closed annulus in A with each component of its boundary a union of arcs and/or boundary segments of A.

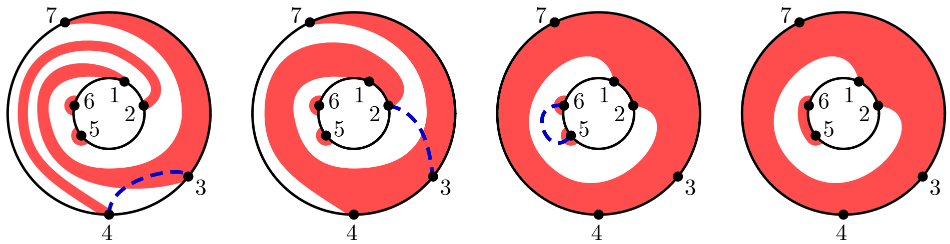
The first two types of embedded blocks are ***degenerate disk blocks***. The last two types of embedded blocks are referred to less specifically as ***annular blocks***.

A ***noncrossing partition*** of A is a collection $\mathcal{P} = \{E_1, \ldots, E_k\}$ of disjoint embedded blocks (the ***blocks of*** $\mathcal{P}$) such that every numbered point is contained in some E_i and at most one E_i is an annular block. Given that the blocks are pairwise disjoint, restricting to at most one annular block serves only to rule out the case of two dangling annular blocks, one containing numbered points on its inner boundary and the other containing numbered points on its outer boundary. Noncrossing partitions are considered up to ambient isotopy. We refer to different, but isotopic, choices of the embedded blocks $E_1, \ldots, E_k$ as different ***embeddings*** of the same noncrossing partition.

Example 3.1. We can now give more context to Figure 1. These are noncrossing partitions of the annulus from Example 2.1. Degenerate blocks are shown with some thickness, to make them visible.

We define a partial order on noncrossing partitions of A, called (in light of Theorem 3.3 below) the ***noncrossing partition lattice***: Noncrossing partitions $\mathcal{P}$ and $\mathcal{Q}$ of A have $\mathcal{P} \leq \mathcal{Q}$ if and only if there exist embeddings of $\mathcal{P}$ and $\mathcal{Q}$ such that every block of $\mathcal{P}$ is contained in some block of $\mathcal{Q}$. We write $\widetilde{NC}_c^A$ for the set of noncrossing partitions of A with this partial order.

Example 3.2. Continuing Example 3.1, we see that the left noncrossing partition shown in Figure 1 is less than the middle noncrossing partition, but that is the only order relation among the three noncrossing partitions shown.



$\mathcal{Q}_1$ $\mathcal{Q}_2$ $\mathcal{Q}_3$ $\mathcal{Q}_4$

Figure 3: $\mathcal{Q}_1 \lessdot \mathcal{Q}_2 \lessdot \mathcal{Q}_3 \lessdot \mathcal{Q}_4$, with simple connectors

Theorem 3.3. *The poset $\widetilde{NC}_c^A$ of noncrossing partitions of an annulus with marked points on both boundaries, n marked points in all, is a graded lattice, with rank function given by n minus the number of non-annular blocks.*

Given $\mathcal{P} \in \widetilde{NC}_c^A$, a ***simple connector for*** $\mathcal{P}$ is an arc or boundary segment κ in the annulus that starts in some block E of $\mathcal{P}$, leaves E, and is disjoint from all other blocks until it enters some block of E' of $\mathcal{P}$, which it does not leave again. (Possibly $E = E'$.) The ***augmentation of*** $\mathcal{P}$ ***along*** κ, written $\mathcal{P} \cup \kappa$, is the smallest noncrossing partition containing $\mathcal{P}$ and α. A more detailed definition is given in [10].

Proposition 3.4. *If $\mathcal{P}, \mathcal{Q} \in \widetilde{NC}_c^A$, then $\mathcal{P}$ is covered by $\mathcal{Q}$ if and only if there exists a simple connector κ for $\mathcal{P}$ such that $\mathcal{Q} = \mathcal{P} \cup \kappa$.*

Example 3.5. Figure 3 shows noncrossing partitions $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, and $\mathcal{Q}_4$ with $\mathcal{Q}_1 \lessdot \mathcal{Q}_2 \lessdot \mathcal{Q}_3 \lessdot \mathcal{Q}_4$. A simple connector is shown for each cover relation.

Suppose $\mathcal{P} \in \widetilde{NC}_c^A$ and λ is a non-self-intersecting curve contained in some block E of $\mathcal{P}$, with each endpoint at a non-numbered point on the boundary E. Then λ is a ***cutting curve*** for $\mathcal{P}$ if it either (when E is annular or non-annular) cuts E into two pieces, each having at least one numbered point, or (when E is annular), cuts E into one non-annular piece. We allow the degenerate case where E is an arc or boundary segment and λ is a non-numbered point in E. Write $\mathcal{P} - \lambda$ for the noncrossing partition obtained from $\mathcal{P}$ by cutting E in this way. A more detailed definition is given in [10].

Proposition 3.6. *If $\mathcal{P}, \mathcal{Q} \in \widetilde{NC}_c^A$, then $\mathcal{P} \lessdot \mathcal{Q}$ if and only if there exists a cutting curve for $\mathcal{Q}$ such that $\mathcal{P} = \mathcal{Q} - e$.*

Example 3.7. Figure 4 shows the same noncrossing partitions $\mathcal{Q}_1 \lessdot \mathcal{Q}_2 \lessdot \mathcal{Q}_3 \lessdot \mathcal{Q}_4$ pictured in Figure 3, this time with the relevant cutting curves shown.

Write $\widetilde{NC}_c^{A,\circ}$ for the subposet of $\widetilde{NC}_c^A$ consisting of noncrossing partitions of A with no dangling annular blocks.

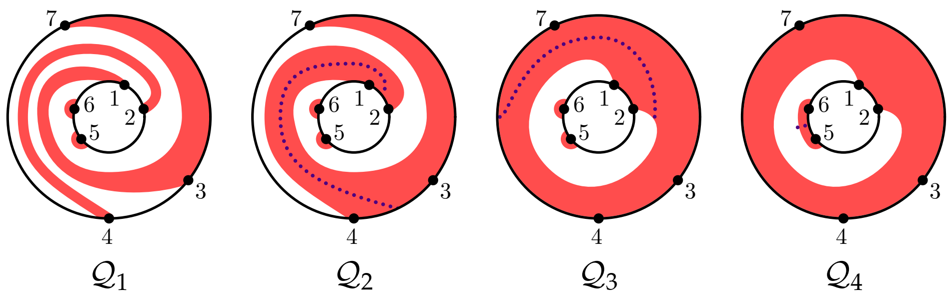


Figure 4: $\mathcal{Q}_1 \lessdot \mathcal{Q}_2 \lessdot \mathcal{Q}_3 \lessdot \mathcal{Q}_4$, with cutting curves

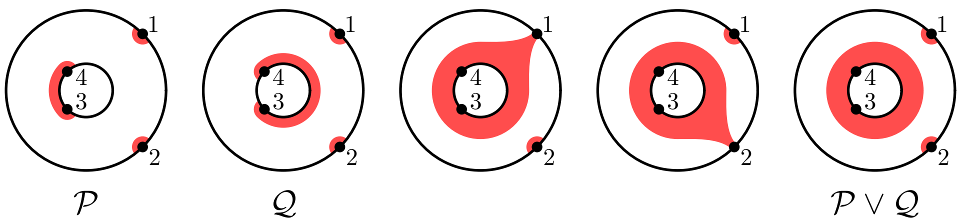


Figure 5: Joins in $\widetilde{NC}_c^A$ may need dangling annular blocks

Proposition 3.8. *Suppose $\mathcal{P}, \mathcal{Q} \in \widetilde{NC}_c^{A,\circ}$. Then $\mathcal{Q}$ covers $\mathcal{P}$ in $\widetilde{NC}_c^{A,\circ}$ if and only if $\mathcal{Q}$ covers $\mathcal{P}$ in $\widetilde{NC}_c^A$.*

Example 3.9. The poset $\widetilde{NC}_c^{A,\circ}$ (noncrossing partitions of the annulus with no dangling annular blocks) can fail to be a lattice. Take $n = 4$ and $c = s_4 s_3 s_1 s_2$. The left two pictures in Figure 5 show noncrossing partitions $\mathcal{P}, \mathcal{Q} \in \widetilde{NC}_c^{A,\circ}$. The next two pictures show minimal upper bounds for $\mathcal{P}$ and $\mathcal{Q}$ in $\widetilde{NC}_c^{A,\circ}$. The rightmost picture shows the join of $\mathcal{P}$ and $\mathcal{Q}$ in $\widetilde{NC}_c^A$, which is not in $\widetilde{NC}_c^{A,\circ}$ because it has a dangling annular block.

The ***Kreweras complement*** is an anti-automorphism $\mathsf{Krew} : \widetilde{NC}_c^A \to \widetilde{NC}_c^A$. For each numbered point i, choose a point i' on the boundary of the annulus A between i and and the next numbered point clockwise of i (for i outer) or counterclockwise of i (for i inner). Given $\mathcal{P} \in \widetilde{NC}_c^A$, first construct a noncrossing partition $\mathcal{P}'$ on the annulus with numbered points $1', \ldots, n'$ whose blocks are the maximal embedded blocks that are disjoint from the nontrivial blocks of $\mathcal{P}$, together with trivial blocks $\{i'\}$ for every i' contained in a block of $\mathcal{P}$. Then $\mathsf{Krew}(\mathcal{P})$ is obtained from $\mathcal{P}'$ by applying a homeomorphism of A that rotates each i' back to i while moving interior points of A as little as possible.

Example 3.10. Figure 6 shows the Kreweras complements of the noncrossing partitions shown in Figure 1.

We now define the map $\mathsf{perm} : \widetilde{NC}_c^A \to S_{\mathbb{Z}}(\text{mod } n)$ that will serve as an isomorphism to $[1, c]_{T \cup L}$. Given $\mathcal{P} \in \widetilde{NC}_c^A$, we define $\mathsf{perm}(\mathcal{P})$ by reading the cycle notation of a permutation from the embedding of the partition in the annulus. Each component

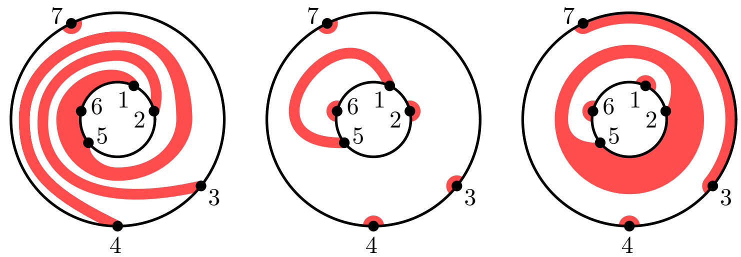


Figure 6: Kreweras complements of the noncrossing partitions from Figure 1
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of the boundary of each block is read as a cycle by following the boundary, keeping the interior of the block on the right. (Each degenerate disk block is first thickened to a disk with one or two numbered points on its boundary, and is thus read as a singleton cycle or transposition.) The entries in the cycle are the numbered points encountered along the boundary, but adding a multiple of n to each, with the multiple of n increases or decreases according to how the block wraps around the annulus. Specifically, the multiple of n changes when we cross the ***date line***, a radial line segment separating n and 1. When a numbered point is reached, we record its value (between 1 and n) plus wn, where w is the number of times we have crossed the date line clockwise minus the number of times we have crossed counterclockwise, since starting to read the cycle. When we return to the numbered point where we started, if we have added or subtracted a nonzero multiple of n, then we continue around the boundary again, obtaining an infinite cycle. Otherwise, if we have recorded the values $a_1, a_2, \ldots, a_k$ around the boundary, we obtain the infinite product $(a_1\ a_2\ \cdots\ a_k)_n$ of cycles. More specifically, a disk block becomes such an infinite product of finite cycles, a dangling annular block becomes a single infinite cycle (either increasing or decreasing), and a non-dangling annular block becomes a pair of infinite cycles, one increasing and one decreasing.

Example 3.11. Labeling the noncrossing partitions shown in Figure 1 (Example 3.1) as $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ from left to right, we apply perm to obtain the following permutations in $\widetilde{S}_7$:

$$\begin{aligned}
\mathsf{perm}(\mathcal{P}_1) &= (1\ \ {-8}\ \ {-4})_7\,(2\ \ {-3})_7\,(5)_7\,(6)_7\\
\mathsf{perm}(\mathcal{P}_2) &= (\cdots\ 1\ \ {-5}\ \ {-6}\ \cdots)\,(\cdots\ 3\ \ 4\ \ 7\ \ 10\ \cdots)\,(5\ \ 6)_7\\
\mathsf{perm}(\mathcal{P}_3) &= (1\ \ {-1}\ \ {-2})_7\,(2)_7\,(3)_7\,(\cdots\ 4\ \ 7\ \ 11\ \cdots)
\end{aligned}$$

Theorem 3.12. *The map* $\mathsf{perm} : \widetilde{NC}_c^A \to S_{\mathbb{Z}}(\mathrm{mod}\ n)$ *is an isomorphism from* $\widetilde{NC}_c^A$ *to the interval* $[1, c]_{T \cup L}$ *in* $S_{\mathbb{Z}}(\mathrm{mod}\ n)$. *It restricts to an isomorphism from* $\widetilde{NC}_c^{A,\circ}$ *to the interval* $[1, c]_T$ *in* $\widetilde{S}_n$.

Theorem 3.13. *If* $\mathcal{P} \in \widetilde{NC}_c^A$, *then* $\mathsf{perm}(\mathsf{Krew}(\mathcal{P})) = \mathsf{perm}(\mathcal{P})^{-1}c$.

4 Noncrossing partitions of a two-orbifold disk

Using a standard technique of "folding" a Coxeter group of type $\widetilde{\mathrm{A}}$, we realize the Coxeter group of type $\widetilde{\mathrm{C}}_{n-1}$ as the group $\widetilde{S}^{\mathrm{s}}_{2n}$ of ***affine signed permutations*** (permutations $\pi : \mathbb{Z} \to \mathbb{Z}$ with $\pi(i+2n) = \pi(i) + 2n$ and $\pi(-i) = -\pi(i)$ for all $i \in \mathbb{Z}$). A Coxeter element c in $\widetilde{S}^{\mathrm{s}}_{2n}$ is encoded by a decomposition of $\{\pm 1, \ldots, \pm(n-1)\}$ into outer and inner points, with i outer if and only if $-i$ is inner.

A corresponding "folding" of planar diagrams realizes the interval $[1, c]_T$ as the set of noncrossing partitions that are symmetric with respect to a certain symmetry ϕ. Viewing pictures of the annulus as perspective drawings of a cylinder, ϕ rotates this cylinder a half-turn about a vertical axis. In the top-row pictures in Figure 2, the two intersections of the axis with the cylinder are marked with small dotted circles. The dashed circle cuts the annulus into two pieces, with the action of ϕ mapping each piece to the other. Thus the quotient of the annulus modulo ϕ is obtained from the annulus by deleting everything inside the dashed circle and identifying each point on the dashed circle with the other point at the same height in the picture. The result is a disk C with two orbifold points, called the ***two-orbifold disk***, which appears in the bottom-row pictures in Figure 2.

An ***arc*** in C is a curve in C that does not intersect itself except possibly at its endpoints, does not intersect the boundary or orbifold points of C, except possibly at its endpoints, and is of one of the following two types: An ***ordinary arc*** has both endpoints on the boundary of C, while an ***orbifold arc*** has one endpoint on the boundary and the other at an orbifold point. An arc may not bound a monogon in C unless that monogon contains one or both orbifold points, and it may not combine with a boundary segment to bound a digon unless that digon contains one or both orbifold points. An ***embedded block*** in C is one of the following:

- a ***trivial block***, meaning a singleton consisting of a numbered point in C;
- an arc (ordinary or orbifold) or boundary segment in C; or
- a ***disk block***, meaning a closed disk in C whose boundary is a union of ordinary arcs and/or boundary segments of C.

The first two types of blocks are ***degenerate disk blocks***. An embedded block can contain one or both orbifold points.

A ***noncrossing partition*** of C is a collection of pairwise disjoint embedded blocks such that every numbered point is contained in one of the embedded blocks. Noncrossing partitions are considered up to ambient isotopy. Write $\widetilde{NC}^{C}_{c}$ for the set of noncrossing partitions of the two-orbifold disk C associated to the Coxeter element c, partially ordered with $\mathcal{P} \le \mathcal{Q}$ if and only if there exist embeddings of $\mathcal{P}$ and $\mathcal{Q}$ such that each block in $\mathcal{P}$ is a subset of a block in $\mathcal{Q}$.

Example 4.1. The noncrossing partitions of the two-orbifold disk on bottom row of Figure 2 exemplify the case where W is of type $\widetilde{C}_6$ and $c = s_6s_4s_3s_0s_1s_2s_5$. The top row shows the corresponding symmetric noncrossing partitions on an annulus.

Theorem 4.2. *$\widetilde{NC}^C_c$ is a graded lattice, with rank function given by*

$$(n-1) - (\#\text{ blocks of } \mathcal{P}) + (\#\text{ orbifold points enclosed by blocks of } \mathcal{P}).$$

Folding allows us to define a map $\mathsf{perm}^C : \widetilde{NC}^C_c \to \widetilde{S}^{\mathrm{s}}_{2n}$ such that the following theorem holds.

Theorem 4.3. *The map perm^C is an isomorphism from $\widetilde{NC}^C_c$ to $[1, c]_T$.*

The map perm^C reads clockwise around blocks of a noncrossing partition to define cycles, using the ***toggle line*** (shown dotted in the bottom-row pictures of Figure 2) and the ***date line*** (a segment down from the bottom orbifold point). We omit the details.

Example 4.4. Labeling the noncrossing partitions shown in Figure 2 as $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ from left to right, we apply perm^C to obtain the following permutations in $\widetilde{C}_6$:

$$\begin{aligned}
\mathsf{perm}^C(\mathcal{P}_1) &= ((\cdots\ 1\ \ 5\ \ 11\ \ 15\ \cdots))\,((2))_{14}\,((4\ \ 6))_{14}\\
\mathsf{perm}^C(\mathcal{P}_2) &= (1\ \ -1)_{14}\,(2\ \ 8\ \ 12\ \ 6)_{14}\,((3\ \ 4))_{14}\,((5))_{14}\\
\mathsf{perm}^C(\mathcal{P}_3) &= ((1\ \ -2))_{14}\,((5\ \ 8\ \ 4\ \ 3))_{14}
\end{aligned}$$

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