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Promotion for fans of Dyck paths

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Abstract. We construct an injection from the set of r-fans of Dyck paths of length n into the set of chord diagrams on $[n]$ that intertwines promotion and rotation. This is done in two different ways, namely as fillings of promotion matrices and in terms of Fomin growth diagrams. Our analysis uses the fact that r-fans of Dyck paths can be viewed as highest weight elements of weight zero in crystals of type B_r, which in turn can be analyzed using virtual crystals. On the level of Fomin growth diagrams, the virtualization process corresponds to the Roby–Krattenthaler blow up construction. Our construction generalizes to vacillating tableaux as well. We give a cyclic sieving phenomenon on r-fans of Dyck paths using the promotion action.

Keywords: Crystal bases, promotion, Fomin growth and chord diagrams, Dyck paths

1 Introduction

Interest in invariant subspaces goes back to Rumer, Teller and Weyl [22], who studied the quantum mechanical description of molecules. In particular, they devised diagrammatic bases for the invariant spaces. For $\mathrm{SL}(n)$, a diagrammatic basis for the invariant space was constructed by Cautis, Kamnitzer and Morrison [2], generalizing Kuperberg's webs [11] for $\mathrm{SL}(2)$ and $\mathrm{SL}(3)$.

The dimension of the invariant subspace of a tensor product $V^{\otimes N}$ of an irreducible representation V of a Lie algebra $\mathfrak{g}$ is equal to the number of highest weight elements of weight zero in $\mathcal{B}^{\otimes N}$, where $\mathcal{B}$ is the crystal basis associated to V [25, 19]. The symmetric group acts on $V^{\otimes N}$ by permuting tensor positions. By Schur–Weyl duality, this action commutes with the action of the Lie group. In particular, the symmetric group acts on the invariant space of $V^{\otimes N}$. It was shown by Westbury [25] that the action of the long cycle corresponds to the action of promotion on highest weight elements of weight zero in $\mathcal{B}^{\otimes N}$. In this setting promotion is defined using Henriques' and Kamnitzer's commutor [5], see [4, 25, 26].

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In general, it is desirable to have a correspondence between highest weight elements of weight zero in $\mathcal{B}^{\otimes N}$ and diagram bases, such as chord diagrams, which intertwine promotion and rotation. For Kuperberg's webs [11], this was achieved by Petersen, Pylyavskyy, Rhoades [18] and Patrias [17] by showing that the growth algorithm of Khovanov and Kuperberg [8] intertwines promotion with rotation. For the vector representation of the symplectic group and the adjoint representation of the general linear group, such a correspondence between highest weight elements of weight zero and chord diagrams which intertwines promotion and rotation was given in [19].

In this paper, we construct an injection from the set of r-fans of Dyck paths of length n into the set of chord diagrams on $[n]$ that intertwines promotion and rotation. There is a natural correspondence between r-fans of Dyck paths and highest weight elements in the tensor product of the spin crystal of type B_r. We present this injection in two different ways: (1) as fillings of promotion matrices [13] (see Section 3.1); (2) in terms of Fomin growth diagrams [3, 10] (see Section 3.2).

While the first description shows that the map intertwines promotion and rotation, the second description shows injectivity. Our proof strategy uses virtualization of crystals (see for example [1]) and results of [19] for oscillating tableaux of weight zero (or equivalently highest weight words of weight zero for the vector representation type C_r):

1. Find a virtual crystal morphism for the spin crystals of type B_r into the r-th tensor power of the crystal of the vector representation of type C_r (Section 2.1).

2. Use virtualization to map a fan of Dyck paths to an oscillating tableau (Section 2.2).

3. Show that this virtualization commutes with promotion and the filling rules.

4. Show that blowing up the filling of the growth diagram corresponds to the filling of the oscillating tableau. In this sense, the blow up on growth diagrams is the analogue of the virtualization on crystals.

An overview of our strategy is shown in Figure 1. Our analysis also generalizes to vacillating tableaux, details of which appear in the long version of this paper [16].

Having the injective map to chord diagrams gives a first step towards a diagrammatic basis for the invariant subspaces. In addition, Fontaine and Kamnitzer [4] as well as Westbury [25] tied the promotion action on highest weight elements of weight zero to the cyclic sieving phenomenon introduced by Reiner, Stanton and White [20]. In Section 4.2, we make this cyclic sieving phenomenon more concrete by providing the polynomial in terms of the energy function. We also conjecture another polynomial, which is the q-deformation of the number of r-fans of Dyck paths, to give a cyclic sieving phenomenon.

The paper is organized as follows. In Section 2, we give a brief review of crystal bases and virtual crystals and provide the virtual crystals for the spin representation of type B_r into type C_r. We also define promotion on crystals via the crystal commutor.

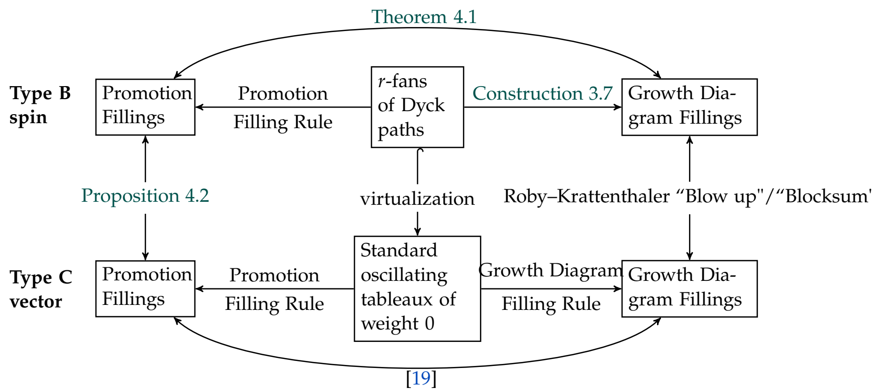

Figure 1: Overview of strategy and results for r-fans of Dyck paths

In Section 3, we give the various filling rules to construct the map to chord diagrams. Section 4 is reserved for the statements of our main results. The long version [16] of this extended abstract contains all proofs and more details on vacillating tableaux.

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2 Crystal bases

Crystal bases form a combinatorial skeleton of representations of quantum groups associated to Lie algebras. They were first introduced by Kashiwara [7] and Lusztig [14]. Axiomatically, for a given root system Φ with index set I and weight lattice Λ, a *crystal* is a nonempty set $\mathcal{B}$ together with maps $e_i, f_i \colon \mathcal{B} \to \mathcal{B} \sqcup \{\emptyset\}$, $\varepsilon_i, \varphi_i \colon \mathcal{B} \to \mathbb{Z}$, $\mathsf{wt} \colon \mathcal{B} \to \Lambda$ for $i \in I$, satisfying certain conditions (see for example [1, Definition 2.13]). The operators e_i and f_i are called *raising* and *lowering operators*. The map wt is the *weight map*. The map ε_i (resp. φ_i) measures how often e_i (resp. f_i) can be applied to the given crystal element. For all crystals considered in this paper, $\varepsilon_i(b) = \max\{k \geqslant 0 \mid e_i^k(b) \neq \emptyset\}$ for $b \in \mathcal{B}$ and similarly for $\varphi_i(b)$. An element $b \in \mathcal{B}$ is called *highest weight* if $e_i(b) = \emptyset$ for all $i \in I$.

Here we define certain crystals for the root systems B_r and C_r explicitly. Let $\mathbf{e}_i \in \mathbb{Z}^r$ be the i-th unit vector with 1 in position i and 0 everywhere else.

Definition 2.1. The *spin crystal* of type B_r, denoted by $\mathcal{B}_{\mathsf{spin}}$, consists of all r-tuples $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_r)$, where $\epsilon_i \in \{\pm\}$. The weight of ϵ is $\mathsf{wt}(\epsilon) = \frac{1}{2}\sum_{i=1}^r \epsilon_i \mathbf{e}_i$. The crystal operator f_r annihilates ϵ unless $\epsilon_r = +$. If $\epsilon_r = +$, f_r acts on ϵ by changing ϵ_r from $+$ to $-$ and leaving all other entries unchanged. The crystal operator f_i for $1 \leqslant i < r$ annihilates ϵ unless $\epsilon_i = +$ and $\epsilon_{i+1} = -$. In the latter case, f_i acts on ϵ by changing ϵ_i to $-$ and ϵ_{i+1} to $+$. The crystal operator e_i is defined similarly.

The crystal $\mathcal{B}_{\mathsf{spin}}$ of type B_3 is depicted in Figure 2.

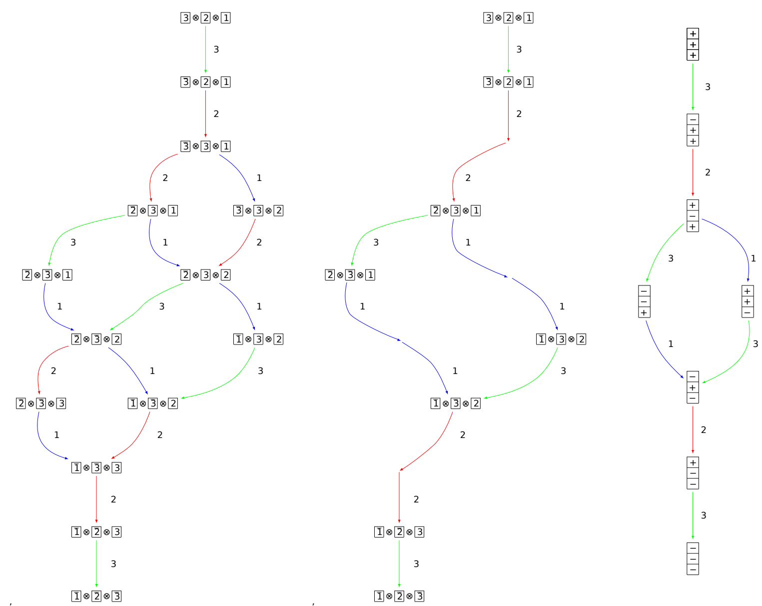


Figure 2: Left: One component of the crystal $\widehat{\mathcal{V}} = \mathcal{C}_{\square}^{\otimes 3}$ of type C_3. Middle: The virtual crystal $\mathcal{V}$ inside $\widehat{\mathcal{V}}$ of type B_3. Right: The spin crystal $\mathcal{B}_{\mathsf{spin}}$ of type B_3.

Definition 2.2. The *crystal for the vector representation* $\mathcal{C}_{\square}$ of type C_r consists of the elements $\{1, 2, \ldots, r, \overline{r}, \ldots, \overline{2}, \overline{1}\}$. The crystal operator f_i for $1 \leqslant i < r$ maps i to $i+1$, maps $\overline{i+1}$ to $\overline{i}$ and annihilates all other elements. The crystal operator f_r maps r to $\overline{r}$ and annihilates all other elements. The crystal operators e_i are defined similarly. Furthermore, $\mathrm{wt}(i) = \mathbf{e}_i$ and $\mathrm{wt}(\overline{i}) = -\mathbf{e}_i$.

A remarkable property of crystals is that they respect *tensor products*. Given two crystals $\mathcal{B}$ and $\mathcal{C}$ associated to the same root system Φ, the tensor product $\mathcal{B} \otimes \mathcal{C}$ as a set is the Cartesian product $\mathcal{B} \times \mathcal{C}$. The weight of $b \otimes c \in \mathcal{B} \otimes \mathcal{C}$ is the sum of the weights $\mathrm{wt}(b \otimes c) = \mathrm{wt}(b) + \mathrm{wt}(c)$. For more information, see [1, Section 2.3].

2.1 Virtual crystals

Stembridge [23] characterized crystals associated to simply-laced root systems in terms of local rules on the crystal graph. Crystals for non-simply-laced root systems can be constructed using virtual crystals [1, Chapter 5].

In this paper, we utilize virtual crystals to construct Fomin growth diagrams and the promotion operators for type B_r using results for type C_r. Hence let us briefly review the set-up for virtual crystals. Let $X \hookrightarrow Y$ be an embedding of Lie algebras such that the fundamental weights ω_i and simple roots α_i map as $\omega_i^X \mapsto \gamma_i \sum_{j \in \sigma(i)} \omega_j^Y$ and $\alpha_i^X \mapsto \gamma_i \sum_{j \in \sigma(i)} \alpha_j^Y$. Here γ_i is a multiplication factor, $\sigma \colon I^X \to I^Y / \operatorname{aut}$ is a bijection and aut is an automorphism on the Dynkin diagram for Y.

Definition 2.3. If there is an embedding of Lie algebras $X \hookrightarrow Y$, then $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a *virtual crystal* for the root system Φ^X if

V1. The ambient crystal $\widehat{\mathcal{V}}$ is a crystal associated to a representation for the root system Φ^Y with crystal operators $\widehat{e}_i$, $\widehat{f}_i$, $\widehat{\varepsilon}_i$, $\widehat{\varphi}_i$ for $i \in I^Y$ and weight function $\widehat{\mathrm{wt}}$.

V2. If $b \in \mathcal{V}$ and $i \in I^X$, then $\widehat{\varepsilon}_j(b)$ has the same value for all $j \in \sigma(i)$ and that value is a multiple of γ_i. The same is true for $\widehat{\varphi}_j(b)$.

V3. The subset $\mathcal{V} \sqcup \{\emptyset\} \subseteq \widehat{\mathcal{V}} \sqcup \{\emptyset\}$ is closed under the virtual crystal operators $e_i := \prod_{j\in\sigma(i)} \widehat{e}_j^{\gamma_i}$ and $f_i := \prod_{j\in\sigma(i)} \widehat{f}_j^{\gamma_i}$ and $\varepsilon_i(b) = \max\{k \geqslant 0 \mid e_i^k(b) \neq \emptyset\}$, $\varphi_i(b) = \max\{k \geqslant 0 \mid f_i^k(b) \neq \emptyset\}$ for $b \in \mathcal{V}$.

The tensor product of two virtual crystals for the same embedding $X \hookrightarrow Y$ is again a virtual crystal (see for example [1, Theorem 5.8]).

We will now apply the theory of virtual crystals to the embedding $B_r \hookrightarrow C_r$. In this setting $I^{C_r} = I^{B_r} = \{1, 2, \ldots, r\}$, $\sigma(i) = \{i\}$, $\gamma_i = 2$ for $1 \leqslant i < r$ and $\gamma_r = 1$. We consider as the ambient crystal $\widehat{\mathcal{V}} = \mathcal{C}_{\square}^{\otimes r}$. Define an ordering $<$ on the set $[r] \cup [\bar{r}]$ as follows $1 < 2 < \cdots < r < \bar{r} < \cdots < \bar{1}$. Denote by $|\cdot|$ the map from $[r] \cup [\bar{r}]$ to $[r]$ that sends letters to their corresponding unbarred values.

Definition 2.4. Let $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ be given by $\mathcal{V} := \{v_r \otimes v_{r-1} \otimes \cdots \otimes v_1 \in \widehat{\mathcal{V}} \mid v_i > v_j \text{ and } |v_i| \neq |v_j| \text{ for all } i > j\}$. Let $f_i = \widehat{f}_i^2$, $e_i = \widehat{e}_i^2$ for $1 \leqslant i < r$ and $f_r = \widehat{f}_r$, $e_r = \widehat{e}_r$.

Definition 2.5. Let $\Psi \colon \mathcal{B}_{\mathsf{spin}} \to \mathcal{V}$ be the map $\Psi(\epsilon_1 \epsilon_2 \cdots \epsilon_r) = v_r \otimes v_{r-1} \otimes \cdots \otimes v_1$, where $v_r > v_{r-1} > \cdots > v_1$ such that if $\epsilon_i = +$ then v contains an i and if $\epsilon_i = -$ then v contains an $\bar{i}$ for all $1 \leqslant i \leqslant r$.

Proposition 2.6. *The map Ψ is a bijective map that intertwines the crystal operators on $\mathcal{B}_{\mathsf{spin}}$ and $\mathcal{V}$. Furthermore, $\mathcal{V}$ is a virtual crystal for the embedding of Lie algebras $B_r \hookrightarrow C_r$.*

An example of the virtual crystal construction for $\mathcal{B}_{\mathsf{spin}}$ is given in Figure 2.

2.2 Highest weights of weight zero

A weight $\lambda \in \Lambda$ is called *minuscule* if $\langle \lambda, \alpha^\vee \rangle \in \{0, \pm 1\}$ for all coroots $\alpha^\vee$. A crystal $\mathcal{B}$ is called *minuscule* if $\mathrm{wt}(b)$ is minuscule for all $b \in \mathcal{B}$. Note that $\mathcal{B}_{\mathsf{spin}}$ is a minuscule crystal (see for example [1, Chapter 5.4]). A weight λ is called *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geqslant 0$ for all $i \in I$. Except for spin weights, dominant weights can be identified with partitions. A *partition* λ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell \geqslant 0$. We identify partitions that differ by trailing zeroes.

Let $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n$ be minuscule crystals. For any highest weight element $u = u_n \otimes \cdots \otimes u_1 \in \mathcal{B}_n \otimes \cdots \otimes \mathcal{B}_1$, we may bijectively associate a sequence of dominant weights $\varnothing = \mu^0, \mu^1, \dots, \mu^n$, where $\mu^q := \sum_{i=1}^q \mathsf{wt}(u_i)$. If μ^n is zero, u is a *highest weight element of weight zero*. Note that the number of highest weight elements of weight zero in a tensor product of crystals is equal to the dimension of the invariant subspace [25, 19].

Definition 2.7 (Sundaram [24])**.** An *r-symplectic oscillating tableau* O of length n and shape μ is a sequence of partitions $\mathsf{O} = (\varnothing = \mu^0, \mu^1, \dots, \mu^n = \mu)$ such that the Ferrers diagrams of two consecutive partitions differ by exactly one cell, and each partition μ^i has at most r nonzero parts.

Next we relate highest weight elements of weight zero in $\mathcal{B}_{\mathsf{spin}}^{\otimes n}$ of type B_r and r-fans of Dyck paths.

Definition 2.8. An *r-fan of Dyck paths* F of length n is a sequence $\mathsf{F} = (\varnothing = \mu^0, \mu^1, \dots, \mu^n = \varnothing)$ of partitions μ^i with at most r parts such that the Ferrers diagram of two consecutive partitions differs by exactly one cell in each part. In other words, μ^i differs from μ^{i+1} by $(\pm 1, \pm 1, \dots, \pm 1)$ for $0 \leqslant i < n$.

Since $\mathcal{B}_{\mathsf{spin}}$ of type B_r is minuscule, by the above discussion $\epsilon = \epsilon_n \otimes \cdots \otimes \epsilon_1 \in \mathcal{B}_{\mathsf{spin}}^{\otimes n}$ is highest weight if and only if $\sum_{i=1}^q \mathsf{wt}(\epsilon_i)$ is dominant for all $1 \leqslant q \leqslant n$. Hence highest weight elements of weight zero can be identified with an r-fan of Dyck paths of length n: the j-th entry of ϵ_i is $+$ if and only if the j-th Dyck path has an up-step at position i. In particular, for a highest weight element ϵ of weight zero, the sequence of dominant weights $\mu^q := \sum_{i=1}^q 2\mathsf{wt}(\epsilon_i)$ for $0 \leqslant q \leqslant n$ defines an r-fan of Dyck paths consistent with Definition 2.8.

Example 2.9. For $r = 3$ and $n = 4$, $\mathsf{F} = ((000), (111), (220), (111), (000))$ is a 3-fan of Dyck paths corresponding to $\epsilon = (-,-,-) \otimes (-,-,+) \otimes (+,+,-) \otimes (+,+,+) \in \mathcal{B}_{\mathsf{spin}}^{\otimes 4}$.

Following Definition 2.5, we obtain an embedding from the set of r-fans of Dyck paths into the set of oscillating tableaux.

Definition 2.10. For an r-fan of Dyck paths $\mathsf{F} = (\varnothing = \lambda^0, \dots, \lambda^n = \varnothing)$ define the oscillating tableau $\iota_{F \to O}(\mathsf{F}) = (\mu^0, \dots, \mu^{rn})$ as follows. Let $v^t = \Psi(\lambda^t - \lambda^{t-1})$ for $1 \leqslant t \leqslant n$ with Ψ as in Definition 2.5. Then $\mu^{tr+s} = \lambda^t + \sum_{i=1}^s \mathsf{wt}(v_i^{t+1})$ for $0 \leqslant t < n$, $0 \leqslant s < r$.

2.3 Promotion via crystal commutor

For finite crystals B_λ of classical type of highest weight λ, Henriques and Kamnitzer [5] introduced the crystal commutor as follows. Let $\eta_{B_\lambda} \colon B_\lambda \to B_\lambda$ be the Lusztig involution, which maps the highest weight vector to the lowest weight vector and interchanges the crystal operators f_i with $e_{i'}$, where $w_0(\alpha_i) = -\alpha_{i'}$ under the longest element w_0. This can be extended to tensor products of such crystals by mapping each connected component to itself using the above. Then the *crystal commutor* is defined as

$$\sigma_{B_\lambda,B_\mu}\colon B_\lambda \otimes B_\mu \to B_\mu \otimes B_\lambda \quad \text{with} \quad b \otimes c \mapsto \eta_{B_\mu \otimes B_\lambda}(\eta_{B_\mu}(c) \otimes \eta_{B_\lambda}(b)).$$

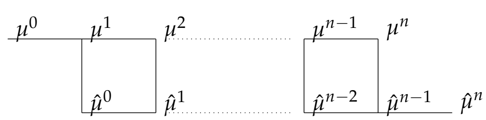
Definition 2.11 ([4, 25, 26])**.** Let C be a crystal and $u \in C^{\otimes n}$ a highest weight element. Then *promotion* pr on u is defined as $\sigma_{C^{\otimes n-1},C}(u)$.

Example 2.12. Promotion is $\sigma_{\mathcal{B}_{\mathsf{spin}}^{\otimes 3},\mathcal{B}_{\mathsf{spin}}}(\epsilon) = (-,-,-) \otimes (-,+,+) \otimes (+,-,-) \otimes (+,+,+)$ for ϵ in Example 2.9.

2.4 Promotion via local rules

Adapting local rules of van Leeuwen [12], Lenart [13] gave a combinatorial realization of the crystal commutor $\sigma_{A,B}$ by constructing an equivalent bijection between the highest weight elements of $A \otimes B$ and $B \otimes A$ respectively. We define Lenart's construction in the case of promotion. The *local rules* of Lenart [13] can be stated as follows: four weight vectors $\lambda, \mu, \kappa, \nu \in \Lambda$ depicted in a square diagram $\begin{array}{cc} \lambda & \nu \\ \kappa & \mu \end{array}$ satisfy the local rule, if $\mu = \mathrm{dom}_W(\kappa + \nu - \lambda)$, where W is the Weyl group of the root system Φ for A and B and $\mathrm{dom}_W(\rho)$ is the dominant weight in the Weyl orbit of ρ.

Let C be a minuscule crystal and $(\emptyset = \mu^0, \mu^1, \dots, \mu^n = \mu)$ the sequence of partitions corresponding to the highest weight element $u \in C^{\otimes n}$. Let $(\emptyset =: \hat{\mu}^0, \hat{\mu}^1, \dots, \hat{\mu}^n := \mu)$ be the sequence of partitions obtained by constructing a skewed grid as in (2.1) and recursively computing $\hat{\mu}^1, \dots, \hat{\mu}^{n-1}$ using Lenart's local rules.



(2.1)

It follows from Lenart [13, Theorem 4.4] that $(\emptyset = \hat{\mu}^0, \hat{\mu}^1, \dots, \hat{\mu}^n = \mu)$ is the unique sequence of partitions corresponding to the highest weight element $\mathsf{pr}(u)$.

3 Chord diagrams

3.1 Promotion-evacuation diagrams

In this section we summarize the method to obtain a map from highest weight words of weight zero to chord diagrams that intertwines promotion and rotation.

Definition 3.1. A *chord diagram* of size n is a graph with n vertices depicted on a circle labelled $1, \dots, n$ in counter-clockwise orientation. The *rotation* of a chord diagram is obtained by rotating all edges clockwise by $\frac{2\pi}{n}$ around the center of the diagram.

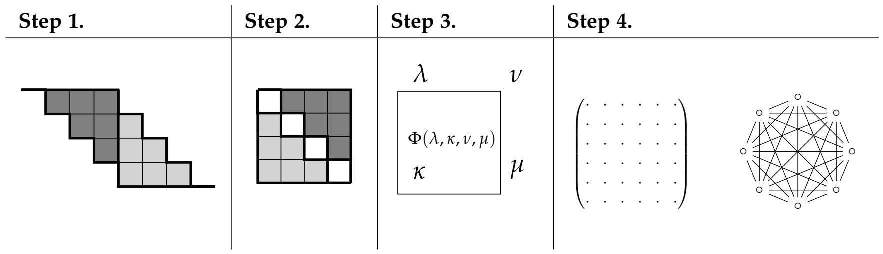


Figure 3: Overview of the steps in our map

In our setting, chord diagrams are undirected graphs with possibly multiple edges between the same two vertices. We can therefore identify a chord diagram with its *adjacency matrix*, which is a symmetric $n \times n$ matrix $M = (m_{ij})_{1 \leqslant i,j \leqslant n}$ with non-negative integer entries m_{ij} equal to the number of edges between vertex i and vertex j. We denote by $\operatorname{rot} M$ the adjacency matrix corresponding to the rotation of the chord diagram.

We now outline the idea to construct a rotation and promotion intertwining map for oscillating tableaux and r-fans of Dyck paths. A visual guideline is given in Figure 3.

Construction 3.2. The construction is given as follows:

Step 1 Iteratively calculate promotion of a highest weight word of weight zero and length n using Lenart's schema (2.1) a total of n times.

Step 2 Group the results into a square grid, called the *promotion matrix*.

Step 3 Fill the cells of the square grid with certain non-negative integers according to a filling rule Φ that only depends on the four corners of the cells in the schema (2.1).

Step 4 Regard the filling as the adjacency matrix of a graph, which is the chord diagram.

Definition 3.3. The *filling rule* for oscillating tableaux and fans of Dyck paths, with cells labelled as in Step 3 of Figure 3, is $\Phi(\lambda, \kappa, \nu, \mu) =$ number of negative entries in $\kappa + \nu - \lambda$.

Remark 3.4. For oscillating tableaux at most one negative entry can occur, so that $\Phi(\lambda, \kappa, \nu, \mu)$ is either 0 or 1. Note that the filling rules are new even in the case of oscillating tableaux as the proofs in [19] did not follow this construction.

Definition 3.5. Denote by M_O (resp. M_F) the function that maps an r-symplectic oscillating tableau (resp. r-fan of Dyck paths) of length n to an $n \times n$ adjacency matrix using Construction 3.2 and the filling rule in Definition 3.3.

Proposition 3.6. *The map M_X for $X \in \{O, F\}$ intertwines promotion and rotation, that is* $\mathsf{M}_X \circ \mathsf{pr} = \operatorname{rot} \circ \mathsf{M}_X$.

3.2 Fomin growth diagrams

Generally speaking, a *Fomin growth diagram* is a means to bijectively map sequences of partitions satisfying certain constraints to non-negative integer fillings of a Ferrers shape, drawn in French notation [3, 10]. To map a 0/1 filling of a Ferrers shape to a sequence of partitions, we iteratively label all corners of the cells of the shape with partitions by certain local forward rules, F1-F6 [10, p. 4]. Conversely, given partitions labelling the right and top corners of a cell, the local backwards rules B1-B6 determine the last partition and the 0/1 filling of the cell [10, p. 5].

To map a non-negative integer filling of a Ferrers shape to a sequence of partitions, we produce a "blow up" construction of the original shape for the Burge variant which contains south-east chains of 1's, as done by [10, Section 4.4]. If a cell is filled with a positive entry m, we replace the cell with an $m \times m$ grid of cells with 1's along the diagonal. If there exist several nonzero entries in one column or row, we arrange the grids of cells so that the 1's form a south-east chain in each column and row.

Since the filling of the blow up growth diagram consists of 1's and 0's, we now apply the forward local rules F1-F6 to obtain the partition labels of all corners. We then "shrink back" the labelled blow up growth diagram to obtain a labelling of the original Ferrers diagram by only using the partitions labelling the intersections that occurred in the original Ferrers diagram. The labelling from the blow up construction is given explicitly by the local forward rules F^40 - F^42 and local backward rules B^40 - B^42 [10, p. 22].

Construction 3.7. Let $\mathsf{F} = (\emptyset = \mu^0, \mu^1, \ldots, \mu^n = \emptyset)$ be an r-fan of Dyck paths. The associated triangular growth diagram is the Ferrers shape $(n-1, n-2, \ldots, 2, 1, 0)$. Label the cells according to the following specification:

1. Label the north-east corners of the cells on the main diagonal from the top-left to the bottom-right with the partitions in F.
2. For each $i \in \{0, \ldots, n-1\}$ label the corner on the first subdiagonal adjacent to the labels μ^i and μ^{i+1} with the partition $\mu^i \cap \mu^{i+1}$.
3. Use backwards rules Burge B^40-B^42 to obtain all other labels and cell fillings.

For oscillating tableaux, Construction 3.7 reduces to rules B1-B6 and F1-F6.

We denote by $\mathsf{G}_F(\mathsf{F})$ (resp. G_O) the symmetric $n \times n$ matrix one obtains from the filling of the growth diagram by putting zeros in the unfilled cells and along the diagonal and completing this to a symmetric matrix. Starting from a filling of a growth diagram one obtains the r-fan by filling the cells of a growth diagram, setting all vectors on corners on the bottom and left border of the diagram to be the empty partition and applying the forwards growth rules Burge F^40-F^42 (resp. F1-F6).

4 Main results

4.1 Results for oscillating tableaux and r-fans of Dyck paths

Our main result states that the fillings of the growth diagrams (Construction 3.7) and the fillings of the promotion matrices (Definition 3.3) coincide. For oscillating tableaux this was not stated explicitly in [19], but can be deduced from the proofs.

Theorem 4.1. *For an oscillating tableau of weight zero* O *and an r-fan of Dyck paths* F, *we have* $\mathsf{G}_O(\mathsf{O}) = \mathsf{M}_O(\mathsf{O})$ *and* $\mathsf{G}_F(\mathsf{F}) = \mathsf{M}_F(\mathsf{F})$. *In particular, the maps* M_O *and* M_F *are injective.*

Proposition 4.2. *For an r-fan of Dyck paths* F, $\iota_{F\to O} \circ \mathsf{pr}_{\mathcal{B}_{\mathsf{spin}}}(\mathsf{F}) = \mathsf{pr}^r_{\mathcal{C}_\square} \circ \iota_{F\to O}(\mathsf{F})$. *In addition,* $\mathsf{M}_F(\mathsf{F}) = \mathsf{blocksum}_r(\mathsf{M}_O(\iota_{F\to O}(\mathsf{F})))$, *where* $\mathsf{blocksum}_r(\mathsf{M}_O(\iota_{F\to O}(\mathsf{F})))$ *is the matrix obtained by replacing* $r \times r$ *blocks in* $\mathsf{M}_O(\iota_{F\to O}(\mathsf{F}))$ *with the sum of their entries.*

4.2 Cyclic sieving

The cyclic sieving phenomenon was introduced by Reiner, Stanton and White [20].

Definition 4.3. Let X be a finite set and C be a cyclic group generated by c acting on X. Let $\zeta \in \mathbb{C}$ be a $|C|^{th}$ primitive root of unity and $f(q) \in \mathbb{Z}[q]$ be a polynomial in q. Then the triple (X, C, f) exhibits the *cyclic sieving phenomenon* if for all $d \geqslant 0$ we have that the size of the fixed point set of c^d (denoted X^{c^d}) satisfies $|X^{c^d}| = f(\zeta^d)$.

We state cyclic sieving phenomena for the promotion action on oscillating tableaux and fans of Dyck paths. For this we need the *local energy function* (see for example [15]) $H\colon \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$. The local energy function is constant on connected components.

For the crystal $\mathcal{C}_\square$ of type C_r, using the ordering $1 < 2 < \cdots < r < \overline{r} < \cdots < \overline{2} < \overline{1}$, we have that $H(a \otimes b) = 0$ if $a \leqslant b$ and $H(a \otimes b) = 1$ if $a > b$. The classical highest weight elements in $\mathcal{B}^{\mathsf{af}}_{\mathsf{spin}} \otimes \mathcal{B}^{\mathsf{af}}_{\mathsf{spin}}$ are $(\epsilon_1, \ldots, \epsilon_r) \otimes (+, \ldots, +)$ with $\epsilon_i = +$ for $1 \leqslant i \leqslant k$ and $\epsilon_i = -$ for $k < i \leqslant r$ for some $0 \leqslant k \leqslant r$. Denoting by $m(\epsilon_1, \ldots, \epsilon_r)$ the number of $-$ in the ϵ_i, we have $H((\epsilon_1, \ldots, \epsilon_r) \otimes (+, \ldots, +)) = \left\lfloor \frac{m(\epsilon_1,\ldots,\epsilon_r)+1}{2} \right\rfloor$. The *energy function* $E\colon \mathcal{B}^{\otimes N} \to \mathbb{Z}$ is defined for $b_1 \otimes \cdots \otimes b_N \in \mathcal{B}^{\otimes N}$ as $E(b_1 \otimes \cdots \otimes b_N) = \sum_{i=1}^{N-1} iH(b_i \otimes b_{i+1})$. Define a polynomial in q using the energy function for highest weight elements in $\mathcal{B}^{\otimes n}$ of weight zero

$$f_{n,r}(q) = q^{c_{n,r}} \sum_{\substack{b \in \mathcal{B}^{\otimes n} \\ \mathsf{wt}(b)=0, e_i(b)=0 \text{ for } 1 \leqslant i \leqslant r}} q^{E(b)},$$

where r is the rank of the root system and $c_{n,r}$ is a constant depending on the type.

Theorem 4.4. *Let* X *be the set of highest weight elements in* $\mathcal{B}^{\otimes n}$ *of weight zero, where* $\mathcal{B}$ *is minuscule. Then* $(X, C_n, f_{n,r}(q))$ *exhibits the cyclic sieving phenomenon, where* C_n *is the cyclic group of order n given by the action of promotion* pr *on* $\mathcal{B}^{\otimes n}$.

Note that $\mathcal{C}_\square$ and $\mathcal{B}_{\mathsf{spin}}$ are minuscule, and hence Theorem 4.4 gives a cyclic sieving phenomenon for oscillating tableaux and fans of Dyck paths. See also [4, 25]. For the type A vector representation, the energy function is the major index and Theorem 4.4 relates to results in [21].

Recall that highest weight elements of weight zero in $\mathcal{B}_{\mathsf{spin}}^{\otimes 2n}$ of type B_r are in bijection with r-fans of Dyck paths of length $2n$. Denote by $D_n^{(r)}$ the set of all r-fans of Dyck paths of length $2n$. We have $|D_n^{(r)}| = \prod_{1\leqslant i\leqslant j\leqslant n-1} \frac{i+j+2r}{i+j}$, see [9]. Define the q-analogue as $g_{n,r}(q) = \prod_{1\leqslant i\leqslant j\leqslant n-1} \frac{[i+j+2r]_q}{[i+j]_q}$ where $[m]_q = 1+q+q^2+\cdots+q^{m-1}$. The following conjecture is equivalent to [6, Conjecture 5.9] on plane partitions and root posets.

Conjecture 4.5. *The triple $(D_n^{(r)}, C_{2n}, g_{n,r}(q))$ exhibits the cyclic sieving phenomenon, where C_{2n} is the cyclic group of order $2n$ that acts on $D_n^{(r)}$ by applying promotion.*

Example 4.6. We have $q^{-4}f_{2,2}(q) = g_{2,2}(q) = q^4+q^2+1$ and

$$\begin{aligned} g_{3,2}(q) &= q^{12}+q^{10}+q^9+2q^8+q^7+2q^6+q^5+2q^4+q^3+q^2+1,\\ q^{-6}f_{3,2}(q) &= q^{10}+q^9+2q^8+q^7+3q^6+q^5+2q^4+q^3+q^2+1. \end{aligned}$$

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