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Deficit and (q, t)-symmetry in triangular Dyck paths

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Abstract. We study the (q, t)-enumeration of triangular Dyck paths considered by Bergeron and Mazin. To do so, we introduce the notion of *triangular* and *sim-sym* tableaux and the *deficit* statistic which is a new interpretation of the *dinv*. We use it to obtain new results and proofs on triangular 2-partitions and an interesting conjecture for a certain lattice interval (q, t, r)-enumeration.

Résumé. Nous étudions la symétrie (q, t) dans l'énumération des chemins de Dyck triangulaires décrits par Bergeron et Mazin. Nous introduisons les notions de *tableau triangulaire* et *sim-sym* ainsi que la statistique du *deficit* qui ré-interprète le *dinv*. Nous utilisons cela pour obtenir de nouveaux résultats et nouvelles démonstrations pour les 2-partages ainsi qu'une conjecture sur un comptage trivarié (q, t, r) des intervalles de certains treillis.

Keywords: triangular partitions, (q, t)-Catalan, Dyck paths

1 Introduction

In [2], Bergeron and Mazin study a certain family of partitions that they call *triangular partitions*, motivated by the results of [4]. A triangular partition is the maximal partition lying under a given line. The sub-partitions of a triangular partition are called the triangular Dyck paths, generalizing classical and rational Dyck paths. The (q, t)-enumeration of paths, such as the (q, t)-Catalan numbers [7], has raised interesting combinatorial questions in recent years, related in particular to representation theory. The Negut formula (see for instance [1, 5, 6]) gives a (q, t)-enumeration of the sub-partitions of any given partition but the signification of each (q, t) monomial is not clear and the coefficients are not always positive. In a recent work generalizing the *shuffle theorem* [4], the authors found out a combinatorial interpretation when the partition is triangular, using two statistics generalizing the classical *area* and *dinv* of a Dyck path. Indeed, in this case the (q, t) polynomials appear as the coefficients of some symmetric functions and are

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by nature symmetric themselves. More generally, Conjecture 7.1.1 from [4] states that the Negut formula gives positive coefficients if the partition lies under a certain convex curve. As expressed in [1], Bergeron further conjectures that this condition is not only sufficient but necessary. Moreover, the (q,t)-enumeration appears to be *Schur positive* in the triangular case [2, Conjecture 1].

In terms of combinatorics, this raises two (very) difficult questions: finding an elementary, combinatorial proof of the (q,t)-symmetry for the enumeration of triangular Dyck paths and proving that they are Schur positive. This includes the classical and rational Dyck path cases, which are special cases of triangular Dyck paths. This is the problem that motivates this extended abstract. Our main results are stated in Theorems 2 and 5 where we prove the conjecture in the (limited) case of 2-partitions and extend the symmetry to a certain interval enumeration. Moreover, we introduce new combinatorial tools such as the *triangular* and *sim-sym* tableaux as well as the *deficit* statistic which explore the combinatorial interpretation of the *dinv* statistic. This led us to state Conjecture 4 in relation to interval enumeration in the generalized Tamari lattices of [10].

The paper is organized as follows. Section 2 recalls the basic notions related to partitions and the definition and main properties of triangular partitions from [2]. In Section 3, we define and explore some new combinatorial tools on triangular partitions, namely the *triangular tableau*, the *deficit* statistic and the *sim-sym* tableaux. The main result of this section is Proposition 3.4 where we prove that the *deficit* statistic actually gives the *dinv*. Section 4 uses these tools to obtain explicit results on 2-partitions: Theorems 2 and 3. Finally, the lattice enumerations are succinctly presented in Section 5 in order for us to state Conjecture 4 and Theorem 5.

2 Background

2.1 Partitions and Young tableaux

Definition 2.1. *A partition[1] λ of size n is a tuple $\lambda_1 \geq \lambda_2 \geq \dots \lambda_k > 0$ such that $|\lambda| := \sum_i^k \lambda_i = n$. For convenience, we consider partitions ending with an infinite number of 0-parts, $\lambda_j := 0$ for $j > k$. A k-partition is a partition $\lambda = (\lambda_1, \dots, \lambda_{k'})$ with $k' \leq k$. We say that a partition μ is a sub-partition of another partition λ if $\mu_i \leq \lambda_i$ for all i.*

As usual, we represent partitions by their *Ferrers diagram* (French style) such that each value λ_i, from bottom to top, corresponds to a line of i square cells as shown on Figure 1.

Definition 2.2. *A standard Young tableau on a partition λ is an integer filling of the Ferrers diagram of λ (we place a number in each cell) such that the numbers are strictly increasing along*

[1] This is actually an integer partition. As there is no ambiguity in the paper, we simply call them partitions.

each row and each column. We identify the cells of λ by their coordinates in the Ferrers diagram (the cell $(0,0)$ is the bottom left-most cell). We consider a tableau θ as a function from the cells of λ to $\{1,\ldots,|\lambda|\}$ and write $\theta(c)$ the value of the cell c in θ.

See Figure 1 for an example. *Schur functions* form a famous basis of symmetric functions indexed by partitions. Any symmetric polynomial can thus be expressed as a sum of Schur functions. A symmetric polynomial is said to be *Schur positive* if all the coefficients in its expansion are positive integers. For the precise definition and properties of Schur functions, we refer the reader to [8].

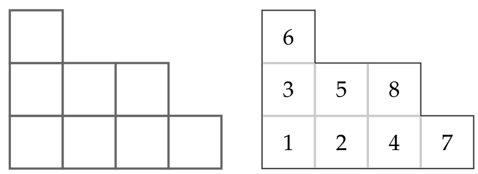

Figure 1: The Ferrers diagram of $(4,3,1)$ and an example of standard Young tableau.

2.2 Triangular partitions

Definition 2.3 (From [2])**.** *A* triangular partition *λ is a partition such that there exist two real numbers r and s with $\lambda_j = \lfloor r - \frac{jr}{s} \rfloor$ for all $j \leq s$, and $\lambda_j = 0$ otherwise.*

We say that the line joining $(r,0)$ and $(0,s)$ is an *r-s-line*. Then λ is the greatest partition lying under the r-s-line. We also say that this line *cuts off* λ . See an example on Figure 2.

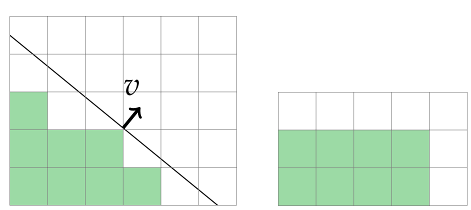

Figure 2: Left: the triangular partition $(4,3,1)$ with one line that cuts it off and its slope vector. Right: a non-triangular partition $(4,4)$.

When there exists an r-s-line cutting off λ with r and s integer values, λ is said to be a *rectangular* partition. It is further said to be *rational* when r and s are coprime. Later on, it will be important to characterize the *slope* of the line. In accordance with [2], this can be done through a single parameter v which characterizes a *slope vector*.

Definition 2.4. *The* slope vector *of an r-s-line is the vector orthogonal to the line and of coordinates $(v, 1-v)$. For simplicity, we write "the slope v". We say that λ admits a slope v if it is cut off by an r-s-line of slope vector $(v, 1-v)$.*

A triangular partition admits an infinite number of slope vectors which corresponds to certain open interval for the value of v. Bergeron and Mazin [2] give a simple way to compute the slope interval using the hooks of the partition. For each cell c of λ, let $a(c)$ be the ***arm*** of the cell, *i.e.* the number of cells lying to the right of c in λ, and $\ell(c)$ be the ***leg*** of c, *i.e.* the number of cells lying above c in λ. Then

$$v^-(c,\lambda) := \frac{\ell(c)}{a(c)+\ell(c)+1} \qquad\qquad v^+(c,\lambda) := \frac{\ell(c)+1}{a(c)+\ell(c)+1}. \tag{2.1}$$

We say that $]v^-(c,\lambda),\ v^+(c,\lambda)[$ is the ***admissible interval*** of the cell c. The slope interval of λ is the intersection of all admissible intervals $]v_\lambda^-, v_\lambda^+[$ where $v_\lambda^- := \max_{c\in\lambda}(v^-(c,\lambda))$ and $v_\lambda^+ := \min_{c\in\lambda}(v^+(c,\lambda))$. In particular, λ is a triangular partition if and only if $v_\lambda^- < v_\lambda^+$ [2, Lemma 1.2].

2.3 Area, sim and (q,t)-enumeration

In this paper, we are interested in the (q,t)-enumeration of sub-partitions of a given triangular partition λ. Following the terminology of [2], we call such objects ***triangular Dyck paths***.

Definition 2.5. *A* triangular Dyck path *is a tuple (λ,μ) such that μ is a sub-partition of λ and λ is a triangular partition. The* area *of a triangular Dyck path is the number of cells belonging to λ but not to μ, given by $\mathrm{area}_\lambda(\mu) := \sum_{i\in\mathbb{N}} \lambda_i - \mu_i$.*

Note that we do not require that μ is also triangular. See Figure 3 for an example. This name is justified in the sense that when λ is the staircase partition, *i.e.* $\lambda = (n-1, n-2, \ldots, 1)$, then the triangular Dyck paths (λ,μ) correspond to actual Dyck paths of length n. Similarly, when λ is a rational partition, then its sub-partitions are rational Dyck paths.

Definition 2.6 (From [2], Section 4.1)**.** *Let (λ,μ) be a triangular Dyck path. A cell c of μ is said to be* similar *if*

$$v^-(c,\mu) \leq \frac{v_\lambda^- + v_\lambda^+}{2} < v^+(c,\mu). \tag{2.2}$$

In other words, the mean slope *of the slope interval of λ belongs to the semi-closed admissible interval of the cell c in μ. Then $\mathrm{sim}_\lambda(\mu)$ is the number of similar cells in the triangular Dyck path (λ,μ).*

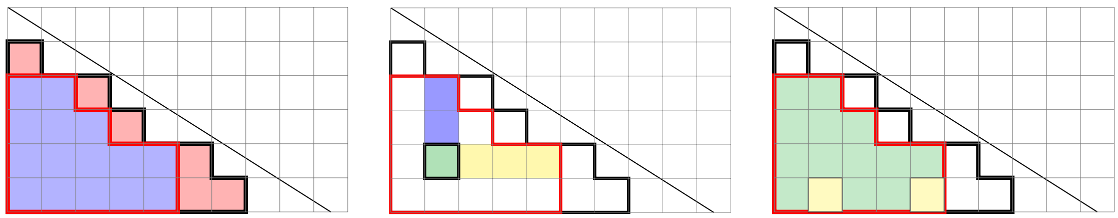

Figure 3: A triangular Dyck path (λ, μ). The left picture shows μ in blue and the area in red with $\text{area}_\lambda(\mu) = 6$. The middle picture illustrates the hook of a given cell in μ. The right picture shows all similar cells in green with $\text{sim}_\lambda(\mu) = 13$.

For example on Figure 3, the mean slope of λ, $\frac{45}{112}$ belongs to the admissible interval of the selected cell given by $v^-(c,\mu) = \frac{1}{3}$ and $v^+(c,\mu) = \frac{1}{2}$. It is important to understand that the similarity of a cell depends on both partitions of the triangular Dyck path. Indeed, the mean slope is computed on λ but the admissible cell interval is computed on μ. If (λ, μ) is a triangular Dyck path such that all the cells of μ are similar, then in particular μ is triangular: a slight perturbation of the mean slope of λ will always cut off μ (see [2, Lemma 4.1]). When all cells of μ are similar cells, we say that (λ, μ) is a *mean-similar* triangular Dyck path.

The *sim* statistic already appears in [4] and is a generalization of the famous *dinv* statistic that can be found in [7] for the classical (q,t)-Catalan enumeration. Indeed, when λ is the staircase partition, then the *sim* of (λ, μ) is actually the *dinv* of the corresponding Dyck path.

We can now define the (q,t) polynomials associated to a triangular partition λ as:

$$A_\lambda(q,t) = \sum_{\mu \subseteq \lambda} q^{\text{area}_\lambda(\mu)}\ t^{\text{sim}_\lambda(\mu)}. \tag{2.3}$$

These polynomials have been shown to be symmetric in q and t via an algebraic proof [4]. However, the question of finding a combinatorial proof is still open even for the classical case. For example, for the partition $\lambda = (3,2)$, we have the following:

$$A_{3,2}(q,t) = q^5 + q^4t + q^3t^2 + q^2t^3 + qt^4 + t^5 + q^3t + q^2t^2 + qt^3 \tag{2.4}$$

As the $A_\lambda(q,t)$ polynomials are symmetric, they can be expressed in terms of Schur functions with the following conjecture.

Conjecture 1 (Conjecture 1 of [2])**.** *The polynomials $A_\lambda(q,t)$ are Schur positive.*

See [2] for many examples of computation, here are $A_{3,2,1}(q,t)$ and $A_{3,2}(q,t)$:

$$A_{3,2,1}(q,t) = s_6(q,t) + s_{4,1}(q,t) + s_{3,1}(q,t), \qquad A_{3,2}(q,t) = s_5(q,t) + s_{3,1}(q,t). \tag{2.5}$$

3 Combinatorial interpretation of similar cells

In this paper, we explore some combinatorial tools related to the aforementioned conjecture and obtain a proof for the case of 2-partitions. The first step is to define a certain Young tableau associated to each triangular partition.

3.1 The triangular Young tableau

Proposition 3.1. *Let λ be a triangular partition. Then for all $0 \leq k \leq |\lambda|$, there exists a unique mean-similar triangular Dyck path (λ, μ) of area k. Besides, if (λ, α) and (λ, μ) are two mean-similar triangular Dyck paths with $|\alpha| \leq |\mu|$, then $\alpha \subseteq \mu$.*

In other words, if (λ, μ) is such that all cells of μ are similar cells, then there is a unique way to remove a cell from μ such that all remaining cells stay similar. This is a consequence of [2, Lemma 4.1]: the mean-slope of λ can always be slightly shifted to obtain an irrational slope. By *moving* this irrational slope towards the origin, you will "touch" the cells of λ one by one, which gives the removal order on the cells of the sub-partitions in order to stay mean-similar, see Figure 4 for an example. This leads to the following definition.

Definition 3.2. *The triangular Young tableau of a triangular partition λ with $|\lambda| = n$ is the unique standard Young tableau such that for all $0 \leq k \leq n$, the cells 1 to k form a sub-partition μ with (λ, μ) a mean-similar triangular Dyck path.*

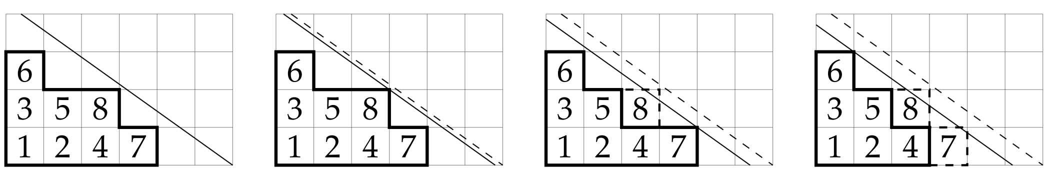

Figure 4: The triangular Young tableaux of $(4, 3, 1)$ with its slope construction.

3.2 Deficit of a triangular Dyck path

Definition 3.3. *Let (λ, μ) be a triangular Dyck path and θ a standard Young tableau of shape λ. We say that there is a θ-inversion in (λ, μ) if there exists a cell $c = (x, y)$ in μ and a cell $c' = (x', y')$ in $\lambda \setminus \mu$ such that $\theta(c) > \theta(c')$. In particular, c cannot be in the same line or column as c'. Then we say that the cell $d = (\min(x, x'), \min(y, y'))$, which is at the hook of c and c', is a θ-deficit cell. We call deficit of the triangular Dyck path (λ, μ) with tableau θ, written $\mathrm{def}_\theta(\mu)$, the number of θ-deficit cells.*

See Figure 5 for an example: 1 is a deficit cell because it is at the hook of 6 and 4, and 2 is a deficit cell because it is at the hook of 5 and 7. Note that it can happen that multiple θ-inversions give the same deficit cell. The following proposition gives a more combinatorial approach to the computation of similar cells.

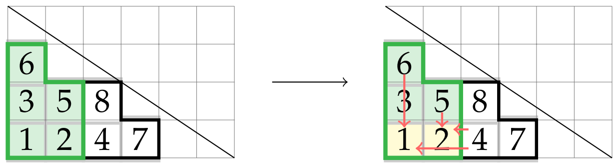

Figure 5: As a sub-partition of $(4,3,1)$, the partition $(2,2,1)$ has two deficit cells 1 & 2.

Proposition 3.4. *Let λ be a triangular partition and θ its triangular Young tableau as in Definition 3.2. Then for any triangular Dyck path (λ,μ), the θ-deficit cells of (λ,μ) are exactly the non-similar cells of μ. In particular, we have $|\lambda| = \mathrm{area}_\lambda(\mu) + \mathrm{sim}_\lambda(\mu) + \mathrm{def}_\theta(\mu)$.*

In the rest of the paper, we write $\mathrm{def}_\lambda(\mu) := \mathrm{def}_\theta(\mu)$ of a triangular Dyck path where θ is the triangular Young tableau.

For the staircase partition (which gives classical Dyck paths), the triangular tableau is the *top-down* tableau where we label the corners of the partition in decreasing order from top to bottom as shown in Figure 6. The deficit then gives a new interpretation for the classical *dinv* statistic.

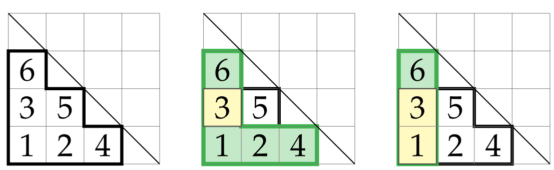

Figure 6: The triangular tableau of the staircase partition $(3,2,1)$ and two examples of sub-partitions (Dyck paths). In the left example, we have $\mu = (3,1,1)$ and $\mathrm{sim}_\lambda(\mu) = 4$. In the second example, we have $\mu = (1,1,1)$ and $\mathrm{sim}_\lambda(\mu) = 1$.

3.3 Sim-sym tableau

The deficit statistic is defined for any tableau θ and just happens to coincide with the non-similar cells when θ is the triangular Young tableau of λ. It is then natural to extend the definition of the *sim* statistic to any Young tableau.

Definition 3.5. *Let θ be a standard Young tableau of triangular shape λ. Then for any triangular Dyck paths (λ, μ), we say that a cell of μ is* θ-similar *if it is not a θ-deficit cell. We write* $\mathrm{sim}_\theta(\mu)$ *the number of θ-similar cells.*

With that definition, again it is natural to extend the definition of (2.3) to this new setting, we define

$$A_\theta(q,t) = \sum_{\mu \subseteq \lambda} q^{\mathrm{area}_\lambda(\mu)}\, t^{\mathrm{sim}_\theta(\mu)} \tag{3.1}$$

where θ is a standard Young tableau of shape λ. These polynomials are not symmetric in general. On the other hand, the triangular Young tableau is not the only tableau giving the symmetry. This led us to introduce the following objects which we believe are of high interest when studying the (q,t)-symmetry.

Definition 3.6. *A* sim-sym *(similar-symmetric) tableau is a Young tableau θ of triangular shape λ such that $A_\theta(q,t) = A_\lambda(q,t)$.*

It is important to note that this equality does not necessarily mean that, for all triangular Dyck paths, one has $\mathrm{sim}_\theta(\mu) = \mathrm{sim}_\lambda(\mu)$. It only means we have a bijection over the triangular Dyck paths of λ which keeps the *area* constant while sending the θ-*sim* statistic to the classical *sim*. Figure 7 gives three examples of sim-sym tableaux for $(4,2,1)$.

The natural question that follows would be to characterize the sim-sym tableaux. This is not an easy question. One way to find different sim-sym tableaux is to choose a different slope to sweep down the partition. Indeed, the triangular Young tableau is defined using a slight deformation of the mean-slope (to ensure that it is irrational). But any irrational slope would actually give a tableau, and computational explorations suggest that these are always sim-sym tableaux. Nevertheless, they are not the only examples. For example, for the partition $(4,2,1)$, we find only two slope-tableaux which are the first two tableaux of Figure 7 (the first one is the mean-slope triangular tableau), but there are actually seven sim-sym tableaux. In the rest of the paper, we characterize sim-sym tableaux on 2-partitions, and explore some interesting conjectures related to the lattice structure of triangular Dyck-paths and sim-sym tableaux.

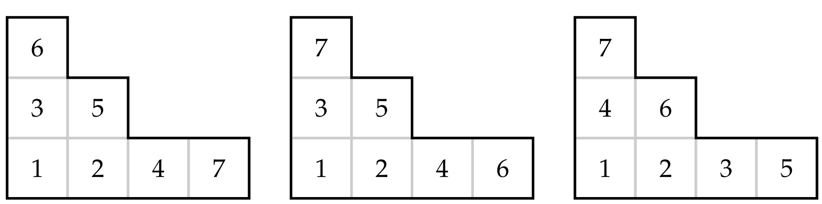

Figure 7: Three sim-sym tableaux of $(4,2,1)$: the first two are slope-tableaux but not the last one.

4 Results on 2-partitions

Using the *deficit* statistic on the triangular tableau, we are able to give a direct proof of Conjecture 1 for 2-partitions.

4.1 Characterization of triangular 2-partitions

We have seen in Section 2.2 that triangular partitions have an easy characterization through their slope vectors. This gives the following result.

Proposition 4.1. *Let $\lambda = (m,n)$ be a 2-partition, λ is triangular if and only if $n \leq \lceil m/2 \rceil$.*

We can also explicitly describe the triangular Young tableau associated with a triangular 2-partition.

Proposition 4.2. *Let $\lambda = (m,n)$ be a triangular partition and θ its triangular Young tableau. Then the values in θ on the shortest row of λ (of size n) are given in decreasing order by $k, k-2, k-4, \dots, k-2(n-1)$ with $k = m+n$ if $n = \lceil m/2 \rceil$, and $k = m+n-1$ if $n < \lceil m/2 \rceil$. As λ only has two (increasing) rows, this entirely characterizes θ.*

In other words, the maximal value of θ will be placed on either of the two rows depending on the length n, then the values will alternate between the two rows until the shortest row is filled. See some examples on Figure 8.

6	8					
1	2	3	4	5	7	9

3	5	
1	2	4

3	5		
1	2	4	6

Figure 8: The triangular tableaux of $(7,2)$, $(3,2)$ and $(4,2)$.

4.2 Explicit (q,t)-enumeration and Schur positivity

We are now able to give a direct proof of the following theorem which was stated in [2]. This formula was actually obtained by [5, Theorem 3.1] which explores a similar (q,t)-enumeration in a different combinatorial context.

Theorem 2 (Formula (6.3) in [2])**.** *For $\lambda = (m,n)$ a triangular 2-partition, we have*

$$A_\lambda(q,t) = \sum_{0 \leq i \leq \min(n, m-n)} s_{m+n-2i,i}(q,t). \tag{4.1}$$

Recall that $s_{k,\ell}(q,t) = \sum_{i=k}^{\ell} q^i t^{k+\ell-i}$. In particular, each monomial $q^i t^j$ appears only once in (4.1). The proof then is purely combinatorial, building an explicit bijection between monomials $q^i t^j$ and triangular Dyck paths which *area* equals to i and *deficit* equals to $m+n-i-j$. In the general case, monomials might appear with coefficients greater than 1, which explains why this proof does not generalize easily.

4.3 Sim-sym tableaux on 2-partitions

The key combinatorial ingredient in the proof of Theorem 2 is the triangular tableau and especially the alternation of values between the two rows. This leads to the following theorem.

Theorem 3 (Characterization of sim-sym tableau on 2-partitions)**.** *Let $\lambda = (m, n)$ be a triangular partition with $n \neq 2$. Then there are exactly $m - 2n + 2$ sim-sym tableaux of shape λ. The values on the shortest row of each tableau are given in decreasing order by $k, k-2, k-4, \ldots, k-2(n-1)$ with $3n - 2 < k \leq m + n$.*

Furthermore, when $n = 2$, we have the aforementioned tableaux as well as the tableau containing values 2 and 5 on its shortest row.

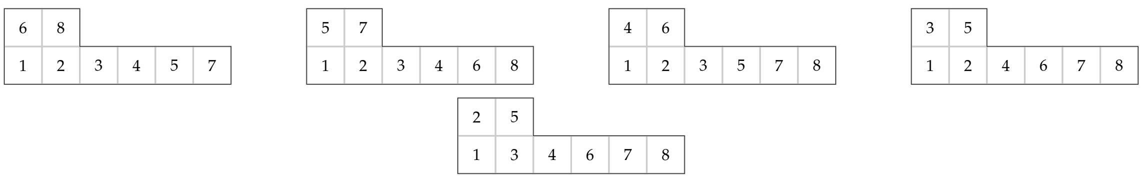

Figure 9: The five different sim-sym tableaux on $(6, 2)$.

5 Lattice and interval enumeration

In some instances, the polynomials $A_\lambda(q, t)$ of (2.3) occur as the expansion in two variables of the characters of a certain GL_2-action. Hence, they are Schur positive, as the Schur functions are the characters of the irreducibles in this context. For all triangular partitions λ, one can define a symmetric function A_λ corresponding to the Hilbert series of alternating component of multivariate diagonal harmonic polynomials [1]. The trivariate case has been studied in particular in the context of the Tamari and m-Tamari lattices [3] with various conjectures emitted in [9]. For example, we have that

$$A_{3,2,1} = s_6 + s_{4,1} + s_{3,1} + s_{1,1,1}. \tag{5.1}$$

When expanded on two variables, the term $s_{1,1,1}(q, t)$ is 0 and we get (2.5), the (q, t)-enumeration of Dyck paths. For the Tamari and m-Tamari cases, the A_λ expanded in three variables q, t, and r are conjectured to enumerate the intervals of the m-Tamari lattices with q counting the *distance* of the interval (length of the maximal chain) and t counting the *dinv* of the maximal element. The signification of the r statistic as well as a combinatorial proof that the polynomials are symmetric in q, t and r, and of their Schur positivity is still open. A wider question would be to find a combinatorial interpretation for the terms appearing in the Schur expansion of the general polynomials A_λ.

Considering the three variables expansion for the triangular case, the first question would be to find a lattice structure "compatible" with the (q,t,r)-enumeration. In [10], the authors describe a generalization of the Tamari lattice for all sub-partitions of any given partition λ. But this lattice does not always give the proper enumeration. In particular, the triangular tableau does not always define a maximal chain in the lattice, which seems to be an important property. This led us to define the *top-down tableau* by iteratively removing the corners of the partition from top to bottom (see last tableau of Figure 7) and to state the following conjecture.

Conjecture 4. *If λ is a triangular partition such that its top-down tableau θ is a sim-sym tableau, then*

$$A_\lambda(q,t,1) = \sum_{(\lambda,\mu_1)\leq(\lambda,\mu_2)} q^{\mathrm{dist}_\lambda(\mu_1,\mu_2)}\ t^{\mathrm{sim}_\theta(\mu_2)} \tag{5.2}$$

where the sum runs over intervals in the λ-Tamari lattice as defined in [10] and $\mathrm{dist}_\lambda(\mu_1,\mu_2)$ is the length of the maximal chain between the triangular Dyck paths (λ,μ_1) and (λ,μ_2).

In the Tamari and m-Tamari cases, the top-down tableau is the triangular tableau and this is the trivariate conjecture found for example in [9]. In the triangular case, there are partitions such as $(4,2,1)$ where the top-down tableau is not the triangular tableau (it is the last one of Figure 7), and actually allows us to find the A_λ enumeration. However, there are cases such as $(4,3,1)$ where the top-down tableau is not sim-sym and the question of finding a compatible lattice structure is still open. Nevertheless, in the case of triangular 2-partitions, the top-down tableau is always sim-sym by Theorem 3. Besides, in this case the λ-Tamari lattice has a very neat structure as shown on Figure 10, which allowed us to perform some recursive enumeration, and prove the conjecture.

Theorem 5. *For λ a triangular 2-partition and θ its top-down tableau, then*

$$A_\lambda(q,t,1) = \sum_{(\lambda,\mu_1)\leq(\lambda,\mu_2)} q^{\mathrm{dist}_\lambda(\mu_1,\mu_2)}\ t^{\mathrm{sim}_\theta(\mu_2)} \tag{5.3}$$

summing over intervals of the λ-Tamari lattice.

Note that in the case of 2-partitions, the Schur expression of A_λ is actually the same as for $A_\lambda(q,t)$, and is thus given by Theorem 2.

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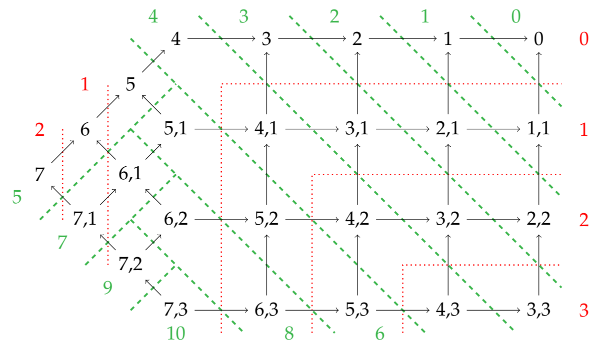

Figure 10: The structure of the λ-lattice on a 2-partition. Elements share the same θ-*deficit* between two red dotted lines and the same θ-*sim* between two green dashed lines.

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