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Combinatorial formulas for shifted dual stable Grothendieck polynomials

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Abstract. The K-theoretic Schur P- and Q-functions GP_λ and GQ_λ may be concretely defined as weight generating functions for semistandard shifted set-valued tableaux. These symmetric functions are the shifted analogues of stable Grothendieck polynomials, and were introduced by Ikeda and Naruse for applications in geometry. Nakagawa and Naruse specified families of dual K-theoretic Schur P- and Q-functions gp_λ and gq_λ via a Cauchy identity involving GP_λ and GQ_λ. They conjectured that the dual power series are weight generating functions for certain shifted plane partitions. We prove this conjecture. We also derive a related generating function formula for the images of gp_λ and gq_λ under the ω involution of the ring of symmetric functions. This confirms a conjecture of Chiu and the second author. Using these results, we verify a conjecture of Ikeda and Naruse that the GQ-functions are a basis for a ring.

Keywords: K-theoretic Schur P- and Q-functions, Cauchy identities, shifted tableaux, set-valued tableaux, plane partitions

1 Introduction

The main results of this extended abstract are explicit combinatorial generating functions for certain families of "dual" power series that were originally defined indirectly by Cauchy identities. The formulas that we establish were conjectured in [2, 20]. We start by giving a summary of the power series involved and the generating functions derived. We then explain an application of our formulas to resolve a conjecture of Ikeda and Naruse from [8]. We also provide a comparison with the work of Lam and Pylyavskyy in the unshifted case [9]. For the full version of these results, with proofs, see [13, 16].

2 Shifted set-valued generating functions

Throughout this section, $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0)$ denotes a strict integer partition. The *shifted Young diagram* of λ is $\mathsf{SD}_\lambda := \{(i, i+j-1) \in [k] \times \mathbb{Z} : 1 \le j \le \lambda_i\}$ where

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$[k] := \{1, 2, \ldots, k\}$. Elements of SD_λ are called *positions* or *boxes*. A *shifted set-valued tableau* of shape λ is a filling T of SD_λ by finite, nonempty subsets of $\frac{1}{2}\mathbb{Z}$. Throughout, we let $i' := i - \frac{1}{2}$ for $i \in \mathbb{Z}$ and refer to half-integers as *primed numbers*.

Let T_{ij} denote the entry assigned by T to box $(i,j) \in \mathsf{SD}_\lambda$, and write $(i,j) \in T$ when (i,j) is in the domain of T. The *diagonal positions* of T are the boxes $(i,j) \in T$ with $i = j$. A shifted set-valued tableau T is *semistandard* if all of the following conditions hold:

(S1) its entries T_{ij} are nonempty finite subsets of $\{1' < 1 < 2' < 2 < \ldots\}$,

(S2) $\max(T_{ij}) \leq \min(T_{i+1,j})$ and $\max(T_{ij}) \leq \min(T_{i,j+1})$ for all relevant $(i,j) \in T$,

(S3) no unprimed number appears in different boxes within the same column, and

(S4) no primed number appears in different boxes within the same row.

We draw shifted tableaux in French notation; for example, both

$$\begin{array}{|c|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & 345 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 2' & 3' & \multicolumn{1}{c}{} \\ \hline 1 & 2' & 2 & 3'3 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 3' & 3 & \multicolumn{1}{c}{} \\ \hline 12 & 2 & 23' & 34 \\ \hline \end{array}$$

are semistandard shifted set-valued tableaux of shape $(4,2,1)$.

Let $\beta, x_1, x_2, x_3, \ldots$ be commuting indeterminates. Define $|T| := \sum_{(i,j)\in T} |T_{ij}|$ and $\mathbf{x}^T := \prod_i x_i^{a_i+b_i}$ where a_i and b_i are the number of times that i and i' appear in T, respectively. Our examples above both have $|T| = 10$ and $\mathbf{x}^T = x_1 x_2^3 x_3^4 x_4 x_5$.

Definition 2.1. Let $\mathsf{ShSVT}_Q(\lambda)$ be the set of all semistandard shifted set-valued tableaux of shape λ and let $\mathsf{ShSVT}_P(\lambda)$ be the subset of such tableaux with no **primed** numbers in any diagonal positions. The *K-theoretic Schur P- and Q-functions* indexed by λ are the formal power series $GP_\lambda := \sum_{T \in \mathsf{ShSVT}_P(\lambda)} \beta^{|T|-|\lambda|}\mathbf{x}^T$ and $GQ_\lambda := \sum_{T \in \mathsf{ShSVT}_Q(\lambda)} \beta^{|T|-|\lambda|}\mathbf{x}^T$.

The functions GP_λ and GQ_λ are sometimes defined by these formulas with $\beta = -1$. This loses no generality since we can recover GP_λ from $\sum_{T \in \mathsf{ShSVT}_P(\lambda)} (-1)^{|T|-|\lambda|}\mathbf{x}^T$ by substituting $x_i \mapsto -\beta x_i$ then dividing by $\beta^{|\lambda|}$. Similar comments apply to GQ_λ.

Both GP_λ and GQ_λ belong to ring $\mathbb{Z}[\beta][\![x_1, x_2, \ldots]\!]$. If $\deg \beta = -1$ then GP_λ and GQ_λ are homogeneous of degree $|\lambda|$, but if $\deg \beta = 0$ then the power series have unbounded degree. Both power series are symmetric in the x_i variables [8, Thm. 9.1]. Setting $\beta = 0$ turns GP_λ and GQ_λ into the classical *Schur P- and Q-functions* P_λ and Q_λ. It follows that as λ ranges over all strict partitions, the sets $\{GP_\lambda\}$ and $\{GQ_\lambda\}$ are linearly independent. While $Q_\lambda = 2^{\ell(\lambda)} P_\lambda$, each GQ_λ is a more complicated (though still finite) $\mathbb{Z}[\beta]$-linear combination of GP_μ's [2, Thm. 1.1].

Ikeda and Naruse introduced GP_λ and GQ_λ in [8] for applications in geometry. Specializations of these symmetric functions represent the structure sheaves of Schubert varieties in the torus-equivariant K-theory of the maximal isotropic Grassmannians of orthogonal and symplectic types [8, Cor. 8.1]; see also [19, 20, 21]. These power series further appear as "stable limits" of K-theory classes of certain orbit closures in [17, 18].

3 Dual functions via Cauchy identities

Our main results concern the following dual forms of GP_λ and GQ_λ.

Definition 3.1. The *dual K-theoretic Schur P- and Q-functions* gp_λ and gq_λ are the unique elements of $\mathbb{Z}[\beta][\![x_1, x_2, \dots]\!]$ indexed by strict partitions λ satisfying the Cauchy identities

$$\sum_\lambda GQ_\lambda(\mathbf{x}) gp_\lambda(\mathbf{y}) = \sum_\lambda GP_\lambda(\mathbf{x}) gq_\lambda(\mathbf{y}) = \prod_{i,j\geq 1} \frac{1-\overline{x_i}y_j}{1-x_iy_j} \quad \text{where } \overline{x} := \frac{-x}{1+\beta x}. \tag{3.1}$$

The power series gp_λ and gq_λ are special cases of Nakagawa and Naruse's *dual universal factorial Schur P- and Q-functions*, which are defined via a more general version of (3.1) [20, Def. 3.2]. Both $\{gp_\lambda\}$ and $\{gq_\lambda\}$ are families of linearly independent functions that are symmetric in the x_i variables [20, Thm. 3.1] and homogeneous if $\deg \beta = 1$ [2, Prop. 6.1]. These properties let one define the following conjugate symmetric functions, which were first considered in [2].

Definition 3.2. Write ω for the $\mathbb{Z}[\beta]$-linear involution of the ring of symmetric functions acting on Schur functions as $\omega(s_\mu) = s_{\mu^\top}$. The *conjugate dual K-theoretic Schur P- and Q-functions* of a strict partition λ are given by $jp_\lambda := \omega(gp_\lambda)$ and $jq_\lambda := \omega(gq_\lambda)$.

4 Shifted plane partition generating functions

Our first main result is a generating function formula for gp_λ and gq_λ that was predicted in [20]. A *shifted plane partition* of strict partition shape λ is a filling of SD_λ by elements of $\{1' < 1 < 2' < 2 < \dots\}$ with weakly increasing rows and columns. Examples include

$$\begin{array}{|c|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{3-4} \multicolumn{2}{c|}{} & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & 2' & 2 & \multicolumn{1}{c}{} \\ \hline 1' & 1' & 1 & 1 & 5' \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 3' & \multicolumn{1}{c}{} \\ \cline{3-4} \multicolumn{2}{c|}{} & 2' & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & 2' & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 1 & 1 & 5 \\ \hline \end{array}$$

which both have shape $(5, 3, 2, 1)$. Given such a filling T, let c_i be the number of distinct columns of T containing i and let r_i be the number of distinct rows of T containing i'. Then define $|\,\mathrm{wt}_{\mathrm{PP}}(T)| = c_1 + r_1 + c_2 + r_2 + \dots$ and $\mathbf{x}^{\mathrm{wt}_{\mathrm{PP}}(T)} := \prod_{i\geq 1} x_i^{c_i+r_i}$. Both examples above have $|\,\mathrm{wt}_{\mathrm{PP}}(T)| = 9$ and $\mathbf{x}^{\mathrm{wt}_{\mathrm{PP}}(T)} = x_1^4 x_2^3 x_3 x_5$.

Theorem 4.1. Let $\mathsf{ShPP}_Q(\lambda)$ be the set of all shifted plane partitions of shape λ, and let $\mathsf{ShPP}_P(\lambda)$ be the subset of such fillings with no **unprimed** diagonal entries. Then

$$gp_\lambda = \sum_{T\in\mathsf{ShPP}_P(\lambda)} (-\beta)^{|\lambda|-|\operatorname{wt}_{\mathrm{PP}}(T)|}\mathbf{x}^{\operatorname{wt}_{\mathrm{PP}}(T)} \text{ and } gq_\lambda = \sum_{T\in\mathsf{ShPP}_Q(\lambda)} (-\beta)^{|\lambda|-|\operatorname{wt}_{\mathrm{PP}}(T)|}\mathbf{x}^{\operatorname{wt}_{\mathrm{PP}}(T)}.$$

This result was conjectured by Nakagawa and Naruse as [20, Conj. 5.1]. These formulas make it possible to compute gp_λ and gq_λ, which is not straightforward from (3.1).

Example 4.2. If $\lambda = (2,1)$ then $\mathsf{ShPP}_P(\lambda)$ consists of

$$\begin{array}{cc} & \boxed{c'} \\ \boxed{a'} & \boxed{b} \end{array},\quad \begin{array}{cc} & \boxed{c'} \\ \boxed{a'} & \boxed{b'} \end{array},\quad \begin{array}{cc} & \boxed{b'} \\ \boxed{a'} & \boxed{a} \end{array},\quad \begin{array}{cc} & \boxed{b'} \\ \boxed{a'} & \boxed{b'} \end{array},\quad \begin{array}{cc} & \boxed{a'} \\ \boxed{a'} & \boxed{a'} \end{array},\quad \text{and} \quad \begin{array}{cc} & \boxed{b'} \\ \boxed{a'} & \boxed{a'} \end{array}$$

for all positive integers $a < b < c$, so Theorem 4.1 asserts that

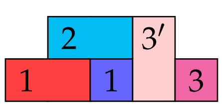
$$gp_{21} = 2\sum_{a<b<c} x_a x_b x_c + \sum_{a<b}(x_a^2 x_b + x_a x_b^2) - \beta\sum_a x_a^2 - \beta\sum_{a<b} x_a x_b = s_{21} - \beta s_2.$$

When $\lambda = (2,1)$, adding primes to the diagonal is a weight-preserving 4-to-1 map $\mathsf{ShPP}_Q(\lambda) \to \mathsf{ShPP}_P(\lambda)$ so Theorem 4.1 also tells us that $gq_{21} = 4s_{21} - 4\beta s_2$.

5 Shifted bar tableaux generating functions

Our second main result is a generating function formula for jp_λ and jq_λ that was predicted in [2]. Continue to let λ be a strict integer partition. Suppose V is a shifted tableau[1] of shape λ with no unprimed entries repeated in any column and no primed entries repeated in any row. Let Π be a partition of the diagram SD_λ into (disjoint, nonempty) subsets of adjacent boxes containing the same entry in V. Each block of Π is a contiguous "bar" of positions in the same row or column, and we refer to the pair $T = (V,\Pi)$ as a *shifted bar tableau* of shape λ.

If V is semistandard in the sense of having weakly increasing rows and columns, then we say that T is also *semistandard*. We draw shifted bar tableaux as pictures like



to represent $T = (V,\Pi) = \left(\begin{array}{ccccc} & \boxed{2} & \boxed{2} & \boxed{3'} & \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{3'} & \boxed{3} \end{array}, \begin{array}{ccccc} & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right).$

These objects are the shifted analogues of what are called *valued-set tableaux* in [9]. The word "valued-set" is just a formal transposition of "set-valued"; we believe that the name "bar tableau" is more intuitive and descriptive.

Given a shifted bar tableau $T = (V,\Pi)$ let $|T| := |\Pi|$ and $\mathbf{x}^T := \prod_{i\geq 1} x_i^{b_i}$ where b_i is the number of blocks in Π containing i or i'. In our example, $|T| = 5$ and $\mathbf{x}^T = x_1^2 x_2 x_3^2$.

[1] That is, a shifted set-valued tableau whose entries are all singleton sets.

Theorem 5.1. Let $\mathsf{ShBT}_Q(\lambda)$ be the set of all semistandard shifted bar tableaux of shape λ. Let $\mathsf{ShBT}_P(\lambda)$ be the subset of such tableaux with no **primed** diagonal entries. Then

$$jp_\lambda = \sum_{T \in \mathsf{ShBT}_P(\lambda)} (-\beta)^{|\lambda|-|T|}\mathbf{x}^T \quad \text{and} \quad jq_\lambda = \sum_{T \in \mathsf{ShBT}_Q(\lambda)} (-\beta)^{|\lambda|-|T|}\mathbf{x}^T.$$

This result was conjectured by Chiu and the second author as [2, Conj. 7.2].

Example 5.2. Suppose $\lambda = (2,1)$. Then $\mathsf{ShBT}_P(\lambda)$ consists of

$$\begin{array}{|c|c|} \hline & c \\ \hline a & b \\ \hline \end{array}\,, \quad \begin{array}{|c|c|} \hline & c \\ \hline a & b' \\ \hline \end{array}\,, \quad \begin{array}{|c|c|} \hline & b \\ \hline a & a \\ \hline \end{array}\,, \quad \begin{array}{|c|c|} \hline & b \\ \hline a & b' \\ \hline \end{array}\,, \quad \text{and} \quad \begin{array}{|cc|} \hline & b \\ \hline \multicolumn{2}{|c|}{a} \\ \hline \end{array}$$

for all positive integers $a < b < c$, so Theorem 5.1 asserts that

$$jp_{21} = 2\sum_{a<b<c} x_a x_b x_c + \sum_{a<b}(x_a^2 x_b + x_a x_b^2) - \beta \sum_{a<b} x_a x_b = s_{21} - \beta s_{11} = \omega(gp_{21}).$$

As in Example 4.2, there is a weight-preserving 4-to-1 map $\mathsf{ShBT}_Q(\lambda) \to \mathsf{ShBT}_P(\lambda)$; this is given by either removing all diagonal primes or applying the map

$$\left\{ \begin{array}{|cc|} \hline & b \\ \hline \multicolumn{2}{|c|}{a} \\ \hline \end{array}\,, \begin{array}{|cc|} \hline & b' \\ \hline \multicolumn{2}{|c|}{a} \\ \hline \end{array}\,, \begin{array}{|c|c|} \hline & b' \\ a & \\ \hline \end{array}\,, \begin{array}{|c|c|} \hline & b' \\ a' & \\ \hline \end{array} \right\} \mapsto \left\{ \begin{array}{|cc|} \hline & b \\ \hline \multicolumn{2}{|c|}{a} \\ \hline \end{array} \right\}.$$

Thus Theorem 5.1 also tells us that $jq_{21} = 4s_{21} - 4\beta s_{11} = \omega(gq_{21})$.

6 Application to conjectures of Ikeda and Naruse

Consider the modules consisting of all infinite $\mathbb{Z}[\beta]$-linear combinations of the functions $\{GP_\lambda\}$ and $\{GQ_\lambda\}$, with λ ranging over all strict partitions. Ikeda and Naruse proved that these modules are both rings [8, Props. 3.4 and 3.5]. This means concretely that $GP_\lambda GP_\mu$ (respectively, $GQ_\lambda GQ_\mu$) always expands as a (possibly infinite) $\mathbb{Z}[\beta]$-linear combination of GP_ν's (respectively, GQ_ν's).

Ikeda and Naruse conjectured that these expansions are actually finite [8, Conj. 3.1 and 3.2], meaning that the **finite** linear spans

$$\mathbf{GP} := \mathbb{Z}[\beta]\text{-span}\{GP_\lambda\} \quad \text{and} \quad \mathbf{GQ} := \mathbb{Z}[\beta]\text{-span}\{GQ_\lambda\} \tag{6.1}$$

(with λ ranging over all strict partition) are also rings. The fact that **GP** is a ring follows from results of Clifford, Thomas, and Yong in [3]; for other proofs see [7, §4], [15, §1.2], and [22, §8]. The problem of showing **GQ** is a ring appears to still be open, however.

Building off [2], we are able to resolve this problem. Specifically, [2, Cor. 7.5] is exactly the assertion that the ring property for the GQ-functions follows from Theorem 5.1 (or more precisely from its skew version, to be given as Theorem 8.4). It follows that:

Theorem 6.1. The $\mathbb{Z}[\beta]$-module **GQ** is a subring of **GP**. Thus, each product $GQ_\lambda GQ_\mu$ is a **finite** $\mathbb{Z}[\beta]$-linear combination of GQ_ν's.

Our proof of this theorem could be adapted to give another proof that **GP** is a ring, but in either case our arguments are nonconstructive. By contrast, [3, Thm. 1.2] gives an explicit Littlewood–Richardson rule for products of *GP*-functions. It is an open problem to find such a rule for the *GQ*-functions, as well as for the *gp*- and *gq*-functions, which span two other subrings of symmetric functions.

7 Comparison with unshifted versions

Our main results are shifted analogues of "classical" theorems, which we summarize here for comparison.

Throughout this section, let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ be an arbitrary integer partition, which is not necessarily strict. The *(unshifted) diagram* of λ is the set of pairs $\mathsf{D}_\lambda := \{(i,j) \in [k] \times \mathbb{Z} : 1 \leq j \leq \lambda_i\}$. A *semistandard (unshifted) set-valued tableau* of shape λ is defined in the same way as the analogous shifted object, except that such a tableau is a filling of D_λ by finite nonempty subsets of $\{1 < 2 < 3 < \dots\}$.

Definition 7.1. Let $\mathsf{SVT}(\lambda)$ be the set of semistandard set-valued tableaux of shape λ. The *stable Grothendieck polynomial* of λ is the power series $G_\lambda := \sum_{T\in\mathsf{SVT}(\lambda)} \beta^{|T|-|\lambda|}\mathbf{x}^T$.

Definition 7.2. The *dual stable Grothendieck polynomials* g_λ are the unique formal power series in $\mathbb{Z}[\beta][\![x_1, x_2, \dots]\!]$ indexed by integer partitions λ satisfying the Cauchy identity

$$\sum_\lambda G_\lambda(\mathbf{x})g_\lambda(\mathbf{y}) = \sum_\lambda s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}) = \prod_{i,j\geq 1}\frac{1}{1-x_iy_j}.$$

Definition 7.3. The *conjugate dual stable Grothendieck polynomial* of λ is $j_\lambda := \omega(g_{\lambda^\top})$.

All three families $\{G_\lambda\}$, $\{g_\lambda\}$, and $\{j_\lambda\}$ are linearly independent symmetric functions which coincide with the usual Schur functions $\{s_\lambda\}$ when $\beta = 0$ [1, 9]. Our definition of j_λ involves a transposition of indices compared to [9, §9.8]; this ensures that $j_\lambda|_{\beta=0} = s_\lambda$.

We define *plane partitions* and *semistandard bar tableaux* of shape λ in the same way as our shifted versions, except the relevant objects are fillings of D_λ by positive integers. Let $\mathsf{PP}(\lambda)$ be the set of plane partitions of shape λ. Let $\mathsf{BT}(\lambda)$ be the set of semistandard bar tableaux of shape λ; these objects are called *valued-set tableaux* in [9, §9].

Theorem 7.4 (Lam and Pylyavskyy [9, §9])**.** For all partitions λ it holds that

$$g_\lambda = \sum_{T\in\mathsf{PP}(\lambda)} (-\beta)^{|\lambda|-|\operatorname{wt}_{\mathsf{PP}}(T)|}\mathbf{x}^{\operatorname{wt}_{\mathsf{PP}}(T)} \quad\text{and}\quad j_\lambda = \sum_{T\in\mathsf{BT}(\lambda)} (-\beta)^{|\lambda|-|T|}\mathbf{x}^T.$$

The stable Grothendieck polynomials G_λ were introduced in Fomin and Kirillov's paper [4] as certain limits of Lascoux and Schützenberger's *Grothendieck polynomials* [11], which are K-theory representatives for Schubert varieties. Buch [1, Thm. 3.1] derived the set-valued tableaux generating function for G_λ given in Definition 7.1, and also proved that the stable Grothendieck polynomials are a $\mathbb{Z}[\beta]$-basis for a ring [1, Cor. 5.5]. For another proof of this ring property, see [23].

Lam and Pylyavskyy defined g_λ and j_λ by the formulas in Theorem 7.4 with $\beta = -1$. They then proved that $\{g_\lambda\}$ is the basis for the ring of symmetric functions dual to $\{G_\lambda\}$ via the Hall inner product [9, Thm. 9.15], and showed that $j_\lambda = \omega(g_{\lambda^\top})$ [9, Prop. 9.25]. For other proofs of the Cauchy identity in Definition 7.2, see [6, 10, 23].

8 Skew versions

As mentioned in Section 6, there are skew generalizations of Theorems 4.1 and 5.1. We write $\mu \subseteq \lambda$ if μ and λ are partitions with $\mu_i \leq \lambda_i$ for all i. If $\mu \subseteq \lambda$ are strict partitions then the *shifted diagram* of λ/μ is the set difference $\mathsf{SD}_{\lambda/\mu} := \mathsf{SD}_\lambda - \mathsf{SD}_\mu$.

Assume $\mu \subseteq \lambda$ are strict partitions. We define shifted set-valued tableaux, plane partitions, and bar tableaux of skew shape λ/μ in exactly the same way as above, only now the relevant objects are fillings of $\mathsf{SD}_{\lambda/\mu}$. The definitions of all related weight statistics like the monomials $\mathbf{x}^T$ are also unchanged. Here is some more relevant notation:

- Let $\mathsf{ShPP}_Q(\lambda/\mu)$ be the set of shifted plane partitions of shape λ/μ, and define $\mathsf{ShPP}_P(\lambda/\mu)$ to be the subset of such fillings with no unprimed diagonal entries.
- Let $\mathsf{ShBT}_Q(\lambda/\mu)$ be the set of semistandard shifted bar tableaux of shape λ/μ, and let $\mathsf{ShBT}_P(\lambda/\mu)$ be the subset of such tableaux with no primed diagonal entries.

We define these sets to be empty if $\mu \not\subseteq \lambda$. These sets will index the terms in generating functions for the skew analogues of gq_λ, gp_λ, jq_λ, and jp_λ, which we now define.

Let $y_1, y_2, \dots$ be another countable set of commuting variables, and for each $f \in \mathbb{Z}[\beta][\![x_1, x_2, \dots]\!]$ define

$$f(\mathbf{x}) := f(x_1, x_2, \dots) = f,\ f(\mathbf{y}) := f(y_1, y_2, \dots),\ \text{and } f(\mathbf{x}, \mathbf{y}) := f(x_1, y_1, x_2, y_2, \dots).$$

If f is symmetric, then we think of $f(\mathbf{x}, \mathbf{y})$ as "$f(x_1, x_2, \dots, y_1, y_2, \dots)$" although it is not clear how to interpret this notation for an arbitrary power series.

Definition 8.1. For strict partitions λ and μ, let $gp_{\lambda/\mu}$ and $gq_{\lambda/\mu}$ be the elements of $\mathbb{Z}[\beta][\![x_1, x_2, \dots]\!]$ with $gp_\lambda(\mathbf{x}, \mathbf{y}) = \sum_\mu gp_\mu(\mathbf{x}) gp_{\lambda/\mu}(\mathbf{y})$ and $gq_\lambda(\mathbf{x}, \mathbf{y}) = \sum_\mu gq_\mu(\mathbf{x}) gq_{\lambda/\mu}(\mathbf{y})$, where the sums are over all strict partitions μ.

Both $gp_{\lambda/\mu}$ and $gq_{\lambda/\mu}$ are symmetric and homogeneous if $\deg \beta = 1$, but the families of such functions are no longer linearly independent. These power series are defined for all strict λ and μ but are nonzero if and only if $\mu \subseteq \lambda$ [2, Props. 6.4 and 6.6].

Definition 8.2. For strict partitions λ and μ, let $jp_{\lambda/\mu}$ and $jq_{\lambda/\mu}$ be the elements of $\mathbb{Z}[\beta][\![x_1, x_2, \ldots]\!]$ with $jp_\lambda(\mathbf{x}, \mathbf{y}) = \sum_\mu jp_\mu(\mathbf{x}) jp_{\lambda/\mu}(\mathbf{y})$ and $jq_\lambda(\mathbf{x}, \mathbf{y}) = \sum_\mu jq_\mu(\mathbf{x}) jq_{\lambda/\mu}(\mathbf{y})$, where the sums are over all strict partitions μ.

Equivalently, one could define these skew analogues by the following formula:

Proposition 8.3 ([2, Eq. (7.4)])**.** It holds that $jp_{\lambda/\mu} := \omega(gp_{\lambda/\mu})$ and $jq_{\lambda/\mu} := \omega(gq_{\lambda/\mu})$.

Our main theorems extend to this skew setting as follows.

Theorem 8.4. For all strict partitions λ and μ, one has

$$gp_{\lambda/\mu} = \sum_{T \in \mathsf{ShPP}_P(\lambda/\mu)} (-\beta)^{|\lambda/\mu| - |\operatorname{wt}_{\mathrm{PP}}(T)|} \mathbf{x}^{\operatorname{wt}_{\mathrm{PP}}(T)}, \quad jp_{\lambda/\mu} = \sum_{T \in \mathsf{ShBT}_P(\lambda/\mu)} (-\beta)^{|\lambda/\mu| - |T|} \mathbf{x}^T,$$
$$gq_{\lambda/\mu} = \sum_{T \in \mathsf{ShPP}_Q(\lambda/\mu)} (-\beta)^{|\lambda/\mu| - |\operatorname{wt}_{\mathrm{PP}}(T)|} \mathbf{x}^{\operatorname{wt}_{\mathrm{PP}}(T)}, \quad jq_{\lambda/\mu} = \sum_{T \in \mathsf{ShBT}_Q(\lambda/\mu)} (-\beta)^{|\lambda/\mu| - |T|} \mathbf{x}^T.$$

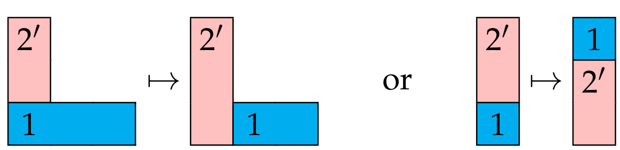
There are also skew versions of GP_λ and GQ_λ, which are symmetric by results in [12].

9 Proof ideas

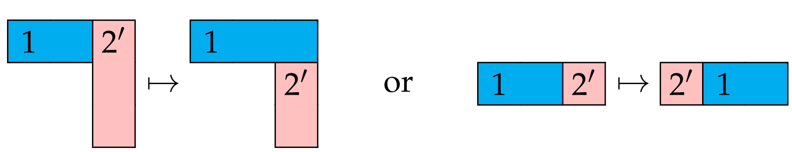
The most difficult part of our proofs of Theorems 4.1, 5.1 and 8.4 is showing that the desired generating functions are symmetric. We derive this explicitly for jp_λ and jq_λ by constructing Bender–Knuth involutions for semistandard shifted bar tableaux. We briefly sketch this construction, which may be of independent interest.

The ith Bender–Knuth involution applied to a semistandard bar tableau T produces a new semistandard bar tableau U in which the number of bars filled with i or i' in T is equal to the number of bars filled with $i+1$ or $(i+1)'$ in U, and vice versa. It is enough to consider the case $i = 1$. The involution proceeds in three steps, using an intermediate object called *sorted bar tableaux*.

Essentially, these intermediate objects appear to be semistandard bar tableaux if the order on the entries were $1' < 2' < 1 < 2$ (rather than $1' < 1 < 2' < 2$); however, some special configurations that violate this description may occur near the diagonal. The first step, converting a semistandard shifted bar tableau to a sorted shifted bar tableau, consists of a number of moves of the form

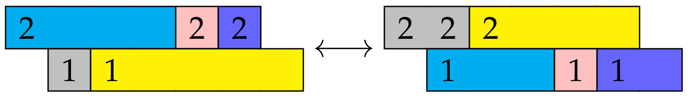


and

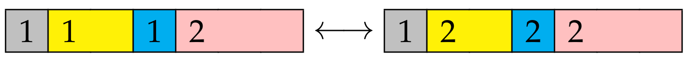


in an appropriate order.

In the second stage, one performs a weight-reversing involution on sorted tableaux. This map is straightforward to describe away from the diagonal. For example, for a collection of bars filled with 1s and 2s forms a configuration like the ones below, we form another two-row-group occupying the same boxes that has a bar division between columns j and $j+1$ in row l (respectively, $l+1$) if and only if a bar division occurs between columns j and $j+1$ in row $l+1$ (respectively, l) in the starting group:



If, instead, a collection of bars filled with 1s and 2s forms a configuration like the ones below, then we form another one-row-group by swapping the number of 1-blocks and 2-blocks (but leaving the bar divisions untouched):



One applies similar moves to the column bars filled with $1'$ and $2'$. These maps obviously reverse the weight, but it is necessary to consider numerous special cases that occur near the diagonal.

Finally, in the third step, one reverses the process from the first step to produce a semistandard shifted bar tableau, whose weight is reversed from the original tableau.

Once symmetry is established, we are able to prove Theorem 8.4 (and so in particular Theorems 4.1 and 5.1) by an inductive algebraic argument. The first step in this argument is to verify Theorem 8.4 when $\lambda = (r)$ has a single nonzero part. This is carried out in [2, §7] by a direct computation.

For the inductive step, it turns out to be sufficient to show that both sides of the identities in Theorem 5.1 satisfy the same Pieri rule when multiplied by $jq_{(r)}$. The Cauchy identity (3.1) lets us derive Pieri rules expanding $jp_\lambda jq_{(r)}$ and $jq_\lambda jq_{(r)}$ from the tableau generating functions for GP_λ and GQ_λ. Our proof that the same rules apply to the desired combinatorial formulas for jp_λ and jq_λ relies on the symmetry of those generating functions: we restrict to a finite number of variables, and interpret a single variable x as recording on one hand the smallest number appearing in the tableaux (corresponding to an inner shape an a skewing operation) but also (on the other hand, by symmetry) as recording the largest number appearing in the tableau (corresponding to a border strip on the outer boundary).

10 Hopf algebras

The power series GP_λ, GQ_λ, $gp_{\lambda/\mu}$, $gq_{\lambda/\mu}$, $jp_{\lambda/\mu}$ and $jq_{\lambda/\mu}$ arise when studying certain Hopf algebras of symmetric functions. We discuss these objects here for more context.

Let R be a commutative ring. Write $\otimes = \otimes_R$ for the corresponding tensor product. We have a familiar notion of an (associative, unital) *R-algebra*: this is an R-module A with an R-linear map $\nabla : A \otimes A \to A$ (called the *product*) and another R-linear map $\iota : R \to A$ (called the *unit*) satisfying a few compatibility axioms.

Dually, an *R-coalgebra* is an R-module with an R-linear map $\Delta : A \to A \otimes A$, (called the *coproduct*) and a linear map $\varepsilon : A \to R$ (called the *counit*) subject to a similar set of axioms; see [5, §1] for the complete definitions. An R-algebra that is also a coalgebra is an *R-bialgebra* if the coproduct and counit maps are algebra homomorphisms.

If A is an R-bialgebra then the set $\mathrm{End}(A)$ of R-linear maps $f : A \to A$ is itself an R-algebra for the product $f * g := \nabla \circ (f \otimes g) \circ \Delta$. The unit element of this *convolution algebra* is the composition $\iota \circ \epsilon$. The bialgebra A is a *Hopf algebra* if the identity map $A \to A$ has a two-sided inverse $\mathtt{S} : A \to A$ (called the *antipode*) in $\mathrm{End}(A)$ relative to $*$.

We specialize to the case when $R = \mathbb{Z}[\beta]$. The objects of interest to us are bialgebras of formal power series in $\mathbb{Z}[\beta][\![x_1, x_2, \ldots]\!]$. In this setting, the counit ε will always be the map setting $x_1 = x_2 = \cdots = 0$, and the product and unit will always be the ones inherited from $\mathbb{Z}[\beta][\![x_1, x_2, \ldots]\!]$. The only maps left to specify are Δ and $\mathtt{S}$ (if it exists).

For example, let $\mathsf{Sym} = \mathbb{Z}[\beta]\text{-span}\{s_\lambda : \lambda \text{ is any partition}\}$ be the $\mathbb{Z}[\beta]$-algebra of bounded-degree symmetric functions. Write $c^\nu_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ for the Littlewood–Richardson coefficients such that $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$. It is well-known [5, §2] that Sym is a Hopf algebra for the coproduct Δ and antipode $\mathtt{S}$ satisfying

$$\Delta(s_\nu) = \sum_{\lambda,\mu \subseteq \nu} c^\nu_{\lambda\mu} s_\mu \otimes s_\lambda \quad \text{and} \quad \mathtt{S}(s_\nu) = (-1)^{|\nu|} s_{\nu^\top} = s_{\nu^\top}(-\mathbf{x}), \tag{10.1}$$

where in the sum λ and μ range over all partitions. It also holds that $s_\nu(\mathbf{x}, \mathbf{y}) = \sum_{\lambda,\mu} c^\nu_{\lambda\mu} s_\mu(\mathbf{x}) s_\lambda(\mathbf{y})$, so the map Δ is the composition of $f \mapsto f(\mathbf{x}, \mathbf{y})$ with the isomorphism $\mathsf{Sym}(\mathbf{x}) \otimes \mathsf{Sym}(\mathbf{y}) \xrightarrow{\sim} \mathsf{Sym} \otimes \mathsf{Sym}$.

Define $\mathbf{gp} = \mathbb{Z}[\beta]\text{-span}\{gp_\lambda\}$ and $\mathbf{gq} = \mathbb{Z}[\beta]\text{-span}\{gq_\lambda\}$ where λ ranges over all strict partitions. These are both submodules of Sym, with $\mathbf{gq} \subseteq \mathbf{gp}$ by results in [2]. The skew and conjugate versions of gp_λ and gq_λ are motivated algebraically by the following:

Theorem 10.1. Both $\mathbf{gp}$ and $\mathbf{gq}$ are Hopf subalgebras of Sym. It holds that

$$\Delta(gp_\nu) = \sum_{\lambda \subseteq \nu} gp_\lambda \otimes gp_{\nu/\lambda} \quad \text{and} \quad \Delta(gq_\nu) = \sum_{\lambda \subseteq \nu} gq_\lambda \otimes gq_{\nu/\lambda},$$

where the sums are over strict partitions. Also, $\mathtt{S}(gp_\nu) = jp_\nu(-\mathbf{x})$ and $\mathtt{S}(gq_\nu) = jq_\nu(-\mathbf{x})$.

Recall the definitions of $\mathbf{GP} \subseteq \mathbf{GQ}$ from (6.1). Although GP_λ and GQ_λ are infinite $\mathbb{Z}[\beta]$-linear combinations of Schur functions, the formula for Δ in (10.1) extends by "continuous linearity" to a well-defined map $\mathbf{GP} \to \mathbf{GP} \otimes \mathbf{GP}$, and the following holds:

Theorem 10.2. Both $\mathbf{GP}$ and $\mathbf{GQ}$ are bialgebras, but neither is a Hopf algebra.

Finally, let $\langle\cdot,\cdot\rangle : \mathsf{Sym}\otimes\mathsf{Sym}\to\mathbb{Z}[\beta]$ be the $\mathbb{Z}[\beta]$-linear form with $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$ for all partitions λ and μ. This extends to a bilinear form on $\mathsf{Sym}\otimes\mathsf{Sym}$ with $\langle f_1\otimes f_2, g_1\otimes g_2\rangle := \langle f_1, g_1\rangle\langle f_2, g_2\rangle$. The formulas (10.1) imply that Sym is self-dual with respect to this form, in the sense that $\langle fg, h\rangle = \langle f\otimes g, \Delta(h)\rangle$ for all $f, g, h \in \mathsf{Sym}$. Analogously:

Theorem 10.3. There is a unique $\mathbb{Z}[\beta]$-bilinear form $[\cdot,\cdot] : \mathbf{gp}\otimes\mathbf{GP}\to\mathbb{Q}[\beta]$ such that

$$[gp_\lambda, GQ_\mu] = [gq_\lambda, GP_\mu] = \delta_{\lambda\mu}, \quad [fg, H] = [f\otimes g, \Delta(H)], \quad \text{and} \quad [f, HK] = [\Delta(f), H\otimes K]$$

for all strict partitions λ, μ and all elements $f, g \in \mathbf{gp}$ and $H, K \in \mathbf{GP}$.

Despite this result, $\mathbf{GP}$ and $\mathbf{GQ}$ are not the duals of $\mathbf{gq}$ and $\mathbf{gp}$, because the restricted forms $[\cdot,\cdot] : \mathbf{gq}\otimes\mathbf{GP}\to\mathbb{Z}[\beta]$ and $[\cdot,\cdot] : \mathbf{gp}\otimes\mathbf{GQ}\to\mathbb{Z}[\beta]$ are technically degenerate. Instead, the duals of $\mathbf{gq}$ and $\mathbf{gp}$ are the "completions" $\widehat{\mathbf{G}}\mathbf{P}$ and $\widehat{\mathbf{G}}\mathbf{Q}$, which respectively consist of all infinite $\mathbb{Z}[\beta]$-linear combinations of GP- and GQ-functions.

These objects are too large to be Hopf algebras (at least, in the category of $\mathbb{Z}[\beta]$-modules), but are examples of what are called *linearly compact Hopf algebras* in [12, 14, 16]. The reason why $\mathbf{GP}$ and $\mathbf{GQ}$ are not Hopf algebras is that the adjoints of the antipodes of $\mathbf{gq}$ and $\mathbf{gp}$ under $[\cdot,\cdot]$ only make sense as maps $\widehat{\mathbf{G}}\mathbf{P}\to\widehat{\mathbf{G}}\mathbf{P}$ and $\widehat{\mathbf{G}}\mathbf{Q}\to\widehat{\mathbf{G}}\mathbf{Q}$.

There are still many open questions related to the symmetric functions in this abstract. For example, Theorems 10.1 and 10.2 imply that products of gp-/gq-/GP-/GQ-functions respectively expand as finite $\mathbb{Z}[\beta]$-linear combinations of gp-/gq-/GP-/GQ-functions. It is expected that the coefficients in these expansions are all actually in $\mathbb{Z}_{\geq 0}[\beta] \subsetneq \mathbb{Z}[\beta]$. At present, however, this is conjectural outside the case of GP-functions; in the GP case, combinatorial interpretations for the coefficients expanding $GP_\lambda GP_\mu$ in $\bigoplus_\nu \mathbb{Z}_{\geq 0}[\beta] GP_\nu$ appear in [3, 7, 22]. For a list of related open problems, see [16, §4.6].

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