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Tautological classes of matroids

Andrew Berget[1], Christopher Eur[*2], Hunter Spink[3], and Dennis Tseng[†4]

[1] *Western Washington University, Bellingham, WA, USA*
[2] *Harvard University, Cambridge, MA, USA*
[3] *Stanford University, Stanford, CA, USA*
[4] *New York, NY, USA*

Abstract. We introduce certain classes on permutohedral varieties which we call "tautological classes of matroids" as a new framework for studying matroids. Using this framework, we unify and extend many recent developments in matroid theory arising from its interaction with algebraic geometry. We achieve this by establishing a cohomological description and a log-concavity property for a 4-variable transformation of the Tutte polynomial, and by establishing an exceptional Hirzebruch-Riemann-Roch formula for permutohedral varieties that bridges K-theory and cohomology theory.

Keywords: Matroids, log-concavity, toric geometry

1 Introduction

Matroid theory has interacted fruitfully with algebraic geometry. See [3] for a survey of recent developments. These developments however had a constraint: They occurred through different geometric models of matroids that were *disjoint*—in the sense that the results and techniques specific to one model could not be easily imported to another.

We introduce "tautological classes of matroids" as a unified framework that overcomes this constraint. The key advantage of this unified approach is that powerful techniques that were previously applicable only within different models can now be used seamlessly in conjunction with each other. This advantage allowed us to recover and extend various recent results in matroid theory, and to resolve open problems. For instance, we establish the following log-concavity for the Tutte polynomial of a matroid.

Theorem 3.3.(iii). Let M be a matroid of rank r on a ground set E, and let $T_{\mathrm{M}}(x,y)$ be its Tutte polynomial. Then, the coefficients of the 4-variable transformation

$$(x+y)^{-1}(y+z)^r(x+w)^{|E|-r}T_{\mathrm{M}}\Big(\frac{x+y}{y+z},\frac{x+y}{x+w}\Big)$$

form a "log-concave unbroken array" (see Definition 3.2).

*ceur@math.harvard.edu. C. Eur was partially supported by NSF DMS-2001854.
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Our framework also opens new doors: For instance, it has already led to the development of the K-theory of matroids [29], the stellahedral geometry of matroids [22], the Gromov-Witten theory of matroids [35], and the tropical geometry of "type B" generalizations of matroids known as delta-matroids [21].

In this extended abstract of our paper [8], in order to emphasize the combinatorial aspects, we present the framework of "tautological classes of matroids" slightly differently from the presentation in [8]. For example, not emphasized in the presentation here is the method of localization in torus-equivariant geometry, even though it is among crucial tools in our proofs. We assume familiarity with the fundamentals of matroids theory, and point to [34, 37] as standard references. We also assume some familiarity with polyhedral geometry, and point to [38] as a reference.

In Section 2, we review in a purely combinatorial manner some previous studies of matroids via algebraic geometry. In Section 3, we construct the "tautological classes" of a matroid, and explain our main results arising from the framework. In Section 4, we explain the underlying algebraic geometry.

Notations. Let $E = \{1, \dots, n\}$ be a finite set of cardinality n, and let $\mathfrak{S}_E$ the permutation group on E. For a subset $S \subseteq E$, we denote by $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i \in \mathbb{R}^E$ the sum of the standard basis vectors indexed by S. Denote by $\langle \cdot, \cdot \rangle$ the standard inner-product on $\mathbb{R}^E$. Let $T = (\mathbb{C}^*)^E = \{(x_1, \dots, x_n) \in \mathbb{C}^E : x_1 \cdots x_n \neq 0\}$ be the standard torus, and let $\mathbb{P}T = T/\mathbb{C}^*$ be the projectivization of T. All algebraic varieties are over $\mathbb{C}$.

2 Background

Many previous studies of matroids via algebraic geometry fall broadly into one of two camps: the "K-theoretic" camp and the "cohomological" camp. More precisely, these correspond to two well-studied geometric rings attached a smooth projective variety X, namely, the Grothendieck K-ring $K(X)$ of vector bundles on X, and the singular cohomology ring $H^\bullet(X)$ of X.

When X is a smooth projective *toric* variety, these two rings admit purely combinatorial descriptions, reviewed in Section 2.1. Specializing to the case of the permutohedral varieties, we highlight in Section 2.2 some previous studies of matroids via these rings.

2.1 A tale of two rings

Let Σ be the normal fan of a simple lattice polytope in $\mathbb{R}^E$ that is unimodular with respect to the lattice $\mathbb{Z}^E$. A *deformation* of Σ is a lattice polytope $P \subset \mathbb{R}^E$ such that the normal fan of P coarsens Σ. We write $P \in \mathrm{Def}(\Sigma)$ in this case. Let X_Σ be the smooth projective toric variety associated to Σ. In this case, the two rings $K(X_\Sigma)$ and $H^\bullet(X_\Sigma)$ have purely combinatorial descriptions as follows.

For a subset $P \subseteq \mathbb{R}^E$, define the function $\mathbf{1}_P : \mathbb{R}^E \to \mathbb{Z}$ by $\mathbf{1}_P(x) = 1$ if $x \in P$ and $\mathbf{1}_P(x) = 0$ if otherwise. Let

$$\mathbb{I}(\Sigma) = \text{the subgroup of } \mathbb{Z}^{(\mathbb{R}^E)} \text{ generated by } \{\mathbf{1}_P : P \in \operatorname{Def}(\Sigma)\}.$$

Let $\overline{\mathbb{I}}(\Sigma)$ be the quotient of $\mathbb{I}(\Sigma)$ by the subgroup generated by $\{\mathbf{1}_P - \mathbf{1}_{P+v} : P \in \operatorname{Def}(\Sigma) \text{ and } v \in \mathbb{Z}^E\}$. This group is closely related to McMullen's polytope algebra [31], and has a ring structure with multiplication determined by $\mathbf{1}_P \cdot \mathbf{1}_{P'} = \mathbf{1}_{P+P'}$, where $P + P'$ denotes the Minkowski sum of polytopes. It has the unit $1 = \mathbf{1}_{\text{the origin}}$. See [22, Appendix A] for details.

Theorem 2.1. [22, Theorem A.9] (cf. [33]) The ring $K(X_\Sigma)$ is isomorphic to $\overline{\mathbb{I}}(\Sigma)$.

For $P \in \operatorname{Def}(\Sigma)$, let $[P]$ denote the class of $\mathbf{1}_P$ in $\overline{\mathbb{I}}(\Sigma) \simeq K(X_\Sigma)$. One can show that $1 - [P]$ is nilpotent in $\overline{\mathbb{I}}(\Sigma)$, and hence $[P]$ admits an inverse. That is, we may consider $[P]^{-1} = \frac{1}{1-(1-[P])} = 1 + (1-[P]) + (1-[P])^2 + \cdots$.

On the other hand, let $PP^\bullet(\Sigma)$ be the ring of piecewise polynomial functions in the n-variables $t_1, \ldots, t_n$ with integral coefficients. That is, on each cone of Σ we have a polynomial in $\mathbb{Z}[t_1, \ldots, t_n]$ such that at an intersection of any two cones, the polynomials agree. It is a graded ring with the i-th graded piece $PP^i(\Sigma)$ consisting of homogeneous degree i piecewise polynomial functions. Let $\overline{PP}^\bullet(\Sigma)$ be the quotient of $PP^\bullet(\Sigma)$ by the ideal generated by the global polynomials (i.e. a piecewise polynomial that is the same polynomial on all cones).

Theorem 2.2. [18, 13] The ring $H^\bullet(X_\Sigma)$ is isomorphic to $\overline{PP}^\bullet(\Sigma)$ as graded rings.[1]

For $P \in \operatorname{Def}(\Sigma)$, let $[h_P]$ denote the class in $\overline{PP}^1(\Sigma)$ of the piecewise polynomial (in fact, piecewise linear) function $h_P \in PP^1(\Sigma)$ defined by $h_P(x) = \min_{y \in P}\langle x, y\rangle$.

The two rings $K(X_\Sigma)$ and $H^\bullet(X_\Sigma)$ have two natural geometric maps to $\mathbb{Z}$, namely, the sheaf Euler characteristic map $\chi : K(X_\Sigma) \to \mathbb{Z}$, and the degree map $\deg_\Sigma : H^\bullet(X_\Sigma) \to \mathbb{Z}$. Under the isomorphisms in Theorems 2.1 and 2.2, these two maps have the following combinatorial characterizations [24]: For any $P \in \operatorname{Def}(\Sigma)$, the maps χ and $\deg_\Sigma$ satisfy

$$\chi([P]) = |P \cap \mathbb{Z}^E| \quad \text{and} \quad \deg_\Sigma([h_P]^n) = \operatorname{volume}(P).$$

Volumes are normalized such that a unit n-dimensional simplex in $\mathbb{R}^E$ has volume 1.

[1] The isomorphism doubles the grading, in the sense that $H^{2i}(X_\Sigma) \simeq \overline{PP}^i(\Sigma)$ for all i.

2.2 The permutohedral fan and matroids

We now specialize to the *permutohedral fan* Σ_E, the normal fan of the *permutohedron*

$$\Pi_E = \text{convex hull of } \{w \cdot (1, \dots, n) : w \in \mathfrak{S}_E\} \subset \mathbb{R}^E.$$

In this case, a lattice polytope $P \subset \mathbb{R}^E$ is a deformation of Σ_E if and only if each edge of P is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in E$. Consequently, we have $-P \in \mathrm{Def}(\Sigma)$ if and only if $P \in \mathrm{Def}(\Sigma)$. (Note that $[-P] \neq -[P]$). Two notable deformations are

$$\Delta = \text{convex hull of } \{\mathbf{e}_i : i \in E\} \quad \text{and} \quad \nabla = \text{convex hull of } \{-\mathbf{e}_i : i \in E\}.$$

The ring $\overline{PP}^\bullet(\Sigma_E)$ has the following explicit presentation.

$$\overline{PP}^\bullet(\Sigma_E) = \frac{\mathbb{Z}[x_S : S \text{ a nonempty proper subset of } E]}{\langle x_S x_{S'} : S \not\subseteq S' \text{ and } S \not\supseteq S'\rangle + \langle \sum_{S \ni i} x_S - \sum_{S' \ni j} x_{S'} : i \neq j \in E\rangle}.$$

We now review some studies of matroids via the rings $\overline{\mathbb{I}}(\Sigma)$ and $\overline{PP}^\bullet(\Sigma)$. We highlight only a small portion of such studies, focusing the Tutte polynomial [36, 17], one of the most famous invariant of matroids. A more extensive review can be found in [8, §1.1] or [3]. Throughout, let M be a matroid of rank r on the ground set E.

Model I: Base polytopes

On the "K-theoretic" camp, the classical result of [26] showed that the *base polytope*

$$P(\mathrm{M}) = \text{convex hull of } \{\mathbf{e}_B : B \subseteq E \text{ a basis of M}\} \subset \mathbb{R}^E$$

is a deformation of Σ_E. Hence, we may consider $[P(\mathrm{M})]$ in the ring $\overline{\mathbb{I}}(\Sigma)$. We highlight the following "K-theoretic" manifestation of the Tutte polynomial of a matroid. To state it, note that $\chi([P(\mathrm{M}) + u\Delta + t\nabla])$, the number of lattice points in $P(\mathrm{M}) + u\Delta + t\nabla$ for u and t nonnegative integers, is a polynomial in u and t by standard Ehrhart theory.

Theorem 2.3. [14, 10] Let $\Psi : \mathbb{Q}[t,u] \to \mathbb{Q}[x,y]$ be defined as the invertible linear map sending $\binom{t}{i}\binom{u}{j} \mapsto x^i y^j$ for all $i,j \geq 0$. Let $T_\mathrm{M}(x,y)$ denote the Tutte polynomial of the matroid M. Then, we have

$$\Psi\Big(\chi([P(\mathrm{M}) + u\Delta + t\nabla])\Big) = (x+y+1)^{-1}(y+1)^r(x+1)^{|E|-r} T_M\Big(\frac{x+y+1}{y+1}, \frac{x+y+1}{x+1}\Big).$$

For another "K-theoretic" manifestation of the Tutte polynomial that also involves lattice point counting, see [23]. A powerful technique in the study of matroids in this "K-theoretic" way is that one can exploit the special polytopal properties of $P(\mathrm{M})$, such as the Hopf monoid structure [2] and valuativity [5, 7].

Model II: Bergman classes

On the "cohomological" camp, a matroid M defines a class $[\Sigma_{\mathrm{M}}] \in \overline{PP}^{n-r}(\Sigma_E)$, called the *Bergman class* of M, defined as follows. If M has loops, we set $[\Sigma_{\mathrm{M}}] = 0$. Suppose now M is loopless. One shows via the theory of Minkowski weights that the assignment

$$x_{S_1} \cdots x_{S_{r-1}} \mapsto \begin{cases} 1 & \text{if } S_1 \subsetneq \cdots \subsetneq S_{r-1} \text{ is a chain of nonempty proper flats of M} \\ 0 & \text{otherwise} \end{cases}$$

gives a well-defined element $w_{\mathrm{M}} \in \mathrm{Hom}(\overline{PP}^{r-1}(\Sigma_E), \mathbb{Z})$ [25, 6]. The dual element w_{M} then defines the class $[\Sigma_{\mathrm{M}}] \in \overline{PP}^{n-r}(\Sigma_E)$, since as a cohomology ring, $\overline{PP}^{\bullet}(\Sigma_E)$ is equipped with the Poincaré duality perfect pairing $\overline{PP}^{n-r}(\Sigma_E) \times \overline{PP}^{r-1}(\Sigma_E) \to \mathbb{Z}$ defined by $(a, b) \mapsto \deg_\Sigma(ab)$.

We highlight the following "cohomological" manifestation of a specialization of the Tutte polynomial known as the *characteristic polynomial* of a matroid.

Theorem 2.4. [28] Let M be a loopless matroid (of rank r), and $T_{\mathrm{M}}(x, y)$ the Tutte polynomial of M. Then, we have

$$\sum_{i=0}^{r-1} \Big(\deg_\Sigma([\Sigma_{\mathrm{M}}][h_\Delta]^{r-i}[h_\nabla]^i) \Big) q^{r-1-i} = T_{\mathrm{M}}(q+1, 0)/(q+1).$$

For other "cohomological" manifestations of the characteristic polynomial that also involves the degree map, see [32] and [4]. A powerful technique in the study of matroids in this "cohomological" way is the major breakthrough known as the "Hodge theory of matroids," paraphrased in the following theorem.

Theorem 2.5. [1, 4] Rings derived from Bergman classes of matroids satisfy the Hodge-Riemann relations. Consequently, the values of $\deg_\Sigma$ involving the Bergman class $[\Sigma_{\mathrm{M}}]$ display a log-concavity behavior.

The authors of [1, 4] combined this Hodge theory of matroids with formulas like Theorem 2.4 to resolve several log-concavity conjectures in matroid theory. For a history and a survey of this development see [20].

Question 2.6. Having reviewed these previous geometric studies of matroids, either through base polytopes in K-theory, or through Bergman classes in cohomology theory, one may naturally ask the following questions:

- The formulas in Theorems 2.3 and 2.4 have a remarkable resemblance to one another. How are they actually related? More generally, how do various manifestations of the Tutte polynomial or its specializations in the theorems and in [23, 32, 4] relate to each other?

- Is there a "cohomological" manifestation of the whole Tutte polynomial itself, rather than its specialization? Moreover, is there a log-concavity behavior for the whole Tutte polynomial?

We fully answer both questions by introducing the tautological classes of matroids.

Remark 2.7. To answer the questions, one may attempt to use the classical Hirzebruch-Riemann-Roch theorem, which in our language states the following. There exist a unique ring isomorphism $ch : \overline{\mathbb{I}}(\Sigma) \otimes \mathbb{Q} \to \overline{PP}^{\bullet}(\Sigma) \otimes \mathbb{Q}$ satisfying $[P] \mapsto \exp([h_P])$ for all $P \in \mathrm{Def}(\Sigma)$, and a class $Td(\Sigma) \in \overline{PP}^{\bullet}(\Sigma) \otimes \mathbb{Q}$ called the *Todd class* of Σ, such that

$$\chi(\xi) = \deg_{\Sigma}\left(ch(\xi) \cdot Td(\Sigma)\right) \quad \text{for all } \xi \in \overline{\mathbb{I}}(\Sigma) \otimes \mathbb{Q}.$$

However, the Todd class often does not admit sufficiently explicit description for combinatorial computations. In fact, the study of Todd classes of permutohedral fans is a research direction of its own [15]. We will instead introduce an exceptional Hirzebruch-Riemann-Roch formula to bridge the K-theory and cohomology theory (Theorem 3.3.(iv)).

3 Main construction and results

We now introduce the tautological classes of matroids and explain the main results. We will follow the construction in [8, Appendix III], which is equivalent to but looks different from the more geometric description in [8, §3]. Geometric motivation behind the constructions and the main results is explained in Section 4.

Let M be a matroid of rank r on ground set E. Define matroids M_i for $i = 0, \ldots, n$ by

$$\text{the set bases of } \mathrm{M}_i = \{B \subseteq E : |B| = i \text{ and } B \text{ contains or is contained in a basis of } \mathrm{M}\}.$$

The sequence $(\mathrm{M}_0, \ldots, \mathrm{M}_n)$ is sometimes called the Higgs factorization of M.

Definition 3.1. The *tautological classes* of a matroid M are elements in $\overline{\mathbb{I}}(\Sigma_E)$ defined as

$$\mathcal{S}_{\mathrm{M}} = \sum_{i=1}^{r} [-P(\mathrm{M}_{i-1})][-P(\mathrm{M}_i)]^{-1} \quad \text{and} \quad \mathcal{Q}_{\mathrm{M}} = \sum_{i=r+1}^{n} [-P(\mathrm{M}_{i-1})][-P(\mathrm{M}_i)]^{-1}.$$

The k-th *Chern class* $c_k(\mathcal{S}_{\mathrm{M}}) \in \overline{PP}^{k}(\Sigma_E)$ is defined as

$$c_k(\mathcal{S}_{\mathrm{M}}) = k\text{-th elementary symmetric polynomial in } \{[h_{-P(\mathrm{M}_{i-1})}] - [h_{-P(\mathrm{M}_i)}]\}_{i=1,\ldots,r}$$

and similarly for $c_k(\mathcal{Q}_{\mathrm{M}})$.

To state the main results, we will need the following definition. Recall that a nonnegative sequence $(a_0, a_1, \dots, a_m)$ is *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq m-1$, and has *no internal zeros* if $a_i a_j > 0 \implies a_k > 0$ for all $0 \leq i \leq k \leq j \leq m$.

Definition 3.2. For a homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_N]$ of degree d with nonnegative coefficients, we say that its coefficients form a *log-concave unbroken array* if, for any $1 \leq i < j \leq N$ and a monomial $x^{\mathbf{m}}$ of degree $d' \leq d$, the coefficients of $\{x_i^k x_j^{d-d'-k} x^{\mathbf{m}}\}_{0 \leq k \leq d-d'}$ in f form a log-concave sequence with no internal zeros.

We now state the main results [8, Theorems A, B, C, D].

Theorem 3.3. Let M be a matroid of rank r on ground set E, and T_{M} its Tutte polynomial.

(i) We have $c_1(\mathcal{Q}_{\mathrm{M}}) = [h_{-P(\mathrm{M})}]$ and $c_{n-r}(\mathcal{Q}_{\mathrm{M}}) = [\Sigma_{\mathrm{M}}]$.

(ii) Define a polynomial $t_{\mathrm{M}}(x,y,z,w)$ by

$$t_{\mathrm{M}}(x,y,z,w) = (x+y)^{-1}(y+z)^r(x+w)^{|E|-r}T_{\mathrm{M}}\Big(\frac{x+y}{y+z}, \frac{x+y}{x+w}\Big).$$

Then, we have an equality

$$\sum_{i+j+k+\ell=n} \Big(\deg_{\Sigma_E}\big([h_\Delta]^i[h_\nabla]^j c_k(\mathcal{S}_{\mathrm{M}})c_\ell(\mathcal{Q}_{\mathrm{M}})\big)\Big)\, x^i y^j (-z)^k w^\ell = t_{\mathrm{M}}(x,y,z,w).$$

(iii) The coefficients of the polynomial $t_{\mathrm{M}}(x,y,z,w)$ form a log-concave unbroken array. (In fact, it is a denormalized Lorentzian polynomial in the sense of [12]).

(iv) There is a unique ring isomorphism $\zeta : \overline{\mathbb{I}}(\Sigma_E) \xrightarrow{\sim} \overline{PP}^\bullet(\Sigma_E)$ satisfying $\zeta([-P(\mathrm{M})]) = \sum_{k\geq 0}(-1)^k c_k(\mathcal{S}_{\mathrm{M}})$ for all matroids M on E. Moreover, the map ζ satisfies

$$\chi(\xi) = \deg_{\Sigma_E}\Big(\zeta(\xi)\cdot(1 + [h_\Delta] + \cdots + [h_\Delta]^{n-1})\Big) \quad \text{for all } \xi \in \overline{\mathbb{I}}(\Sigma_E).$$

Some remarks about the main results and their implications follow:

- By appropriately combining parts (i), (ii), and (iv), one easily recovers the various geometric manifestations of the Tutte polynomial such as Theorem 2.3, Theorem 2.4, [23, Theorem 5.2], [32, Theorem 5.4], and [4, Theorem 1.2]. This answers the first question in Question 2.6. The proof of the key formula (ii) does not depend on any of these previous results.

- Part (iii) answers the second question in Question 2.6, and in fact contains as special cases the log-concavity results previously proven in [1] and [4]. See [8, Discussion after Theorem B] for details. The proof of (iii) is an illustration of how our framework can exploit powerful tools in both of the models of matroids reviewed in Section 2.2, since the two key tools in the proof are valuativity (from model I) and the Hodge theory of matroids (from model II).

- Part (iv) differs from the classical Hirzebruch-Riemann-Roch theorem (Remark 2.7) despite the resemblance: The isomorphism in ζ is over integral coefficients and differs from the Chern character map ch, and the class $(1 + [h_\Delta] + \cdots + [h_\Delta]^{n-1})$ differs from the Todd class $Td(\Sigma_E)$.

4 Underlying geometry

We explain the underlying algebraic geometry for those familiar with toric geometry. Let X_E be the projective smooth toric variety associated to the fan Σ_E. It contains the projectivized torus $\mathbb{P}T$ as its open dense torus, and thus the standard torus T acts on X_E. Let the *inverse standard action* of T on $\mathbb{C}^E$ be given by $(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) = (t_1^{-1} x_1, \dots, t_n^{-1} x_n)$.

A *realization* of a matroid M of rank r is an r-dimensional subspace $L \subseteq \mathbb{C}^E$ such that the set of bases of M equals

$$\{B \subseteq E : \text{the composition } L \hookrightarrow \mathbb{C}^E \twoheadrightarrow \mathbb{C}^B \text{ is an isomorphism}\}.$$

When M has a realization $L \subseteq \mathbb{C}^E$, the two models of matroids reviewed in Section 2.2 arise geometrically in the following way.

(I) Let $Gr(r; E)$ be the Grassmannian of r-dimensional subspaces in $\mathbb{C}^E$, with T acting via the inverse standard action. The moment polytope of the torus-orbit-closure $\overline{T \cdot L} \subseteq Gr(r; E)$ is exactly the polytope $-P(\mathrm{M})$. As $-P(\mathrm{M}) \in \mathrm{Def}(\Sigma_E)$, we have a canonical map $X_E \to \overline{T \cdot L}$. Pulling back to X_E the line bundle $\mathcal{O}(1)$ on $Gr(r; E)$ via the composition $X_E \to \overline{T \cdot L} \hookrightarrow Gr(r; E)$, we obtain the line bundle whose K-class is equal to $[-P(\mathrm{M})]$ under the isomorphism of Theorem 2.1.

(II) The *wonderful compactification*, studied in [19], is the closure W_L of $\mathbb{P}L \cap \mathbb{P}T$ inside X_E. As a subvariety of X_E, the variety W_L defines a cohomology class $[W_L] \in H^\bullet(X_E)$, which equals the Bergman class $[\Sigma_\mathrm{M}]$ under the isomorphism of Theorem 2.2.

Tautological classes of matroids are modeled after the following vector bundles constructed from realizations of matroids. Let $\underline{\mathbb{C}}^E$ denote the T-equivariant vector bundle $X_E \times \mathbb{C}^E$, where T acts on $\mathbb{C}^E$ via the inverse standard action.

Definition 4.1. Given a realization $L \subseteq \mathbb{C}^E$ of M, define two T-equivariant vector bundles $\mathcal{S}_L$ and $\mathcal{Q}_L$ on X_E by

$$\mathcal{S}_L = T\text{-equivariant subbundle of } \underline{\mathbb{C}}^E \text{ whose fiber at the identity of } \mathbb{P}T \text{ is } L, \quad \text{and}$$
$$\mathcal{Q}_L = T\text{-equivariant quotient bundle of } \underline{\mathbb{C}}^E \text{ whose fiber at the identity of } \mathbb{P}T \text{ is } \mathbb{C}^E/L.$$

Equivalently, consider the map $X_E \to \overline{T \cdot L} \hookrightarrow Gr(r;E)$ in (I). Pulling back the tautological subbundle $\mathcal{S}$ and the quotient bundle $\mathcal{Q}$ of $Gr(r;E)$ to X_E along this map gives the vector bundles $\mathcal{S}_L$ and $\mathcal{Q}_L$.

The K-classes of these vector bundles only depend on the matroid M that L realizes. More precisely, as elements in $K(X_E)$, we have $[\mathcal{S}_L] = \mathcal{S}_{\mathrm{M}}$ and $[\mathcal{Q}_L] = \mathcal{Q}_{\mathrm{M}}$ [8, Proposition 3.7]. The description of the classes $\mathcal{S}_{\mathrm{M}}$ and $\mathcal{Q}_{\mathrm{M}}$ given here arises as follows. Given a realization $L \subseteq \mathbb{C}^E$ of M, we may consider a flag $(0 = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n = \mathbb{C}^E)$ of subspaces where L_i is a general i-dimensional subspace containing L. This gives rise to filtrations of the vector bundles $\mathcal{S}_L$ and $\mathcal{Q}_L$. The resulting description of their K-classes constitute Definition 3.1.

We now discuss the geometry behind some of the main results listed in Theorem 3.3.

Among the most important invariants of vector bundles are their Chern classes, which loosely speaking measure the dependency loci of global sections. When M has a realization $L \subseteq \mathbb{C}^E$, the statement $c_{n-r}(\mathcal{Q}_{\mathrm{M}}) = [\Sigma_{\mathrm{M}}]$ in Theorem 3.3.(i) is the reflection of the following geometric statement [8, Theorem 7.10].

Theorem 4.2. Let $s_{\mathbf{1}}$ be the constant $(1,1,\ldots,1)$ global section of $\underline{\mathbb{C}}^E$, which defines a section s of $\mathcal{Q}_L$ via $\underline{\mathbb{C}}^E \twoheadrightarrow \mathcal{Q}_L$. Then, the wonderful compactification W_L is exactly the vanishing locus of the section s.

The theorem implies that the normal bundle $\mathcal{N}_{W_L/X_E}$ is isomorphic to the restriction $\mathcal{Q}_L|_{W_L}$. Using this, one can further relate the restriction $\mathcal{S}_L|_{W_L}$ to the log-tangent sheaf $\mathcal{T}_{W_L}(-\log \partial W_L)$ of the wonderful compactification, and consequently recover the results of [32]. See [8, Section 8] for details.

When a vector bundle is globally generated, like the bundle $\mathcal{Q}_L$, one expects the Chern classes of $\mathcal{Q}_L$ to display "positivity behaviors" in the sense of [30]. The log-concavity statement of Theorem 3.3.(iii), when M has a realization L, is a manifestation of such expectation.

A geometric way to characterize the map ζ in Theorem 3.3.(iv) is that $\zeta([\mathcal{O}_{W_L}]) = [W_L]$ for any subspace $L \subseteq \mathbb{C}^E$, where $\mathcal{O}_{W_L}$ denotes the structure sheaf of the wonderful compactification W_L [8, Corollary 10.6]. However, phrased in this way, it is not even clear why such a ring map ζ should be well-defined. One geometric explanation for Theorem 3.3.(iv) in this light, as well as a generalization to wonderful compactifications, is given in [29].

Updates and outlook

Let us list some works that the framework of tautological classes of matroids has led to since the appearance of [8]. Theorem 4.2 is a central observation behind the development of the Gromov-Witten theory of matroids in [35]. A generalization of Theorem 3.3.(iv) can be found in [29], where the authors use the generalization to develop the K-theory of matroids. An "augmented" variant of the tautological classes of matroids was introduced and studied in [22], leading to a new log-concavity property of matroids and a better understanding of matroid Schubert varieties studied in [11]. For "type B" generalizations of matroids known as delta-matroids, an "isotropic" variant of tautological classes was introduced and studied in [21], leading to a log-concavity result for delta-matroids.

We speculate that many interesting properties of $\mathcal{S}_{\mathrm{M}}$ and $\mathcal{Q}_{\mathrm{M}}$ remain undiscovered. For instance, one may replace "elementary symmetric polynomials" in Definition 3.1 by Schur polynomials, yielding Schur classes of tautological classes of matroids. Do these have nice numerical or combinatorial properties as did the Chern classes in Theorem 3.3 [8, Question 1.4]? See also [9, Conjecture 9.13, Remark 9.15] for a related conjecture about their positivity, as well as generalizations to double Grothendieck polynomials and equivariant K-theory.

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