Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #24, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

A Realization of Poset Associahedra

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Abstract. Given any connected poset P we give a simple realization of Galashin's poset associahedron $\mathscr{A}(P)$ as a convex polytope in $\mathbb{R}^P$. The realization is inspired by the description of $\mathscr{A}(P)$ as a compactification of the configuration space of order-preserving maps $P \to \mathbb{R}$. In addition, we discuss several combinatorially interesting examples of poset associahedra.

Keywords: Poset, associahedron, cyclohedron, realization, configuration space, compactification

1 Introduction

Given a finite connected poset P, the poset associahedron $\mathscr{A}(P)$ is a simple, convex polytope of dimension $|P| - 2$ introduced by Galashin [5]. Poset associahedra arise as a natural generalization of Stasheff's associahedra [6, 12, 17, 18], and were originally discovered by considering compactifications of the configuration space of order-preserving maps $P \to \mathbb{R}$. These compactifications are generalizations of the Axelrod–Singer compactification of the configuration space of a line [1, 8, 15]. Galashin constructed poset associahedra by performing stellar subdivisions on the polar dual of Stanley's *order polytope* [16], but did not provide an explicit realization.

Poset associahedra bear resemblance to graph associahedra, where the face lattice of each is described by a *tubing criterion*. However, neither class is a subset of the other. When Carr and Devadoss introduced graph associahedra in [3], they distinguish between *bracketings* and *tubings* of a path, where the idea of bracketings does not naturally extend to any simple graph. In the case of poset associahedra, the idea of bracketings *does* extend to every connected poset.

In this paper, we provide a simple realization of $\mathscr{A}(P)$ as an intersection of half spaces, inspired by the compactification description and by a similar realization of graph associahedra due to Devadoss [4]. In independent work [10], Mantovani, Padrol, and Pilaud found a realization of poset associahedra as sections of graph associahedra. The

*andrewsack@math.ucla.edu. This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-2034835. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

authors of [10] also generalize from posets to oriented building sets (which combine a building set with an oriented matroid).

Various poset associahedra have already been studied including *permutohedra*, *associahedra*, and *operahedra* [9]. We study two more classes of posets that give rise to previously unstudied polytopes with intriguing combinatorics.

2 Tubes and tubings

2.1 Background

We start by defining the poset associahedron.

Definition 2.1. Let $(P, \preceq)$ be a finite poset. We make the following definitions:

- A subset $\tau \subseteq P$ is *connected* if it is connected as an induced subgraph of the Hasse diagram of P.
- $\tau \subseteq P$ is *convex* if whenever $a, c \in \tau$ and $b \in P$ such that $a \preceq b \preceq c$, then $b \in \tau$.
- A *tube* of P is a connected, convex subset $\tau \subseteq P$ such that $2 \leq |\tau|$.
- A tube τ is *proper* if $|\tau| \leq |P| - 1$.
- Two tubes $\sigma, \tau \subseteq P$ are *nested* if $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$. Tubes σ and τ are *disjoint* if $\tau \cap \sigma = \emptyset$.
- For disjoint tubes σ, τ we say $\tau \prec \sigma$ if there exists $a \in \tau, b \in \sigma$ such that $a \prec b$.
- A *proper tubing* T of P is a set of proper tubes of P such that any pair of tubes is nested or disjoint and the transitive closure of the relation $\prec$ is a partial order on T. That is, whenever $\tau_1, \ldots, \tau_k \in T$ with $\tau_1 \prec \cdots \prec \tau_k$ then $\tau_k \not\prec \tau_1$. This is referred to as the *acyclic tubing condition.*
- A proper tubing T is *maximal* if adding any tube to T is not a proper tubing.

Figure 1 shows examples and non-examples of proper tubings.

Definition 2.2. For a finite poset P, the *poset associahedron* $\mathscr{A}(P)$ is a simple, convex polytope of dimension $|P| - 2$ whose face lattice is isomorphic to the set of proper tubings ordered by reverse inclusion. That is, if F_T is the face corresponding to T, then $F_S \subset F_T$ if one can make S from T by adding tubes. Vertices of $\mathscr{A}(P)$ correspond to maximal tubings of P.

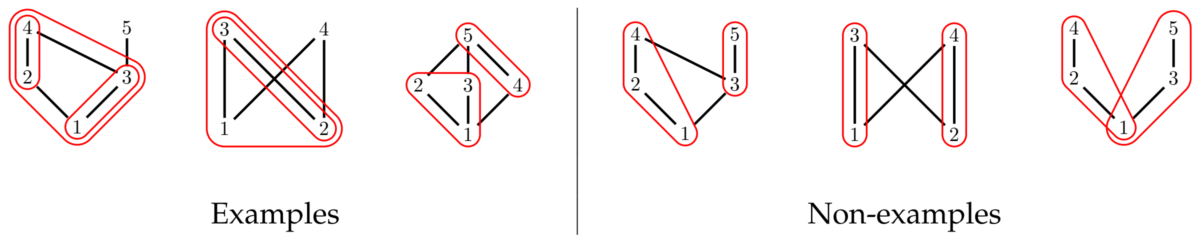

Figure 1: Examples and non-examples of proper tubings.

2.2 Realization

We realize $\mathscr{A}(P)$ as an intersection of half-spaces. We work in the ambient space $\mathbb{R}^P_{\Sigma=0}$, the space of real functions on P that sum to 0. For a subset $\tau \subseteq P$, define a linear function α_τ on $\mathbb{R}^P_{\Sigma=0}$ by

$$\alpha_\tau(p) := \sum_{\substack{i \prec j \\ i,j \in \tau}} p_j - p_i.$$

Here the sum is taken over all covering relations contained in τ. We define the half-space h_τ and the hyperplane H_τ by

$$h_\tau := \left\{ p \in \mathbb{R}^P_{\Sigma=0} \mid \alpha_\tau(p) \geq n^{2|\tau|} \right\} \qquad \text{and}$$
$$H_\tau := \left\{ p \in \mathbb{R}^P_{\Sigma=0} \mid \alpha_\tau(p) = n^{2|\tau|} \right\}.$$

The following is our main result:

Theorem 2.3. *If P is a finite, connected poset, the intersection of H_P with h_τ for all proper tubes τ gives a realization of $\mathscr{A}(P)$.*

2.3 An interpretation of tubings

When P is a chain, $\mathscr{A}(P)$ recovers the classical associahedron. There is a simple interpretation of proper tubings that explains all of the conditions above in terms of *generalized words.*

We can understand the classical associahedron as follows: Let $P = ([n], \leq)$ be a chain. We can think of the chain as a word we want to multiply together with the rule that two elements can be multiplied if they are connected by an edge. A maximal tubing of P is a way of disambiguating the order in which one performs the multiplication. If a pair of adjacent elements x and y have a pair of brackets around them, they contract along the edge connecting them and replace x and y by their product.

Similarly, we can understand the Hasse diagram of an arbitrary poset P as a *generalized word* we would like to multiply together. Again, we are allowed to multiply two elements if they are connected by an edge, but when multiplying elements, we contract

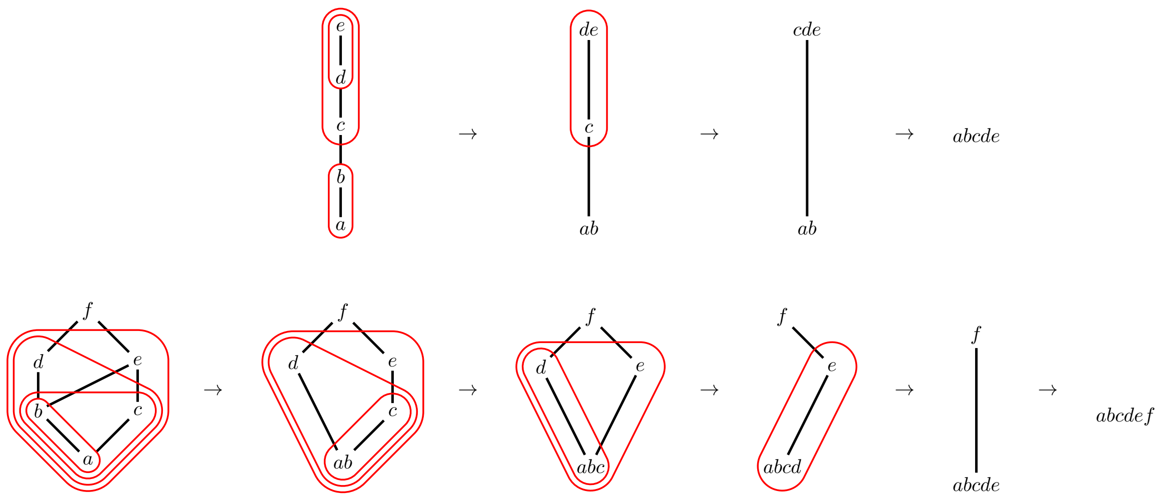

Figure 2: Multiplication of a word and of a generalized word

along the edge connecting them and then take the transitive reduction of the resulting directed graph. That is, we identify the two elements and take the resulting quotient poset. A maximal tubing is again a way of disambiguating the order of the multiplication. See Figure 2 for an illustration of this multiplication order. This perspective is discussed in relation to operahedra in [9, Section 2.1] when the Hasse diagram of P is a rooted tree.

3 Configuration spaces and compactifications

We turn our attention to the relationship between poset associahedra and configuration spaces. For a poset P, the *order cone*

$$\mathscr{L}(P) := \left\{ p \in \mathbb{R}^P_{\Sigma=0} \mid p_i \leq p_j \text{ for all } i \preceq j \right\}$$

is the set of order preserving maps $P \to \mathbb{R}$ whose values sum to 0.

Fix a constant $c \in \mathbb{R}^+$. The *order polytope*, first defined by Stanley [16] and extended by Galashin [5], is the $(|P| - 2)$-dimensional polytope

$$\mathscr{O}(P) := \{ p \in \mathscr{L}(P) \mid \alpha_P(p) = c \}.$$

Remark 3.1. When P is *bounded*, that is, has a unique maximum $\hat{1}$ and minimum $\hat{0}$, this construction is projectively equivalent to Stanley's order polytope where we replace the conditions of the coordinates summing to 0 and $\alpha_P(p) = c$ with $p_{\hat{0}} = 0$ and $p_{\hat{1}} = 1$, see [5, Remark 2.5].

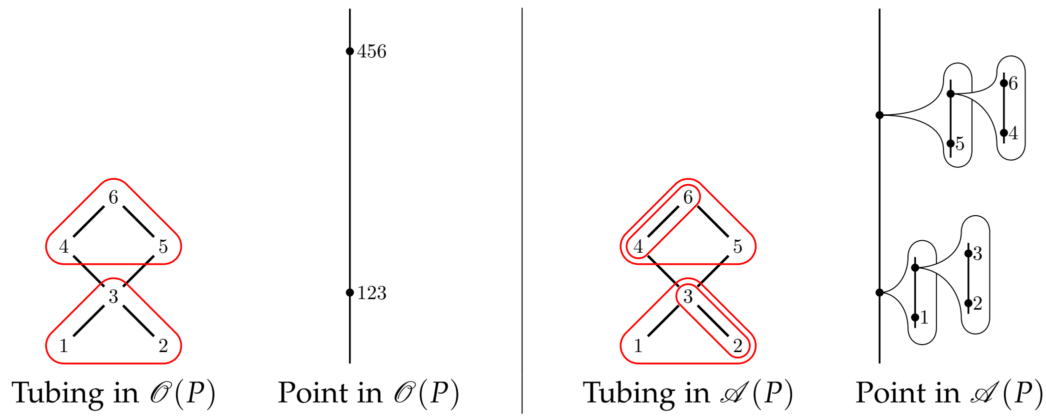

Figure 3: A vertex in $\mathscr{O}(P)$ vs. $\mathscr{A}(P)$

Galashin [5] obtains the poset associahedra by an alternative compactification of $\mathscr{O}^\circ(P)$, the interior of $\mathscr{O}(P)$. We describe this compactification informally, as it serves as motivation for the realization in Theorem 2.3.

A point is on the boundary of $\mathscr{O}(P)$ when any of the inequalities in the order cone achieve equality. The faces of $\mathscr{O}(P)$ are in bijection with proper tubings of P such that all tubes are disjoint. Let T be such a tubing. If p is in the face corresponding to T and $\tau \in T$ then $p_i = p_j$ for $i, j \in \tau$.

We can think of the point p in the face corresponding to T as being "what happens in $\mathscr{O}(P)$" when for each $\tau \in T$, the coordinates are infinitesimally close. However, by taking all coordinates in τ to be equal, we lose information about their relative ordering. In $\mathscr{A}(P)$, we still think of the coordinates in τ as being infinitesimally close, but we are still interested in their configuration. Upon zooming in, this is parameterized by the order polytope of the subposet $(\tau, \preceq)$. We iterate this process, allowing points in τ to be infinitesimally closer, and so on. We illustrate this in Figure 3. This idea is a common explanation of the Axelrod–Singer compactification of $\mathscr{O}^\circ(P)$ when P is a chain, see [1, 8, 15].

The idea of the realization in Theorem 2.3 is to replace the notions of *infinitesimally close* and *infinitesimally closer* with being *exponentially close* and *exponentially closer.* For $p \in \mathscr{L}(P)$, α_τ acts a measure of how close the coordinates of $p|_\tau$ are. We can make this precise with the following definition and lemma.

Definition 3.2. For $S \subseteq P$ and $p \in \mathbb{R}^P$, define the *diameter* of p relative to S by

$$\operatorname{diam}_S(p) = \max_{i,j \in S} |p_i - p_j|.$$

That is, $\operatorname{diam}_S(p)$ is the diameter of $\{p_i : i \in S\}$ as a subset of $\mathbb{R}$.

Lemma 3.3. *Let $\tau \subseteq P$ be a tube and let $p \in \mathscr{L}(P)$. Then*

$$\operatorname{diam}_\tau(p) \leq \alpha_\tau(p) \leq \frac{n^2}{4} \operatorname{diam}_\tau(p).$$

In particular, for $p \in \mathscr{L}(P)$, if $p \in H_\tau$, then $\{p_i \mid i \in \tau\}$ is clustered tightly together compared to any tube containing τ. If $p \in h_\tau$, then $\{p_i \mid i \in \tau\}$ is spread far apart compared to any tube contained in τ.

4 Realizing the poset associahedron

We are now prepared to sketch the proof of Theorem 2.3. Define

$$\mathscr{A}(P) := \bigcap_{\sigma \subset P} h_\sigma \cap H_P$$

where the intersection is over all tubes of P. Theorem 2.3 follows as a result of three lemmas:

Lemma 4.1. *If T is a maximal tubing, then*

$$v^T := \bigcap_{\tau \in T \cup \{P\}} H_\tau$$

is a point.

Lemma 4.2. *If T is a collection of tubes that do not form a proper tubing, then*

$$\bigcap_{\tau \in T} H_\tau \cap \mathscr{A}(P) = \emptyset.$$

Lemma 4.3. *If T is a maximal tubing and $\tau \notin T$ is a proper tube, then $\alpha_\tau(v^T) > n^{2|\tau|}$. That is, v^T lies in the interior of h_τ.*

Lemma 4.1 follows from a standard induction argument.

Proof sketch of Lemma 4.2. If T is not a proper tubing, then there are two cases:

(1) There is a pair of non-nested and non-disjoint tubes τ_1, τ_2 in T.

(2) There is a sequence of disjoint tubes $\tau_1, ..., \tau_k$ such that $\tau_1 \prec \cdots \prec \tau_k \prec \tau_1$.

For $S \subseteq P$, define the *convex hull* of S as

$$\mathrm{conv}(\sigma) := \{b \in P \mid \exists a, c \in S : a \leq b \leq c\}.$$

Take $\sigma = \mathrm{conv}(\tau_1 \cup \cdots \cup \tau_k)$. One can show that σ is a tube, so Lemma 3.3 tells us that for each τ_i, $\mathrm{diam}_{\tau_i}(p)$ is very small compared to $n^{2|\sigma|}$. As the tubes either intersect or are cyclic, one can show this forces $\mathrm{diam}_\sigma(p)$ to also be small, so $\alpha_\sigma(p) < n^{2|\sigma|}$. $\square$

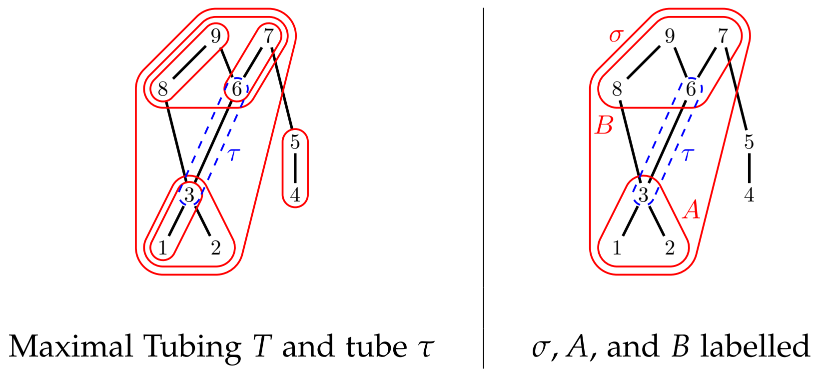

Figure 4: An example illustrating the proof of Lemma 4.3.

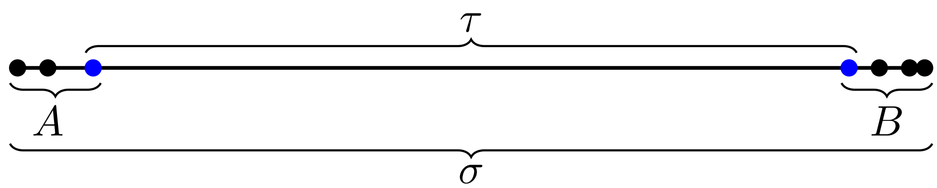

Figure 5: If $\mathrm{diam}_A(p)$ and $\mathrm{diam}_B(p)$ are small and $\mathrm{diam}_\sigma(p)$ is large, then $\mathrm{diam}_\tau(p)$ is large.

Proof sketch of Lemma 4.3. Define the *convex hull* of τ *relative* to T by

$$\mathrm{conv}_T(\tau) := \min\{\sigma \in T \mid \tau \subset \sigma\}.$$

T partitions σ into a lower set A and an upper set B where A and B are either tubes or singletons. Furthermore, A and B both intersect τ. See Figure 4 for an example illustrating this.

By Lemma 3.3, $\mathrm{diam}_A(v^T)$ and $\mathrm{diam}_B(v^T)$ are both very small compared to $\mathrm{diam}_\sigma(v^T)$. Then for any $a \in A, b \in B$, $|v^T_a - v^T_b|$ must be large. As τ intersects both A and B, $\mathrm{diam}_\tau(v^T)$ must be large and hence $v^T \in h_\tau$. See Figure 5 for an illustration of this. □

Remark 4.4. A similar approach for realizing graph associahedra is taken by Devadoss [4]. For a graph $G = (V, E)$, Devadoss realizes the graph associahedron of G by taking the supporting hyperplane for a graph tube τ to be

$$\left\{ p \in \mathbb{R}^V \mid \sum_{i \in \tau} p_i = 3^{|\tau|} \right\}.$$

One difference is that Devadoss realizes graph associahedra by cutting off slices of a simplex whereas we cut off slices of an order polytope. When the Hasse diagram of P is a tree, the poset associahedron is combinatorially equivalent to the graph associahedron

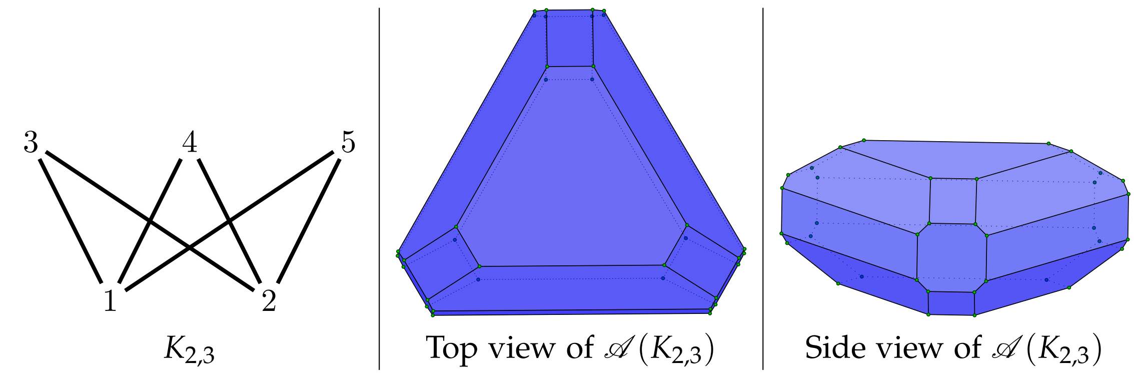

Figure 6: $K_{2,3}$ and $\mathscr{A}(K_{2,3})$

of the line graph of the Hasse diagram. In this case, the two realizations have linearly equivalent normal fans. If the Hasse diagram of P is a path graph, then both realizations have linearly equivalent normal fans to the realization of the associahedron due to Shnider and Sternberg [17].

5 Examples

Several classes of posets produce combinatorially interesting polytopes. Recall that for a simple d-dimensional polytope P, the f-vector, h-vector, and γ-vector of P are $(f_0, \dots, f_d)$, $(h_0, \dots, h_d)$, and $(\gamma_0, \dots, \gamma_{\lfloor d/2 \rfloor})$, where f_i is the number of i-dimensional faces and

$$\sum_{i=0}^{d} f_i t^i = \sum_{i=0}^{d} h_i (t+1)^i,$$
$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}.$$

The h-vector and γ-vector frequently encode interesting combinatorial data such as Eulerian numbers, Narayana numbers, and binomial coefficients. We consider all polytopes in this section only up to combinatorial equivalence. For clarity, we write $h_i(P)$ when P is not clear from context.

5.1 Posets with Hasse diagram $K_{m,n}$

Let $K_{m,n}$ be the poset whose Hasse diagram is the complete bipartite graph $K_{m,n}$ where for each $1 \le i \le m < j \le m+n$ we have $i \prec j$. For example, $\mathscr{A}(K_{1,n})$ is the classical permutohedron. Let $\mathfrak{S}_n$ be the set of permutations of size n. Following the notation of [13], we recall the following definitions. For $w \in \mathfrak{S}_n$, the *descent statistic* is

$$\mathrm{des}(w) := \#\{(i, i+1) \mid w(i) > w(i+1)\}.$$

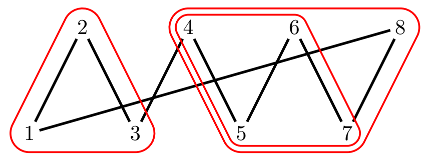

Figure 7: A tubing of a cyclic fence.

A *double descent* is a pair of consecutive descents. We say that w has a *final descent* if $w(n-1) > w(n)$. For $\mathscr{A}(K_{1,n})$, the h-vector is given by the *Eulerian numbers* [12]

$$h_i(A(K_{1,n})) = \#\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = i\}.$$

Let $\hat{\mathfrak{S}}_n$ be the set of permutations of size n without any consecutive descents or final descent. Then

$$\gamma_i(A(K_{1,n})) = \#\{w \in \hat{\mathfrak{S}}_n \mid \mathrm{des}(w) = i\}.$$

These properties generalize to $\mathscr{A}(K_{m,n})$. In particular,

$$h_i(\mathscr{A}(K_{m,n})) = \#\{w \in \mathfrak{S}_{m+n} \mid \mathrm{des}(w) = i, w(1) \leq m, w(m+n) > m\} \quad \text{and}$$
$$\gamma_i(\mathscr{A}(K_{m,n})) = \#\{w \in \hat{\mathfrak{S}}_{m+n} \mid \mathrm{des}(w) = i, w(1) \leq m, w(m+n) > m\}.$$

5.2 Posets whose Hasse diagram is a path graph

When the Hasse diagram is isomorphic to the path graph P_n on n vertices, $\mathscr{A}(P_n)$ recovers the classical associahedron. Let DP_n be the set of Dyck paths with n up-steps. A *peak* is an up-step immediately followed by a down-step. Here, h_i is given by the *Narayana numbers* [12]

$$\begin{aligned} h_i(\mathscr{A}(P_n)) &:= N(n-1,i) \\ &= \frac{1}{n-1}\binom{n-1}{i}\binom{n-1}{i-1} \\ &= \{w \in DP_{n-1} \mid \#\mathrm{peaks}(w) = i+1\}. \end{aligned}$$

5.3 Cyclic fences

The *fence*, F_n, is the poset with n elements and alternating covering relations

$$1 \prec 2 \succ 3 \prec 4 \succ \dots$$

As the Hasse diagram is a path, $\mathscr{A}(F_n)$ is the classical associahedron. We define the *cyclic fence* CF_{2n} to be the fence F_{2n} with the additional relation $1 \prec 2n$, see Figure 7.

One may expect that $\mathscr{A}(CF_{2n})$ is equivalent to the *cyclohedron* [2, 11, 14, 17], a cyclic variant of the associahedron. However, this is not the case! The n-dimensional cyclohedron $\mathscr{C}_n$ has $\binom{2n}{n}$ vertices, but $\mathscr{A}(CF_{2(n+1)})$ has $4^n\binom{2n}{n}$ vertices. Despite this, these polytopes are related!

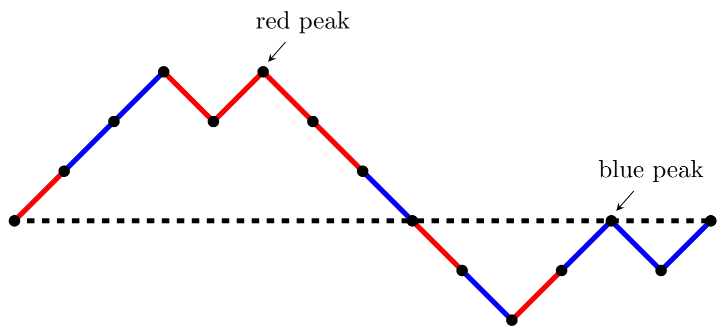

Figure 8: A colored balanced path with 1 red peak and 1 blue peak.

A *balanced path* is a sequence of up-steps and down-steps with an equal number of each. Let BP_n be set of balanced paths with n up-steps. In [14], Simion observes that $|BP_n| = \binom{2n}{n}$ and for the cyclohedron,

$$h_i(\mathscr{C}_n) = \#\{w \in BP_n \mid \#\,\text{peaks}(w) = i\} = \binom{n}{i}^2.$$

We define a *colored balanced path* to be a balanced path where each step is colored red or blue, and let CBP_n be the set of all colored balanced paths with n up-steps. Define a *red peak* to be a red up-step immediately followed by a red down-step and similarly define a *blue peak*, see Figure 8. Then $|CBP_n| = 4^n\binom{2n}{n}$, and for $\mathscr{A}(CF_{2(n+1)})$,

$$h_i(\mathscr{A}(CF_{2(n+1)})) = \#\{w \in CBP_n \mid \#\text{red peaks}(w) - \#\text{blue peaks}(w) = i - n\}$$

$$\gamma_i(\mathscr{A}(CF_{2(n+1)})) = 4^i\binom{n}{i}^2.$$

6 Open questions

Question 6.1. Define a *colored Dyck path* to be a Dyck path where each step is colored red or blue, and let CDP_n be the set of colored Dyck paths with n up-steps. We can define

$$h_i = \#\{w \in CDP_n \mid \#\text{red peaks}(w) - \#\text{blue peaks}(w) = i - n\}.$$

and calculate $\gamma_i = 4^i N(n-1, i)$. Observe that similarly to the case of colored balanced paths, the γ-vector is 4^i times the h-vector of non-colored paths. Is $(h_0, \ldots, h_{2n})$ the h-vector of a polytope related to the associahedron?

Question 6.2. Galashin conjectured that the γ-vector of a poset associahedron is always non-negative, despite poset associahedra not being flag simple in general. We strengthen this by conjecturing that the polynomial $\sum f_i t^i$ is real-rooted. If poset associahedra are indeed γ-positive, it would be interesting to find a combinatorial interpretation of the γ-vector.

Question 6.3. Postnikov, Reiner, and Williams [13] found a statistic on maximal tubings of graph associahedra of chordal graphs where

$$\sum_T t^{\mathrm{stat}(T)} = \sum h_i t^i.$$

It would be interesting to find a similar statistic on maximal tubings of poset associahedra. For a simple polytope P, one can orient the edges of P according to a generic linear form and take $\mathrm{stat}(v) = \mathrm{outdegree}(v)$ [19, §8.2]. It may be possible to use our realization to find the desired statistic.

Question 6.4. While $h_i(\mathscr{A}(CF_{2(n+1)})$ is given by a peak statistic, we do not know a bijective proof of this fact. In particular, we would like a bijection that preserves $\mathrm{stat}(T)$ from Question 6.3. There is a known bijection between maximal tubings of $CF_{2(n+1)}$ and CDP_n although it is complicated, see [7].

Acknowledgements

The author is grateful to Pavel Galashin for his many helpful comments and suggestions, to Stefan Forcey for fruitful conversations, and to the anonymous referees for useful comments.

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