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Balanced Shifted Tableaux

Jiyang Gao*1, Shiliang Gao†2, and Yibo Gao‡3

[1] *Department of Mathematics, Harvard University, Cambridge, MA 02138*
[2] *Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801*
[3] *Department of Mathematics, University of Michigan, Ann Arbor, MI 48104*

Abstract. We introduce balanced shifted tableaux, as an analogue of balanced tableaux of Edelman and Greene, from the perspective of root systems of type B and C. We show that they are equinumerous to standard Young tableaux of the corresponding shifted shape by presenting an explicit bijection.

Keywords: balanced tableaux, standard Young tableaux, Edelman-Greene insertion

1 Introduction

In their seminal paper [2], Edelman and Greene introduced *balanced tableaux* and showed that they are equinumerous to standard Young tableaux of the same shape. They defined the *Edelman-Greene insertion* which yields a bijective proof of the reduced words of the longest permutation being equinumerous to standard Young tableaux of staircase shape, a result due originally to Stanley [10]. Fomin, Greene, Reiner and Shimozono [3] later generalized this enumeration result to diagrams and related the story to Schubert polynomials.

Shifted tableaux, just as Young tableaux, are also algebraically and combinatorially meaningful (see for example [9, 12]). In this paper, we define *balanced shifted tableaux* (Definition 2.3), as an analogue to balanced tableaux, from the perspective of root systems of type B and C. The following is our main theorem, which says that balanced shifted tableaux are equinumerous to standard shifted tableaux.

Theorem 1.1. *For a shifted shape λ, the number of standard Young tableaux of shape λ equals the number of balanced shifted tableaux of shape λ.*

We prove Theorem 1.1 by presenting an explicit bijection between the two sets of objects, $\mathrm{SYT}(\lambda)$ and $\mathrm{BS}(\lambda)$. Specifically, we have the following chain of bijections:

$$\mathrm{SYT}(\lambda) \longleftrightarrow \mathrm{SYT}(Z(d,r))|_\lambda \longleftrightarrow \mathrm{Red}(w^{(d,r)})|_\lambda \longleftrightarrow \mathrm{BS}(Z(d,r))|_\lambda \longleftrightarrow \mathrm{BS}(\lambda),$$

*jgao@math.harvard.edu.
†sgao23@illinois.edu. Shiliang Gao was partially supported by NSF Graduate Research Fellowship under grant No. DGE-1746047.
‡gaoyibo@umich.edu.

where we address each step separately. We defer the definition of $\mathrm{SYT}(\lambda)$ and $\mathrm{BS}(\lambda)$ to Section 2 and the definition of $\mathrm{SYT}(Z(d,r))|_\lambda$, w^λ and $\mathrm{BS}(Z(d,r))|_\lambda$ to Section 5. Here, $\mathrm{SYT}(\lambda) \to \mathrm{SYT}(Z(d,r))|_\lambda$ and $\mathrm{BS}(\lambda) \to \mathrm{BS}(Z(d,r))|_\lambda$ are the procedures to pad a tableau from shape λ to a large trapezoid $Z(d,r)$, while the middle steps utilize type B Edelman-Greene insertion defined by Kraśkiewicz [6]. Our strategy largely follows the framework of Edelman and Greene [2], with the main difference that double staircases, which are the analogues of staircases in type B, are no longer sufficient for padding purposes.

2 Definitions and Preliminaries

2.1 Strict partitions and shifted tableaux

A *strict partition* λ is a sequence of strictly decreasing positive integers $(\lambda_1 > \lambda_2 > \cdots > \lambda_d > 0)$, where d is the number of (nonzero) parts of λ. We denote $|\lambda| = \sum_{i=1}^{d} \lambda_i$ as the *size* of λ. For a strict partition λ its corresponding *shifted shape*, consists of λ_i boxes in row i, shifted $d-i+1$ steps to the left. More specifically, the shifted shape is the diagram

$$D(\lambda) := \{(i, j{-}d{+}i{-}1) \mid 1 \le i \le d,\ 1 \le j \le \lambda_i\}.$$

For simplicity of notation, we also use λ to denote its shape $D(\lambda)$. Note that for a shifted shape, its columns $-(d-1), \ldots, 0$ form a staircase shape of length d flipped horizontally. For a shifted shape λ, define a *shifted tableau* T to be a filling of $D(\lambda)$ with non-negative integers. For any shifted tableau T, let $\mathrm{sh}(T)$ denote its underlying shifted shape.

Throughout the paper, we fix the number d, that is the length of all the shifted shapes we are going to consider. We also write $\bar{i}$ to mean $-i$.

Definition 2.1. A shifted tableau T of shape λ is called a *standard Young tableau* if it is a filling of $1, 2, \ldots, |\lambda|$ that is increasing in rows and columns.

The set of standard Young tableaux of shape λ is denote $\mathrm{SYT}(\lambda)$ and its cardinality is denoted f^λ. The number f^λ can be computed via the hook length formula as we explain here. For a box $(i,j) \in \lambda$ with $j \ge 0$, its *hook* $H(i,j)$ consists of all the boxes in row i to the right of (i,j), all the boxes in column j below (i,j) and the box (i,j) itself. For a box $(i,\bar{j}) \in \lambda$ with $j > 0$, its *hook* $H(i,\bar{j})$ consists of all the boxes in row i to the right of $(i,\bar{j})$, all the boxes in column $\bar{j}$ below $(i,\bar{j})$, the box $(i,\bar{j})$ itself and all the boxes in row $d-j+1$. Let $h(i,j) = |H(i,j)|$ be the size of the hook.

Theorem 2.2. *[11] For a shifted shape λ, $f^\lambda = |\lambda|! \,/ \prod_{x\in\lambda} h(x)$.*

To define an analogous notion of balanced tableaux, as in [2], for shifted shapes, we need some more notions. For a filling B of shape λ, its *extended filling* $\tilde{B}$ is a filling of the extended shape

$$\tilde{\lambda} = \lambda \cup \{(1,\bar{d}), (2,\overline{d{-}1}), \ldots (d,\bar{1})\}$$

which agrees with B on λ and equals $B(i,0)$ on the newly added box $(i, -(d+1-i))$. The *extended hook* is defined as $\tilde{H}(i,j) = H(i,j)$ for $j \geq 0$, and $\tilde{H}(i,\bar{j}) = H(i,j) \cup \{(d+1-j, \bar{j})\}$ for $j > 0$. See Example 2.5 for visualization.

For a box $(i,j) \in \lambda$, we also define its *rank function* $\mathrm{rk}(i,j)$. If $j \geq 0$, let $\mathrm{rk}(i,j)$ be the number of boxes in row i of $H(i,j)$, and let $\mathrm{rk}(i,\bar{j})$ be 2 plus the number of boxes in $H(i,j)$ with positive column index. More formally,

$$\mathrm{rk}(i,j) = \begin{cases} \lambda_i - d + i - j & \text{if } j \geq 0, \\ \lambda_i - d + i + \lambda_{d+1+j} + j + 1 & \text{if } j < 0. \end{cases}$$

We can now introduce our main object of study:

Definition 2.3. A shifted tableau B of shape λ is called a *balanced shifted tableau* if it is a filling of $1, 2, \ldots, |\lambda|$ such that $B(i,j)$ is the $\mathrm{rk}(i,j)$-th largest entry in the extended hook $\tilde{H}(i,j)$ of $\tilde{B}$ for all $(i,j) \in \lambda$. Define $\mathrm{BS}(\lambda)$ to be the set of balanced shifted tableaux of shape λ.

Remark 2.4. We remark that our definition of balanced tableaux is different from the balanced filling in Section 6 of [4] and the standard w-tableau as in [7]. One can derive a result similar to Theorem 1.1 using their definition, but we note that the two results are fundamentally different. The difference can be interpreted loosely as studying the balanced tableaux of dominant permutation v.s. Grassmannian permutation in the framework of [3].

Example 2.5. Let $\lambda = (6,2,1)$ and consider the balanced shifted tableau in Figure 1. The hook $H(1,-1)$ contains the colored boxes so $h(1,-1) = 7$, while the extended hook $\tilde{H}(1,-1)$ contains one more box at coordiante $(3,-1)$, which is circled and filled with 1. As this hook contains 3 boxes with positive column index, we have $\mathrm{rk}(1,-1) = 5$. The balanced condition is now satisfied at coordinate $(1,-1)$ as 3 is indeed the 5-th largest numbers among the numbers in the extend hook, $9,5,2,4,3,7,1,1$.

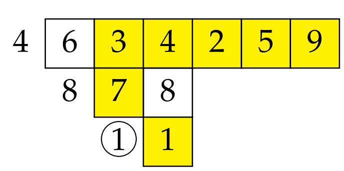

Figure 1: A balanced shifted tableau of shape $(6,2,1)$

2.2 Root systems and Weyl groups

Readers are referred to [5] for detailed exposition on root systems and Weyl groups. Let $\Phi \subset V \simeq \mathbb{R}^d$ be a finite crystallographic root system of rank d, with a chosen set of positive roots Φ^+ which corresponds to a set of simple roots $\Delta = \{\alpha_0, \alpha_1, \dots, \alpha_{d-1}\}$. Let s_α be the reflection across the hyperplane normal to α, and write s_i for the simple reflections s_{α_i}. Let $W(\Phi) \subset \mathrm{GL}(V)$ be the finite Weyl group, defined to be generated by $s_0, \dots, s_{d-1}$.

For $w \in W(\Phi)$, let $\ell(w)$ denote its Coxeter length, which equals the size of its (left) inversion set $\mathrm{Inv}(w) := \Phi^+ \cap w\Phi^-$. For any sequence $\mathbf{a} = (a_1, a_2, \dots, a_{\ell(w)})$, we say $\mathbf{a}$ is a reduced word of w if $w = s_{a_1}s_{a_2}, \dots, s_{a_{\ell(w)}}$. Let $\mathrm{Red}(w)$ be the set of reduced words of w. For each reduced word $\mathbf{a} \in \mathrm{Red}(w)$, its *(total) reflection order* is an ordering $\mathrm{ro}(\mathbf{a}) = \gamma_1, \dots, \gamma_{\ell(w)}$ of $\mathrm{Inv}(w)$ where $\gamma_j = s_{a_1} \cdots s_{a_{j-1}}\alpha_j \in \Phi^+$. Let

$$\mathrm{ro}(w) = \{\mathrm{ro}(\mathbf{a}) : \mathbf{a} \in \mathrm{Red}(w)\}.$$

The following proposition is classical and very useful, which follows immediately from the biconvexity classification of inversion sets. See for example Proposition 3 of [1].

Proposition 2.6. *Let $\gamma = \gamma_1, \dots, \gamma_{\ell(w)}$ be an ordering of $\mathrm{Inv}(w)$. Then $\gamma \in \mathrm{ro}(w)$ if and only if for all the triples $\alpha, \beta, \alpha+\beta \in \Phi^+$ such that $\alpha, \alpha+\beta \in \mathrm{Inv}(w)$,*

1. *if $\beta \notin \mathrm{Inv}(w)$, then α appears before $\alpha+\beta$ in this sequence;*
2. *and if $\beta \in \mathrm{Inv}(w)$, then $\alpha+\beta$ appears in the middle of α and β.*

We are primarily concerned with root systems of type B_n, and adopt the following convention, where e_i is the i-th coordinate vector:

- $\Phi(B_n) = \{\pm e_j \pm e_i \mid 1 \le i < j \le n\} \cup \{\pm e_i \mid 1 \le i \le n\}$;
- $\Phi^+(B_n) = \{e_j \pm e_i \mid 1 \le i < j \le n\} \cup \{e_i \mid 1 \le i \le n\}$;
- $\Delta = \{\alpha_0 = e_1, \alpha_1 = e_2 - e_1, \dots, \alpha_{n-1} = e_n - e_{n-1}\}$;
- $W(B_n) = \{\text{permutation } w \text{ on } 1, \dots, n, \bar{1}, \dots, \bar{n} \mid w(i) = -w(\bar{i}),\ \forall i\}$.

The type B_n Weyl group $W(B_n)$ is called the group of *signed permutations*. For a signed permutation w, its one-line notation is written as $w(1)w(2)\cdots w(n)$. For example, $w = 3\bar{4}2\bar{1} \in W(B_4)$ means that $w(1) = 3$, $w(2) = -4$, $w(3) = 2$ so that $w(-3) = -2$ and $w(4) = -1$. A reduced word of $w \in W(B_n)$ can be viewed as going from $\mathrm{id} = 12\cdots n$ to w by swapping adjacent entries (and their negatives) one step at a time, while the corresponding reflection order records $e_j - e_i$ if the values j and i are swapped (and records e_i if i and $\bar{i}$ are swapped).

Example 2.7. Consider $w = 1\bar{3}42 \in W(B_4)$ with a reduced word $\mathbf{a} = 21031 \in \mathrm{Red}(w)$. We compute its reflection order to be $e_3 - e_2, e_3 - e_1, e_3, e_4 - e_2, e_3 + e_1$, which can be seen as follows:

$$1234 \xrightarrow{e_3 - e_2} 1324 \xrightarrow{e_3 - e_1} 3124 \xrightarrow{e_3} \bar{3}124 \xrightarrow{e_4 - e_2} \bar{3}142 \xrightarrow{e_3 + e_1} 1\bar{3}42.$$

3 Bijection between $\mathrm{BS}(Z(d,r))$ and $\mathrm{Red}(w^{(d,r)})$ via reflection order

A crucial shape for our analysis is the *trapezoid*

$$Z(d,r) := (r+2d-1, r+2d-3, \dots, r+3, r+1)$$

with height d and base lengths $r+2d-1$ and $r+1$. In particular, $Z(d,0)$ is the *double staircase* and every shifted shape is contained in some trapezoid of the same height.

Set $e_{-j} = -e_j$ for all $j > 0$ and $e_0 = 0$, and consider the labeling $f : Z(d,r) \longrightarrow \Phi^+(B_{d+r})$ where

$$f(i,j) = \begin{cases} e_{d+1-i} - e_j & \text{if } j \leq 0, \\ e_{d+1-i} + e_{j+d} & \text{if } 0 < j \leq r, \\ e_{d+1-i} - e_{j-r} & \text{if } j > r. \end{cases} \tag{3.1}$$

Define the permutation $w^{(d,r)} \in W(B_{d+r})$ associated to $Z(d,r)$ by

$$w^{(d,r)}(i) := \begin{cases} d+i & \text{if } 0 < i \leq r, \\ \overline{i-r} & \text{if } i > r. \end{cases} \tag{3.2}$$

Proposition 3.1. *For all $d > 0$ and $r \geq 0$, $f(Z(d,r)) = \mathrm{Inv}(w^{(d,r)})$.*

The labeling f can also be extended to a labeling $\tilde{f} : \tilde{Z}(d,r) \to \Phi^+(B_{d+r})$ where

$$\tilde{Z}(d,r) = Z(d,r) \cup \{(1,\bar{d}), (2,\overline{d-1}), \dots (d,\bar{1})\}$$

is the extended shape of $Z(d,r)$ with d extra boxes as defined in Section 2. The extended labeling is given by

$$\tilde{f}(i,j) = \begin{cases} 2e_{d+1-i} & \text{if } j = \overline{d+1-i}, \\ f(i,j) & \text{otherwise.} \end{cases}$$

Example 3.2. For $d = 3$ and $r = 2$, we have $Z(d,r) = (7,5,3)$ and $w^{(3,2)} = 45\bar{1}\bar{2}\bar{3} \in W(B_5)$. See Figure 2 for the extended labeling $\tilde{f}$ in this case.

The filling of a balanced shifted tableau $B \in \mathrm{BS}(Z(d,r))$ can be viewed as a map $B : Z(d,r) \to \mathbb{N}$ by sending a box to its entry. Then the composition $Bf^{-1} : \mathrm{Inv}(w^{(d,r)}) \to \mathbb{N}$ encodes an ordering of the roots in $\mathrm{Inv}(w^{(d,r)})$. We will show that this actually gives a reflection order in $\mathrm{ro}(w^{(d,r)})$.

Proposition 3.3. *The map $B \mapsto Bf^{-1}$ is a bijection between $\mathrm{BS}(Z(d,r))$ and $\mathrm{ro}(w^{(d,r)})$, and thus induces a bijection between $\mathrm{BS}(Z(d,r))$ and $\mathrm{Red}(w^{(d,r)})$.*

$2e_3$	e_3+e_2	e_3+e_1	e_3	e_4+e_3	e_5+e_3	e_3-e_1	e_3-e_2
	$2e_2$	e_2+e_1	e_2	e_4+e_2	e_5+e_2	e_2-e_1	
		$2e_1$	e_1	e_4+e_1	e_5+e_1		

Figure 2: The extended labeling $\tilde{f}$ of $\tilde{Z}(3,2)$

Since there is a natural bijection between $\mathrm{ro}(w)$ and $\mathrm{Red}(w)$, Proposition 3.3 implies:

Corollary 3.4. *The map* $B \mapsto \mathrm{ro}^{-1}(Bf^{-1})$ *is a bijection between* $\mathrm{BS}(Z(d,r))$ *and* $\mathrm{Red}(w^{(d,r)})$.

Example 3.5. Assume we started with the following balanced tableau of shape $Z(3,2)$.

$$B = \begin{array}{|c|c|c|c|c|c|c|} \hline 4 & 8 & 7 & 10 & 13 & 5 & 15 \\ \hline \multicolumn{1}{c|}{} & 3 & 2 & 6 & 9 & 1 & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 11 & 12 & 14 & \multicolumn{2}{c}{} \\ \cline{3-5} \end{array}$$

The corresponding reflection order Bf^{-1} is given as follows:

$$12345 \xrightarrow{e_2-e_1} 21345 \xrightarrow{e_2} \bar{2}1345 \xrightarrow{e_2+e_1} 1\bar{2}345 \xrightarrow{e_3+e_2} 13\bar{2}45 \xrightarrow{e_3-e_1} 31\bar{2}45$$

$$\xrightarrow{e_4+e_2} 314\bar{2}5 \xrightarrow{e_3} \bar{3}14\bar{2}5 \xrightarrow{e_3+e_1} 1\bar{3}4\bar{2}5 \xrightarrow{e_5+e_2} 1\bar{3}45\bar{2} \xrightarrow{e_4+e_3} 14\bar{3}5\bar{2}$$

$$\xrightarrow{e_1} \bar{1}4\bar{3}5\bar{2} \xrightarrow{e_4+e_1} 4\bar{1}\bar{3}5\bar{2} \xrightarrow{e_5+e_3} 4\bar{1}5\bar{3}\bar{2} \xrightarrow{e_5+e_1} 45\bar{1}\bar{3}\bar{2} \xrightarrow{e_3-e_2} 45\bar{1}\bar{2}\bar{3}.$$

Therefore, we can read off a reduced word $\mathbf{a}$ of $w^{(3,2)} = 45\bar{1}\bar{2}\bar{3}$ as

$$\mathbf{a} = \mathrm{ro}^{-1}(Bf^{-1}) = 101213014201324 \in \mathrm{Red}(w^{(3,2)}).$$

4 Bijection between $\mathrm{SYT}(Z(d,r))$ and $\mathrm{Red}(w^{(d,r)})$ via the Kraśkiewicz's insertion

We will follow the notations as recorded in Section 1.3 of [8]. For a shifted tableau T of shape $\lambda = (\lambda_1, \ldots, \lambda_d)$, define $\pi(T) = T_d T_{d-1}, \ldots, T_1$ to be the reading word of T obtained by reading left to right along rows and from bottom to top, where T_i represents the i-th row. For a unimodal sequence of integers

$$\mathbf{R} = (r_1 > r_2 > \ldots > r_k < r_{k+1} < \ldots < r_m),$$

we define the decreasing part of **R** to be $\mathbf{R}^{\downarrow} = (r_1 > r_2 > \ldots > r_k)$, and the increasing part of **R** to be $\mathbf{R}^{\uparrow} = (r_{k+1} < r_{k+2} < \ldots < r_m)$. Note that we include the minimal integer of the sequence in $\mathbf{R}^{\downarrow}$.

Let $w \in W(B_n)$ and $\mathbf{a} = a_1 a_2 \ldots a_{\ell(w)} \in \mathrm{Red}(w)$, we define the *Kraśkiewicz's insertion algorithm* recursively. Set $(P^{(0)}, Q^{(0)}) := (\emptyset, \emptyset)$, for any $i \in [\ell(w)]$, define the insertion

$$(P^{(i-1)}, Q^{(i-1)}) \leftarrow a_i =: (P^{(i)}, Q^{(i)})$$

as follows:

Step 1: Set **R** to be the first row of $P^{(i-1)}$ and $a = a_i$.

Step 2: Insert a into **R** as follows:

- Case 1 ($\mathbf{R}a$ is unimodal): Append a to the right of **R** to obtain $P^{(i)}$. Then define $Q^{(i)}$ from $Q^{(i-1)}$ by adding i to the unique box in $P^{(i)}/P^{(i-1)}$. Stop.
- Case 2 ($\mathbf{R}a$ is not unimodal): Let b be the smallest number in $\mathbf{R}^{\uparrow}$ such that $b \geq a$.
  - Case 2.1 ($a = 0$ and **R** contains 101 as a subsequence): We leave **R** unchanged and return to start of Step 2 with $a = 0$ and **R** equals the next row.
  - Case 2.2 (otherwise): Replace b with a. Set $c = b$ if $a \neq b$ or $c = b + 1$ if $a = b$.

  We now insert c into $\mathbf{R}^{\downarrow}$. Let d be the largest integer such that $d \leq c$. This number always exists since $\mathbf{R}^{\downarrow}$ contains the smallest number in the row. Replace d with c. Set $a' = d$ if $c \neq d$ or $a' = d + 1$ if $c = d$.

Step 3: Repeat Step 2 with $a = a'$ and **R** the next row.

Define $P(\mathbf{a}) = P^{(\ell(w))}$ to be the *insertion tableau* and $Q(\mathbf{a}) = Q^{(\ell(w))}$ to be the *recording tableau*.

Example 4.1. Let $w = w^{(3,2)} = 45\bar{1}\bar{2}\bar{3}$ as in Example 3.2. Consider the reduced word $\mathbf{a} = 010121012342312 \in \mathrm{Red}(w)$. Following the above insertion algorithm, we obtain

$$P^{(0)} = \emptyset \xrightarrow{0} \begin{ytableau} 0 \end{ytableau} \xrightarrow{1} \begin{ytableau} 0 & 1 \end{ytableau} \xrightarrow{0} \begin{ytableau} 1 & 0 \\ \none & 0 \end{ytableau} \xrightarrow{1} \begin{ytableau} 1 & 0 & 1 \\ \none & 0 \end{ytableau} \xrightarrow{2} \begin{ytableau} 1 & 0 & 1 & 2 \\ \none & 0 \end{ytableau} \xrightarrow{1} \begin{ytableau} 2 & 0 & 1 & 2 \\ \none & 0 & 1 \end{ytableau} \xrightarrow{0} \begin{ytableau} 2 & 1 & 0 & 2 \\ \none & 1 & 0 \\ \none & \none & 0 \end{ytableau} \xrightarrow{1}$$

$$\begin{ytableau} 2 & 1 & 0 & 1 \\ \none & 1 & 0 & 1 \\ \none & \none & 0 \end{ytableau} \xrightarrow{2} \begin{ytableau} 2 & 1 & 0 & 1 & 2 \\ \none & 1 & 0 & 1 \\ \none & \none & 0 \end{ytableau} \xrightarrow{3} \begin{ytableau} 2 & 1 & 0 & 1 & 2 & 3 \\ \none & 1 & 0 & 1 \\ \none & \none & 0 \end{ytableau} \xrightarrow{4} \begin{ytableau} 2 & 1 & 0 & 1 & 2 & 3 & 4 \\ \none & 1 & 0 & 1 \\ \none & \none & 0 \end{ytableau} \xrightarrow{2} \begin{ytableau} 3 & 1 & 0 & 1 & 2 & 3 & 4 \\ \none & 1 & 0 & 1 & 2 \\ \none & \none & 0 \end{ytableau} \xrightarrow{3}$$

$$\begin{ytableau} 4 & 1 & 0 & 1 & 2 & 3 & 4 \\ \none & 1 & 0 & 1 & 2 & 3 \\ \none & \none & 0 \end{ytableau} \xrightarrow{1} \begin{ytableau} 4 & 2 & 0 & 1 & 2 & 3 & 4 \\ \none & 2 & 0 & 1 & 2 & 3 \\ \none & \none & 0 & 1 \end{ytableau} \xrightarrow{2} \begin{ytableau} 4 & 3 & 0 & 1 & 2 & 3 & 4 \\ \none & 3 & 0 & 1 & 2 & 3 \\ \none & \none & 0 & 1 & 2 \end{ytableau}.$$

We then have

$$P(\mathbf{a}) = \begin{array}{|c|c|c|c|c|c|c|} \hline 4 & 3 & 0 & 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 0 & 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 0 & 1 & 2 & \multicolumn{2}{c}{} \\ \cline{3-5} \end{array}, \quad Q(\mathbf{a}) = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 9 & 10 & 11 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 & 8 & 12 & 13 & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 7 & 14 & 15 & \multicolumn{2}{c}{} \\ \cline{3-5} \end{array},$$

and the reading word $\pi(P(\mathbf{a})) = 012301234301234$.

Definition 4.2. For a shifted tableau T with m rows, we say T is a *standard decomposition tableau* of $w \in W(B_n)$ if

1. $\pi(T) = T_m T_{m-1}, \ldots, T_1$ is a reduced word of w,

2. T_i is a unimodal subsequence of maximal length in $T_m T_{m-1} \ldots T_i$.

Define the set of all such tableaux to be $\mathrm{SDT}(w)$.

Theorem 4.3 (Theorem 5.2, [6])**.** *The Kraśkiewicz's insertion gives a bijection between* $\{\mathbf{a} \in \mathrm{Red}(w)\}$ *and the pairs of tableaux* $(P(\mathbf{a}), Q(\mathbf{a}))$ *where* $P(\mathbf{a}) \in \mathrm{SDT}(w)$ *and* $Q(\mathbf{a})$ *is a standard tableau of the same shape.*

Lemma 4.4. $\mathrm{SDT}(w^{(d,r)})$ *consists of exactly one shifted tableau. In other words, any* $\mathbf{a} \in \mathrm{Red}(w^{(d,r)})$ *has the same P tableau.*

Corollary 4.5. *By restricting to the recording tableau, Kraśkiewicz's insertion gives a bijection* $\mathbf{a} \mapsto Q(\mathbf{a})$ *between* $\mathrm{Red}(w^{(d,r)})$ *and* $\mathrm{SYT}(Z(d,r))$*.*

Combining Corollary 3.4 and Corollary 4.5, we derive Theorem 1.1 for the trapezoid shape $Z(d,r)$.

5 The general case

In this Section, we prove our main result, Theorem 1.1, in full generality. Fix a strict partition $\lambda \subset Z(d,r)$ such that $\lambda_d > 0$ and set $N = |\lambda|$. Let $\ell = |Z(d,r)| = \ell(w^{(d,r)})$ and set $\mu_0 = \sigma_0 = 0$, $\mu_i = Z(d,r)_i - \lambda_i$ and $\sigma_i = \sum_{k=1}^{i} \mu_k$ for all $i \in [d]$. Recall our main framework

$$\mathrm{SYT}(\lambda) \longleftrightarrow \mathrm{SYT}(Z(d,r))|_\lambda \longleftrightarrow \mathrm{Red}(w^{(d,r)})|_\lambda \longleftrightarrow \mathrm{BS}(Z(d,r))|_\lambda \longleftrightarrow \mathrm{BS}(\lambda).$$

The first arrow of bijection is immediate (Definition 5.1). In each of the subsequent subsections, we prove one remaining bijection respectively. A complete example is given at the end of this section in Example 5.11. Readers are encouraged to refer to this example for intuition.

5.1 Bijection between $\mathrm{SYT}(Z(d,r))|_\lambda$ and $\mathrm{Red}(w^{(d,r)})|_\lambda$

Definition 5.1. For any tableau $T \in \mathrm{SYT}(\lambda)$, define $T^+ \in \mathrm{SYT}(Z(d,r))$ to be the tableau obtained from T by assigning $N+1, \ldots, \ell$ to the cells in $Z(d,r) \setminus \lambda$ from left to right along rows and from top to bottom. Define $\mathrm{SYT}(Z(d,r))|_\lambda$ to be the set of all such T^+ obtained from some $T \in \mathrm{SYT}(\lambda)$.

Example 5.2. Let $\lambda = (6,2,1) \subset Z(3,2)$ and

$$T = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 & 9 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & \multicolumn{3}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 8 & \multicolumn{3}{c}{} \\ \cline{3-3} \end{array}, \text{ then } T^+ = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 & 9 & 10 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & 11 & 12 & 13 & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 8 & 14 & 15 & \multicolumn{2}{c}{} \\ \cline{3-5} \end{array}.$$

Definition 5.3. Define the word $\mathbf{a}^\lambda = \mathbf{a}_1^\lambda \cdots \mathbf{a}_d^\lambda$ where

$$\mathbf{a}_i^\lambda = d + r - i - \mu_i + 1, \ldots, d + r - i. \tag{5.1}$$

Define $\mathrm{Red}(w^{(d,r)})|_\lambda$ to be the set of reduced words ending with $\mathbf{a}^\lambda$.

Proposition 5.4. *The bijection $\mathbf{a} \mapsto Q(\mathbf{a})$ between $\mathrm{Red}(w^{(d,r)})$ and $\mathrm{SYT}(Z(d,r))$ in Corollary 4.5 induces a well-defined bijection between $\mathrm{Red}(w^{(d,r)})|_\lambda$ and $\mathrm{SYT}(Z(d,r))|_\lambda$.*

A key ingredient in the proof of Proposition 5.4 is the existence of the *reverse* of Kraśkiewicz's insertion.

Lemma 5.5 (Lemma 1.25, [8])**.** *Given $(P(\mathbf{a}), Q(\mathbf{a}))$ for some $\mathbf{a} = a_1 a_2 \cdots a_{\ell(w)} \in \mathrm{Red}(w)$ and $w \in W(B_n)$, let Q' be obtained by removing the largest entry in $Q(\mathbf{a})$. Then there is a unique $a \in [0, n-1]$ and a unique $P' \in \mathrm{SDT}(ws_a)$ such that $P' \leftarrow a = P$, and $\mathrm{sh}(P') = \mathrm{sh}(Q')$. In fact, we have $a = a_{\ell(w)}$.*

5.2 Bijection between $\mathrm{BS}(\lambda)$ and $\mathrm{BS}(Z(d,r))|_\lambda$

Recall some notations from the beginning of this section: $\mu_i = Z(d,r)_i - \lambda_i$, $\sigma_i = \sum_{k=1}^{i} \mu_k$ and $N = |\lambda|$.

Definition 5.6. Define $\mathrm{BS}(Z(d,r))|_\lambda$ to be the set of balanced tableaux T of shape $Z(d,r)$ such that for all $i \in [d]$ and any $k \in [N + \sigma_{i-1} + 1, N + \sigma_i]$, k appears in row i of T.

Lemma 5.7. *Let $B \in \mathrm{BS}(\lambda)$ and fix some $i \in [d]$ such that either $i = 1$ or $\lambda_{i-1} \geq \lambda_i + 3$. Denote $\lambda^\#$ the shifted diagram obtained from λ by adding a box in the i-th row. Let j be the column index of the box $\lambda^\# \setminus \lambda$. Let $B^\#$ be the tableau obtained from B by*

1. *interchange column j and $j+1$ of B,*

2. *define* $B^{\#}(i,j) = N+1$.

Then $B^{\#}$ *is a balanced tableau and the following map is a bijection:*

$$\begin{aligned} f_i : \mathrm{BS}(\lambda) &\longrightarrow \{T \in \mathrm{BS}(\lambda^{\#}) : T(i,j) = N+1\} \\ B &\longmapsto B^{\#}. \end{aligned} \tag{5.2}$$

Lemma 5.8. $\mathrm{BS}(Z(d,r))|_\lambda$ *is the image of* $\mathrm{BS}(\lambda)$ *under the composition of maps* $F = (f_d)^{a_d} \circ (f_{d-1})^{a_{d-1}} \circ \cdots \circ (f_1)^{a_1}$ *with each* f_i *defined as in* (5.2)*. As a result,* F *is a bijection between* $\mathrm{BS}(\lambda)$ *and* $\mathrm{BS}(Z(d,r))|_\lambda$.

5.3 Bijection between $\mathrm{BS}(Z(d,r))|_\lambda$ and $\mathrm{Red}(w^{(d,r)})|_\lambda$

Proposition 5.9. *Let* $\mathbf{a} \in \mathrm{Red}(w^{(d,r)})$ *be a reduced word. Then* $\mathrm{ro}(\mathbf{a})$ *gives a balanced tableau in* $\mathrm{BS}(Z(d,r))|_\lambda$ *if and only if the ending segment of* $\mathbf{a}$ *is the same as* $\mathbf{a}^\lambda$ *as in Definition 5.3. Consequently, this induces a bijection between* $\mathrm{BS}(Z(d,r))|_\lambda$ *and* $\mathrm{Red}(w^{(d,r)})|_\lambda$.

Combining Proposition 5.4, Lemma 5.8 and Proposition 5.9, we get a bijection between $\mathrm{SYT}(\lambda)$ and $\mathrm{BS}(\lambda)$.

Proposition 5.10. *The bijection* $\mathrm{SYT}(\lambda) \longleftrightarrow \mathrm{BS}(\lambda)$ *in Theorem 1.1 does not depend on the parameter* r.

Example 5.11. We now work out an example in the case where $\lambda = (6,2,1)$, $d = 3$ and $r = 2$. Assume we start with a balanced tableau $B \in \mathrm{BS}(\lambda)$ shown here:

$$B = \begin{array}{cccccc} 6 & 3 & 4 & 1 & 5 & 9 \\ & 7 & 8 & & & \\ & & 2 & & & \end{array}.$$

We can complete it to $B^+ \in \mathrm{BS}(Z(3,2))$ using the algorithm in Lemma 5.8

$$B = \begin{array}{cccccc} 6 & 3 & 4 & 1 & 5 & 9 \\ & 7 & 8 & & & \\ & & 2 & & & \end{array} \to \begin{array}{ccccccc} 6 & 3 & 4 & 1 & 5 & 9 & 10 \\ & 7 & 8 & & & & \\ & & 2 & & & & \end{array} \to \begin{array}{ccccccc} 6 & 3 & 4 & 5 & 1 & 9 & 10 \\ & 7 & 8 & 11 & & & \\ & & 2 & & & & \end{array} \to$$

$$\begin{array}{ccccccc} 6 & 3 & 4 & 5 & 9 & 1 & 10 \\ & 7 & 8 & 11 & 12 & & \\ & & 2 & & & & \end{array} \to \begin{array}{ccccccc} 6 & 3 & 4 & 5 & 9 & 10 & 1 \\ & 7 & 8 & 11 & 12 & 13 & \\ & & 2 & & & & \end{array} \to \begin{array}{ccccccc} 6 & 3 & 4 & 9 & 5 & 10 & 1 \\ & 7 & 8 & 12 & 11 & 13 & \\ & & 2 & 14 & & & \end{array} \to$$

$$\begin{array}{ccccccc} 6 & 3 & 4 & 9 & 10 & 5 & 1 \\ & 7 & 8 & 12 & 13 & 11 & \\ & & 2 & 14 & 15 & & \end{array} = B^+$$

Now B^+ gives a reflection order of $w^{(d,r)}$ as follows

$$\begin{aligned}12345 &\xrightarrow{e_3-e_2} 13245 \xrightarrow{e_1} \bar{1}3245 \xrightarrow{e_3+e_1} 3\bar{1}245 \xrightarrow{e_3} \bar{3}\bar{1}245 \xrightarrow{e_3-e_1} \bar{1}\bar{3}245\\ &\xrightarrow{e_3+e_2} \bar{1}2\bar{3}45 \xrightarrow{e_2+e_1} 2\bar{1}\bar{3}45 \xrightarrow{e_2} \bar{2}\bar{1}\bar{3}45 \xrightarrow{e_4+e_3} \bar{2}\bar{1}4\bar{3}5 \xrightarrow{e_5+e_3} \bar{2}\bar{1}45\bar{3}\\ &\xrightarrow{e_2-e_1} \bar{1}\bar{2}45\bar{3} \xrightarrow{e_4+e_2} \bar{1}4\bar{2}5\bar{3} \xrightarrow{e_5+e_2} \bar{1}45\bar{2}\bar{3} \xrightarrow{e_4+e_1} 4\bar{1}5\bar{2}\bar{3} \xrightarrow{e_5+e_1} 45\bar{1}\bar{2}\bar{3}.\end{aligned}$$

We can read off the reduced word $\mathbf{a} = 201012103412312 \in \mathrm{Red}(w^{(3,2)})$. We can confirm that the reduced word ends with $\mathbf{a}^\lambda = 412312$. Now we perform the Kraśkiewicz's insertion on $\mathbf{a}$ described in Section 4 and we get

$$P(\mathbf{a}) = \begin{array}{|c|c|c|c|c|c|c|}\hline 4 & 3 & 0 & 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 0 & 1 & 2 & 3 \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 0 & 1 & 2 \\ \cline{3-5}\end{array}, \quad Q(\mathbf{a}) = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 & 9 & 10 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & 11 & 12 & 13 \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & 14 & 15 \\ \cline{3-5}\end{array},$$

Finally, let $T^+ = Q(\mathbf{a}) \in \mathrm{SYT}(w^{(3,2)})$, and $T \in \mathrm{SYT}(\lambda)$ is obtained from T^+ by deleting the largest entries until $|\lambda|$ entries are left:

$$T^+ = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 & 9 & 10 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & 11 & 12 & 13 \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & 14 & 15 \\ \cline{3-5}\end{array}, \quad T = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 & 9 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 \\ \cline{3-3}\end{array}.$$

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