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The tropical critical points of an affine matroid

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Abstract. We prove that the maximum likelihood degree of a matroid M equals its beta invariant $\beta(M)$. For an element e of M that is neither a loop nor a coloop, this is defined to be the degree of the intersection of the Bergman fan of (M, e) and the inverted Bergman fan of $N = (M/e)^\perp$. Equivalently, for a generic vector $\mathsf{w} \in \mathbb{R}^{E-e}$, this is the number of ways to find weights $(0, \mathsf{x})$ on M and y on N with $\mathsf{x} + \mathsf{y} = \mathsf{w}$ such that on each circuit of M (resp. N), the minimum x-weight (resp. y-weight) occurs at least twice.

Keywords: matroid, Bergman fan, maximum likelihood degree

1 Introduction

During the Workshop on Nonlinear Algebra and Combinatorics from Physics at the Center for the Mathematical Sciences and Applications at Harvard University in April 2022, Sturmfels [15] posed one of those combinatorial problems that is deceivingly simple to state, but whose answer requires a deeper understanding of the objects at hand.

Problem 1.1. *[15] (Combinatorial version) Let M be a matroid on ground set E. Let e be an element that is neither a loop nor a coloop. Let M/e be the contraction of M by e and let $N = (M/e)^\perp$ be its dual matroid. Given a vector $\mathsf{w} \in \mathbb{R}^{E-e}$, find weight vectors $(0, \mathsf{x}) \in \mathbb{R}^E$ on M (where e has weight 0) and $\mathsf{y} \in \mathbb{R}^{E-e}$ on N such that*

- *on each circuit of M, the minimum x-weight occurs at least twice,*
- *on each circuit of N, the minimum y-weight occurs at least twice, and*
- $\mathsf{w} = \mathsf{x} + \mathsf{y}$.

Can this always be done? What is the number of solutions for generic w?

Theorem 1.2. *(Geometric Version) Problem 1.1 can always be solved. If $\mathsf{w} \in \mathbb{R}^{E-e}$ is generic, the number of solutions equals the* beta invariant $\beta(M)$ *of the matroid.*

We now restate Theorem 1.2 in tropical terms; see relevant definitions in Section 2.3.

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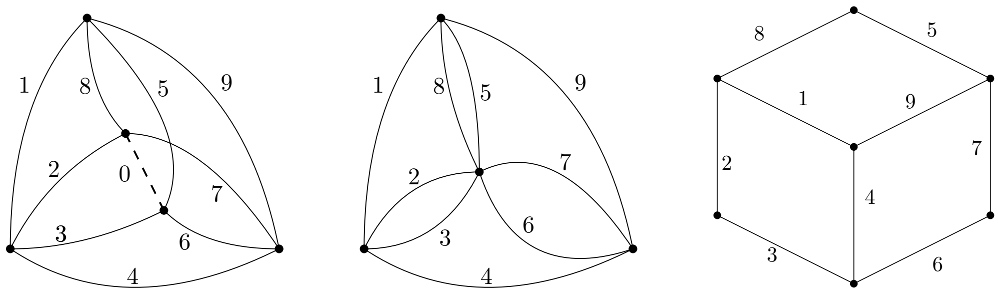

Figure 1: A graph G, its contraction $G/0$, and its dual $H = (G/0)^{\perp}$.

Theorem 1.3. *Let M be a matroid on E, and let $e \in E$ be an element that is neither a loop nor a coloop. Let (M, e) be the affine matroid of M with respect to e, and let $N = (M/e)^{\perp}$. Then the degree of the intersection of the Bergman fan $\Sigma_{(M,e)}$ and the inverted Bergman fan $-\Sigma_N$ is*

$$\deg(\Sigma_{(M,e)} \cdot -\Sigma_N) = \beta(M)\,.$$

Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, and Telen [1] first encountered (a special case of) Problem 1.1 in their study of the maximum likelihood estimation for linear discrete models. Using algebro-geometric results of Huh and Sturmfels [10], which built on work of Varchenko [16], they proved Theorem 1.2 and Theorem 1.3 for matroids realizable over the real numbers.

We prove the equivalent Theorem 1.2 and Theorem 1.3 for all matroids. Following the original motivation, we call the answer to Problem 1.1 the maximum likelihood degree of a matroid; our main result is that it equals the beta invariant.

We first prove Theorem 1.2 combinatorially, relying on the tropical geometric fact that all generic w give the same intersection degree. We show that when the entries of w are super-increasing with respect to some order $<$ on E, the solutions to Problem 1.1 are naturally in bijection with the β-nbc bases of the matroid with respect to $<$. We then sketch a proof of Theorem 1.3 that relies on the theory of tautological classes of matroids of Berget, Eur, Spink, and Tseng [6]. This is an extended abstract of our results in [5].

2 Notation and preliminaries

2.1 The lattice of set partitions

A set partition $\boldsymbol{\lambda}$ of a set E is a collection of subsets, called blocks, of E, say $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_\ell\}$, that cover E and the pairwise intersection is empty. We write $\boldsymbol{\lambda} \models E$. We let $|\boldsymbol{\lambda}| = \ell$ be the number of blocks of $\boldsymbol{\lambda}$. If $e \in E$ and $\boldsymbol{\lambda} \models E$, we write $\boldsymbol{\lambda}(e)$ for the block of $\boldsymbol{\lambda}$ that contains e.

We define the *linear space of a set partition* $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_\ell\} \models E$ to be

$$\begin{aligned} \mathsf{L}(\boldsymbol{\lambda}) &:= \operatorname{span}\{\mathsf{e}_{\lambda_1}, \ldots, \mathsf{e}_{\lambda_\ell}\} \subseteq \mathbb{R}^E \\ &= \{\mathsf{x} \in \mathbb{R}^E \mid x_i = x_j \text{ whenever } i, j \text{ are in the same block of } \boldsymbol{\lambda}\}, \end{aligned}$$

where $\{\mathsf{e}_i : i \in E\}$ is the standard basis of $\mathbb{R}^E$, $\mathsf{e}_S = \sum_{s \in S} e_s$ for $S \subseteq E$. Notice that $\dim \mathsf{L}(\boldsymbol{\lambda}) = |\boldsymbol{\lambda}|$. The map $\boldsymbol{\lambda} \mapsto \mathsf{L}(\boldsymbol{\lambda})$ is a bijection between the set partitions of E and the flats of the *braid arrangement*, which is the hyperplane arrangement in $\mathbb{R}^E$ given by the hyperplanes $x_i = x_j$ for $i \neq j$ in E.

If $\boldsymbol{\lambda} \models 0 \sqcup E$ then we write $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} = \{\mathsf{x} \in \mathbb{R}^E : (0, \mathsf{x}) \in \mathsf{L}(\boldsymbol{\lambda}) \subseteq \mathbb{R}^{0 \sqcup E}\}$.

2.2 The intersection graph of two set partitions

The following construction from [3] will play an important role.

Definition 2.1. *Let* $\boldsymbol{\lambda} \models 0 \sqcup E$ *and* $\boldsymbol{\mu} \models E$ *be set partitions. The* intersection graph $\Gamma = \Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *is the bipartite graph with vertex set* $\boldsymbol{\lambda} \sqcup \boldsymbol{\mu}$ *and edge set* E*, where the edge* e *connects the parts* $\lambda(e)$ *of* $\boldsymbol{\lambda}$ *and* $\mu(e)$ *of* $\boldsymbol{\mu}$ *containing* e*. On this graph, the vertex corresponding to* $\boldsymbol{\lambda}(0)$ *is marked with a hollow point.*

The intersection graph may have several parallel edges connecting the same pair of vertices. Notice that the label of a vertex in Γ is just the set of labels of the edges incident to it. Therefore we can remove the vertex labels, and simply think of Γ as a bipartite multigraph on edge set E. This is illustrated in Figure 2.

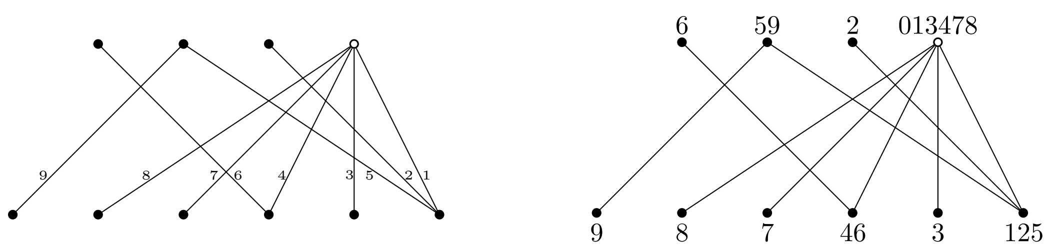

Figure 2: The intersection graph of $\{6, 59, 2, 013478\} \models [0, 9]$ and $\{9, 8, 7, 46, 3, 125\} \models [9]$, omitting brackets for easier legibility. **Left:** The elements of $[0, 9]$ are labelling the edges. **Right:** the vertices are labelled by parts of the set partitions.

Lemma 2.2. *Let* $\boldsymbol{\lambda} \models 0 \sqcup E$ *and* $\boldsymbol{\mu} \models E$ *be set partitions and* $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *be their intersection graph.*

1. *If* $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *has a cycle, then* $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu})) = \emptyset$ *for generic*[1] $\mathsf{w} \in \mathbb{R}^E$.

[1]This means that this property holds for all w outside of a set of measure 0.

2. *If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is disconnected, then $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is not a point for any $\mathsf{w} \in \mathbb{R}^E$.*

3. *If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree, then $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is a point for any $\mathsf{w} \in \mathbb{R}^E$.*

Proof. Let $\mathsf{x} \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} \in \mathsf{L}(\mu)$ such that $\mathsf{x} + \mathsf{y} = \mathsf{w}$. Write $x_{\lambda(i)} := x_i$ and $y_{\mu(j)} := y_j$ for simplicity. The subspace $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is cut out by the equalities

$$\begin{aligned} x_{\lambda(i)} + y_{\mu(i)} &= w_i \qquad \text{for } i \in E, \\ x_{\lambda(0)} &= 0. \end{aligned}$$

This system has $|E| + 1$ equations and $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$ unknowns. The linear dependences among these equations are controlled by the cycles of the graph $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. More precisely, the first $|E|$ linear functionals $\{x_{\lambda(i)} + y_{\mu(i)} \,:\, i \in E\}$ gives a realization of the graphical matroid of $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. The last equation is clearly linearly independent from the others.

If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ has a cycle with edges $i_1, i_2, \ldots, i_{2k}$ in that order, then the above equalities imply that $w_{i_1} - w_{i_2} + w_{i_3} - \cdots - w_{i_{2k}} = 0$. For a generic w, this equation does not hold, so $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu})) = \emptyset$.

If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is disconnected, let A be the set of edges in a connected component not containing the vertex $\lambda(0)$. If $\mathsf{x} \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} \in \mathsf{L}(\boldsymbol{\mu})$ satisfy $\mathsf{x} + \mathsf{y} = \mathsf{w}$ and $x_0 = 0$, then $\mathsf{x} + r\mathsf{e}_A \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} - r\mathsf{e}_A \in \mathsf{L}(\boldsymbol{\mu})$ also satisfy those equations for any real number r. Therefore $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is not a point.

Finally, if $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree, then its number of vertices is one more than the number of edges, that is, $|E| + 1 = |\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$, so the system of equations has equally many equations and unknowns. Also, these equations are linearly independent since $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree. It follows that the system has a unique solution. □

When $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree, we call $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ an *arboreal pair*.

Lemma 2.3. *Let $\boldsymbol{\lambda} \models 0 \sqcup E$ and $\boldsymbol{\mu} \models E$ be an arboreal pair of set partitions and let $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ be their intersection tree. Let $\mathsf{w} \in \mathbb{R}^E$. The unique vectors $\mathsf{x} \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} \in \mathsf{L}(\boldsymbol{\mu})$ such that $\mathsf{x} + \mathsf{y} = \mathsf{w}$ and $x_0 = 0$ are given by*

$$\begin{aligned} x_{\lambda_i} &= w_{e_1} - w_{e_2} + \cdots \pm w_{e_k} \qquad \textit{where } e_1e_2\ldots e_k \textit{ is the unique path from } \lambda_i \textit{ to } \lambda(0) \\ y_{\mu_j} &= w_{f_1} - w_{f_2} + \cdots \pm w_{f_l} \qquad \textit{where } f_1f_2\ldots f_l \textit{ is the unique path from } \mu_j \textit{ to } \lambda(0) \end{aligned}$$

for any i and j.

Proof. This follows readily from the fact that, for each $1 \leq i \leq k$, the values of $x_{\lambda(e_i)}$ and $y_{\mu(e_i)}$ on the vertices incident to edge i have to add up to w_{e_i}. □

Example 2.4. *The set partitions $\boldsymbol{\lambda} = \{6, 59, 2, 013478\} \models [0,9]$, $\boldsymbol{\mu} = \{9, 8, 7, 46, 3, 125\} \models [9]$ form an arboreal pair, whose intersection tree is shown in Figure 2. We have, for example,*

$y_9 = w_9 - w_5 + w_1$ because the path from $\mu(9) = \{9\}$ to $\lambda(0) = \{013478\}$ uses edges $9,5,1$ in that order. The remaining values are:

$$x_6 = w_6 - w_4, \quad x_{59} = w_5 - w_1, \quad x_2 = w_2 - w_1, \quad x_{13478} = 0,$$
$$y_9 = w_9 - w_5 + w_1, \quad y_8 = w_8, \quad y_7 = w_7, \quad y_{46} = w_4, \quad y_3 = w_3, \quad y_{1235} = w_1.$$

Definition 2.5. *A vector* $\mathsf{w} \in \mathbb{R}^{n+1}$ *is* super-increasing *if* $\omega_{i+1} > 3\omega_i > 0$ *for* $1 \le i \le n$.

Lemma 2.6. *Let* w *be super-increasing. For any* $1 \le a < b \le n+1$ *and any choice of* ϵ_i*s and* δ_i*s in* $\{-1, 0, 1\}$*, we have* $\omega_a + \sum_{i=1}^{a-1} \epsilon_i \omega_i < \omega_b + \sum_{j=1}^{b-1} \delta_j \omega_j$.

Proof. After verifying inductively that $\omega_c > 2 \sum_{i=1}^{c-1} \omega_i$ for all c, we see that

$$\omega_a + \sum_{i=1}^{a-1} \epsilon_i \omega_i \le \omega_a + \sum_{i=1}^{a-1} \omega_i < \frac{3}{2}\omega_a < \frac{1}{2}\omega_b < \omega_b - \sum_{j=1}^{b-1} \omega_j \le \omega_b + \sum_{j=1}^{b-1} \delta_j \omega_j$$

as desired. □

Definition 2.7. *Given a super-increasing vector* $\mathsf{w} \in \mathbb{R}^{n+1}$ *and a real number* x*, we will say* x is near w_i *and write* $x \approx w_i$ *if* $w_i - (w_1 + \cdots + w_{i-1}) \le x \le w_i + (w_1 + \cdots + w_i)$*. Note that if* $x \approx w_i$ *and* $y \approx w_j$ *for* $i < j$ *then* $x < y$.

2.3 Matroids, Bergman fans, and tropical geometry

We assume familiarity with basic notions in matroid theory; for definitions and proofs, see [13, 17]. We also state here some facts from tropical geometry that we will need; see [11, 12] for a thorough introduction.

Let M be a matroid on E. The *dual matroid* $M^\perp$ is the matroid on E whose set of bases is $\{B^\perp \mid B \text{ is a basis of } M\}$, where $B^\perp := E - B$.

The following lemma is useful to how M and $M^\perp$ interact; see [13, Proposition 2.1.11].

Lemma 2.8. *[2, Lemma 3.14] Let* M *be a matroid, and let* F *be a flat of* M *and* G *be a flat of* $M^\perp$*. Then* $|F \cup G| \ne |E| - 1$.

Definition 2.9. *Fix a linear order* $<$ *on* M*. A* broken circuit *is a set of the form* $C - \min_< C$ *where* C *is a circuit of* M*. An* nbc-basis *of* M *is a basis of* M *that contains no broken circuits. A* βnbc-basis *of* M *is an nbc-basis* B *such that* $B^\perp \cup 0 \setminus 1$ *is an nbc-basis of* $M^\perp$.

Theorem 2.10. *[7] The number of* β*nbc-bases of* M *is the* beta invariant $\beta(M)$ *which is given by* $\beta(M) = |\chi'_M(1)|$*, where* χ_M *is the* characteristic polynomial *of* M*:*

$$\chi_M(t) = \sum_{X \subseteq E} (-1)^{|X|} t^{r(M) - r(X)}.$$

Assume that $B = \{b_1 > \cdots > b_r\}$ is a basis of the matroid M. We define the flats $F_i := \operatorname{cl}_M\{b_1, \ldots, b_i\}$ and the flag $\mathcal{F}_M(B) := \{F_i\}$. The following characterization of nbc-basis will be useful.

Lemma 2.11. *Let M be a matroid of size $n+1$ and rank $r+1$, and B a basis of M. Then B is an nbc-basis of M if and only if $b_i = \min F_i$ for $i = 1, \ldots, r+1$.*

Proof. Omitted. □

An *affine matroid* (M, e) on E is a matroid M on E with a chosen element $e \in E$.

Definition 2.12. *[14] The* Bergman fan *of a matroid M on E is*

$$\Sigma_M = \{\mathsf{x} \in \mathbb{R}^E \mid \min_{c \in C} x_c \text{ is attained at least twice for any circuit } C \text{ of } M\}.$$

The Bergman fan of an affine matroid (M, e) *on E is*

$$\Sigma_{(M,e)} = \{\mathsf{x} \in \mathbb{R}^{E-e} \mid (0, \mathsf{x}) \in \Sigma_M\}.$$

The motivation for this definition comes from tropical geometry. A subspace $V \subset \mathbb{R}^E$ determines a matroid M_V, and the tropicalization of V is precisely the Bergman fan of M_V. Similarly, an affine subspace $W \subset \mathbb{R}^{E-e}$ determines an affine matroid (M_W, e) consisting of a matroid M_W on E and a special element e, which represents the hyperplane at infinity. The tropicalization of W is the Bergman fan $\Sigma_{(M_W,e)}$.

Theorem 2.13. *[4] The Bergman fan of a matroid M is a tropical fan equal to the union of the cones*

$$\begin{aligned} \sigma_{\mathcal{F}} &= \textit{cone}(\mathsf{e}_{F_1}, \ldots, \mathsf{e}_{F_{r+1}}) \\ &= \{\mathsf{x} \in \mathbb{R}^E \mid x_a > x_b \text{ whenever } a \in F_i \text{ and } b \notin F_i \text{ for some } 1 \le i \le r+1\} \end{aligned}$$

for the complete flags $\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E\}$ of flats of M.

If Σ_1 and Σ_2 are tropical fans of complementary dimensions, then Σ_1 and $\Sigma_2 + v$ intersect transversally at a finite set of points for generic vectors $v \in \mathbb{R}^n$. Each intersection point p is equipped with a weight $w(p)$ that depends on the respective intersecting cones. It is a general fact that the quantity

$$\deg(\Sigma_1 \cdot \Sigma_2) := \sum_{p \in \Sigma_1 \cap (v + \Sigma_2)} w(p)$$

is constant for generic v; this is the degree of the intersection [12, Proposition 4.3.3, 4.3.6].

In all the intersections that arise in this paper, one can verify that the weight $w(p)$ is equal to 1 for every intersection point p. Therefore the degree of the intersection will be simply the number of intersection points:

$$\deg(\Sigma_{(M,e)} \cdot -\Sigma_N) := |\Sigma_{(M,e)} \cap (v - \Sigma_N)|$$

for generic $v \in \mathbb{R}^{E-e}$.

3 A combinatorial proof of the main theorem

Let M be a matroid on $[0,n]$ of rank $r+1$ such that 0 is not a loop nor a coloop. Then $M/0$ has rank r, and $N = (M/0)^{\perp}$ has rank $n-r$. For any basis B of M containing 0, $B^{\perp} = [0,n] - B$ is a basis of $N = (M/0)^{\perp}$. Conversely, every basis of N equals $B^{\perp}$ for a basis B of M containing 0.

The following Lemma constructs an intersection point in $\Sigma_M|_{x_0=0} \cap (\mathsf{w} - \Sigma_N)$ for each β-nbc basis B of M.

Lemma 3.1. *Let M be a matroid on $E = [0,n]$ of rank $r+1$ such that 0 is not a coloop, and let $N = (M/0)^{\perp}$. Let $\mathsf{w} \in \mathbb{R}^n$ be super-increasing. For any β-nbc basis B of M, there exist unique vectors $(0,\mathsf{x}) \in \sigma_{\mathcal{F}_M(B)}$ and $\mathsf{y} \in \sigma_{\mathcal{F}_N(B^{\perp})}$ such that $\mathsf{x}+\mathsf{y} = \mathsf{w}$.*

Proof. First we show that the set partitions $\boldsymbol{\pi}$ of $\mathcal{F}_M(B)$ and $\boldsymbol{\pi}^{\perp}$ of $\mathcal{F}_N(B^{\perp})$ form an arboreal pair. Since they have sizes $|B| = r+1$ and $|B^{\perp}| = n-r$, respectively, their intersection graph has $n+1$ vertices and n edges. Therefore it is sufficient to prove that the intersection graph $\Gamma_{\boldsymbol{\pi},\boldsymbol{\pi}^{\perp}}$ is connected.

Assume contrariwise, and let A be a connected component not containing the edge 1. Let $a > 1$ be the smallest edge in A. Then a is the smallest element of its part $\boldsymbol{\pi}(a)$ in $\boldsymbol{\pi}$, and since B is nbc in M, this implies $a \in B$. Similarly, since $B^{\perp}$ is nbc in N, this also implies $a \in B^{\perp}$. This is a contradiction.

It follows from Lemma 2.2 that there exist unique $(0,\mathsf{x}) \in \mathsf{L}(\boldsymbol{\pi})$ and $\mathsf{y} \in \mathsf{L}(\boldsymbol{\pi}^{\perp})$ such that $\mathsf{x}+\mathsf{y} = \mathsf{w}$. It remains to show that $(0,\mathsf{x}) \in \sigma_{\mathcal{F}}$ and $\mathsf{y} \in \sigma_{\mathcal{F}^{\perp}}$.

Lemma 2.3 provides formulas for x and y in terms of the paths from the various vertices of the tree of $\Gamma_{\boldsymbol{\pi},\boldsymbol{\pi}^{\perp}}$ to $\pi(0)$. To understand those paths, let us give each edge e an orientation as follows:

$$\begin{aligned} \pi(e) \longrightarrow \pi^{\perp}(e) \quad &\text{if} \quad \min \pi(e) > \min \pi^{\perp}(e), \\ \pi(e) \longleftarrow \pi^{\perp}(e) \quad &\text{if} \quad \min \pi(e) < \min \pi^{\perp}(e). \end{aligned}$$

We never have $\min \pi(e) = \min \pi^{\perp}(e)$, because as above, that would imply $e \in B \cap B^{\perp}$.

We claim that every vertex other than $\pi(0)$ has an outgoing edge under this orientation. Consider a part $\pi_i \neq \pi(0)$ of $\boldsymbol{\pi}$; let $\min \pi_i = b$. Edge b connects $\pi_i = \pi(b)$ to $\pi^{\perp}(b) \ni b$, and we cannot have $\min \pi^{\perp}(b) > b = \min \pi(b)$, so we must have $\pi_i \to \pi^{\perp}(b)$. The same argument works for any part $\pi_j^{\perp}$ of $\boldsymbol{\pi}^{\perp}$.

Now, every element $b \in B$ is minimum in $\pi(b)$, so there is a directed path that starts at $\pi(b)$ and can only end at $\pi(0)$ whose first edge is b. Furthermore, by the definition of the orientation, the labels of the edges decrease along this path. Thus in the alternating sum $x_b = w_b \pm \cdots$ given by Lemma 2.3, the first term dominates, and $x_b \approx w_b$. Similarly, $y_c \approx w_c$ for all $c \in B^{\perp}$.

Therefore, if we write $B = \{b_1 > \cdots > b_r > b_{r+1} = 0\}$, since w is super-increasing, it follows that $x_{b_1} > x_{b_2} > \cdots > x_{b_r} > x_{b_{r+1}} = 0$, so indeed $(0,\mathsf{x}) \in \sigma_{\mathcal{F}}$. Similarly, if we write $B^{\perp} = E - B = \{c_1 > \cdots > c_{n-r} > c_{n-r+1} = 1\}$, then $y_{c_1} > y_{c_2} > \cdots > y_{c_{n-r+1}}$, so $\mathsf{y} \in \sigma_{\mathcal{F}^{\perp}}$. The desired result follows. □

Example 3.2. *The graphical matroid M of the graph G in Figure 1 has six β-nbc bases: 0256, 0257, 0259, 0368, 0378, and 0379. Let us compute the intersection point in $\Sigma_{(M,0)} \cap (\mathsf{w} - \Sigma_N)$ associated to* 0257 *for the super-increasing vector* $\mathsf{w} = (10^0, 10^1, \ldots, 10^8) \in \mathbb{R}^9$.

For $B = 0257$, we have $B^{\perp} = 134689$. Then

$$\begin{aligned} \mathcal{F}_M(B) &= \{\emptyset \subsetneq 7 \subsetneq 57 \subsetneq 2457 \subsetneq 0123456789\} \\ \mathcal{F}_N(B^{\perp}) &= \{\emptyset \subsetneq 9 \subsetneq 89 \subsetneq 689 \subsetneq 46789 \subsetneq 346789 \subsetneq 123456789\}, \end{aligned}$$

give rise to the corresponding set compositions

$$\boldsymbol{\pi} = 7|5|24|013689, \qquad \boldsymbol{\pi}^{\perp} = 9|8|6|47|3|125.$$

This is indeed an arboreal pair, as evidenced by their intersection graph in Figure 3.

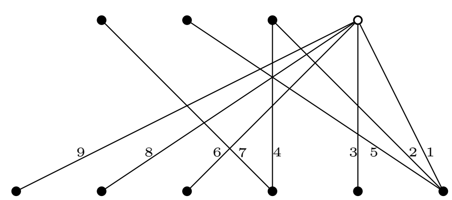

Figure 3: The intersection graph of $\boldsymbol{\pi} = 7|5|24|13689$ and $\boldsymbol{\pi}^{\perp} = 9|8|6|47|3|125$.

Lemma 3.1 gives us the unique points $(0, \mathsf{x}) \in \mathcal{F}_{\boldsymbol{\pi}}$ and $\mathsf{y} \in \mathcal{F}_{\boldsymbol{\tau}}$ such that $\mathsf{x} + \mathsf{y} = \mathsf{w}$; they are given by the paths to the special vertex $\pi(0)$ in the intersection tree $\Gamma_{\boldsymbol{\pi},\boldsymbol{\pi}^{\perp}}$. For example $x_7 = 10^6 - 10^3 + 10^1 - 10^0 = 999009$ and $y_4 = 10^3 - 10^1 + 10^0 = 991$ are given by the paths 7421 and 421 from $\pi(7) = \pi_1$ and $\pi^{\perp}(7) = \pi^{\perp}_4$ to $\pi(0)$, respectively. In this way we obtain:

$\mathsf{x} =$	0	9	0	9	9999	0	999009	0	0
$\mathsf{y} =$	1	1	100	991	1	100000	991	10000000	100000000
$\mathsf{w} =$	1	10	100	1000	10000	100000	1000000	10000000	100000000

and x is in the intersection $\Sigma_{(M,0)} \cap (\mathsf{w} - \Sigma_N)$.

The following lemma indicates that any intersection point between $\Sigma_{(E,e)}$ and $v - \Sigma_N$ is an intersection between the cones described above. That is, cones corresponding to a β-nbc basis.

Lemma 3.3. *Let M be a matroid on $E = [0, n]$ of rank $r + 1$, such that 0 is not a loop nor a coloop, and $N = (M/0)^{\perp}$. Let $\mathsf{w} \in \mathbb{R}^n$ be generic and super-increasing. Let*

$$\begin{aligned} \mathcal{F} &= \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E\} \\ \mathcal{G} &= \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{n-r-1} \subsetneq G_{n-r} = E - 0\} \end{aligned}$$

be complete flags of the matroids M and N, respectively, such that $\Sigma_{(M,0)}$ and $\mathsf{w} - \Sigma_N$ intersect at $\sigma_{\mathcal{F}}$ and $\mathsf{w} - \sigma_{\mathcal{G}}$. Then there exists a β-nbc basis B such that $\mathcal{F} = \mathcal{F}_M(B)$ and $\mathcal{G} = \mathcal{F}_N(B^{\perp})$.

Proof. By Lemma 2.2, the set compositions $\boldsymbol{\pi}$ and $\boldsymbol{\tau}$ of $\mathcal{F}$ and $\mathcal{G}$ form an arboreal pair. In particular, $\pi_a \cap \tau_b = (F_a - F_{a-1}) \cap (G_b - G_{b-1})$ cannot have more than one element for any a and b. We proceed in several steps.

1. Our first step will be to show that in the intersection tree $\Gamma_{\boldsymbol{\pi},\boldsymbol{\tau}}$, the top right vertex π_{r+1} contains 0 and 1, the bottom right vertex τ_{n-r} contains 1, and thus the edge 1 connects these two rightmost vertices.

Each G_i is a flat of $N = M^{\perp} - 0$, so $G_i^{\bullet} := \mathrm{cl}_{M^{\perp}}(G_i) \in \{G_i, G_i \cup 0\}$ is a flat of $M^{\perp}$. Consider the flag of flats of $M^{\perp}$

$$\mathcal{G}^{\bullet} := \{\emptyset = G_0^{\bullet} \subsetneq G_1^{\bullet} \subsetneq \cdots \subsetneq G_{n-r-1}^{\bullet} \subsetneq G_{n-r}^{\bullet} = E\},$$

where $G_{n-r}^{\bullet} = E$ because 0 is not a coloop of $M^{\perp}$ and $G_0^{\bullet} = \emptyset$ because 0 is not a loop of $M^{\perp}$. Let M be the minimal index such that $0 \in G_M^{\bullet}$, so

$$\mathcal{G}^{\bullet} := \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{M-1} \subsetneq G_M \cup 0 \subsetneq \cdots \subsetneq G_{n-r-1} \cup 0 \subsetneq G_{n-r} \cup 0 = E\},$$

Consider the unions of the flat F_r with the coflats in $\mathcal{G}^{\bullet}$; let j be the index such that

$$F_r \cup G_{j-1}^{\bullet} \neq E, \qquad F_r \cup G_j^{\bullet} = E$$

Since it is the union of a flat and a coflat, the former cannot have size $|E| - 1$, so $(F_r \cup G_j^{\bullet}) - (F_r \cup G_{j-1}^{\bullet}) = (E - F_r) \cap (G_j^{\bullet} - G_{j-1}^{\bullet})$ has size at least 2. But $\mathcal{F}$ and $\mathcal{G}$ are arboreal so $\pi_{r+1} \cap \tau_j = (E - F_r) \cap (G_j - G_{j-1})$ has size at most 1. This has two consequences:

a) $G_j^{\bullet} = G_j \cup 0$ and $G_{j-1}^{\bullet} = G_{j-1}$, that is, $j = M$.

b) $0 \in E - F_r = \pi_{r+1}$.

Similarly, consider the unions of the coflat $G_{n-r-1}^{\bullet}$ with the flats in $\mathcal{F}$; let i be the index such that

$$F_{i-1} \cup G_{n-r-1}^{\bullet} \neq E, \qquad F_i \cup G_{n-r-1}^{\bullet} = E.$$

Analogously, we get that $(F_i - F_{i-1}) \cap (E - G_{n-r-1}^{\bullet})$ has size at least 2, whereas $\pi_i \cap \tau_{n-r} = (F_i - F_{i-1}) \cap (E - 0 - G_{n-r-1})$ has size at most 1. This has three consequences:

c) $G_{n-r-1}^{\bullet} = G_{n-r-1}$, that is, $M = n - r$.

d) $0 \in F_i - F_{i-1}$, which in light of b) implies that $i = r + 1$.

e) $(F_i - F_{i-1}) \cap (E - 0 - G_{n-r-1}) = \pi_{r+1} \cap \tau_{n-r} = \{e\}$ for some element $e \in E - 0$. But $e \in \pi_{r+1}$ means that $x_e = 0$ is minimum among all x_is for any $(0, \mathsf{x}) \in \sigma_{\mathcal{F}}$, and $e \in \tau_{n-r}$ means that y_e is minimum among all y_is for any $\mathsf{y} \in \sigma_{\mathcal{G}}$. Since $\mathsf{w} = \mathsf{x} + \mathsf{y}$ for some such x and y, $w_e = x_e + y_e$ is minimum among all w_is, and since w is super-increasing, $e = 1$.

It follows that in the intersection tree $\Gamma_{\pi,\tau}$, the top right vertex π_{r+1} contains 0 and 1 by d) and e), the bottom right vertex τ_{n-r} contains 1 by e), and thus 1 connects them.

2. Next we claim that for any path in the tree $\Gamma_{\pi,\tau}$ that ends with the edge 1, the first edge has the largest label.[2] Assume contrariwise, and consider a containment-minimal path P that does not satisfy this property; its edges must be labelled $e < f > f_2 > \cdots > f_k$ sequentially. If edge e goes from $\pi(e)$ to $\tau(e)$, Lemma 2.3 gives $x_e = w_e - w_f \pm (\text{terms smaller than } w_f) \approx -w_f < 0 = x_1$, contradicting that $(0, \mathsf{x}) \in \sigma_{\mathcal{F}}$. If e goes from bottom to top, we get $y_e = w_e - w_f \pm (\text{terms smaller than } w_f) \approx -w_f < w_1 = y_1$, contradicting that $\mathsf{y} \in \sigma_{\mathcal{G}}$.

3. Now define

$$\begin{aligned} b_i &:= \min(F_i - F_{i-1}) \quad \text{for} \quad i = 1, \ldots, r+1, \\ c_j &:= \min(G_j - G_{j-1}) \quad \text{for} \quad j = 1, \ldots, n-r. \end{aligned}$$

Then $B := \{b_1, \ldots, b_{r+1}\}$ and $C := \{c_1, \ldots, c_{n-r}\}$ are bases of M and N, and $\mathcal{F} = \mathcal{F}_M(B)$ and $\mathcal{G} = \mathcal{F}_N(C)$. We claim that B is β-nbc and $C = B^{\perp}$.

To do so, we first notice that the path from vertex $\pi_i = F_i - F_{i-1}$ (resp. $\tau_j = G_j - G_{j-1}$) to edge 1 must start with edge b_i (resp. c_j): if it started with some larger edge $b' \in F_i - F_{i-1}$, then the path from edge b_i to edge 1 would not start with the largest edge. This has two consequences:

f) The sets B and C are disjoint. If we had $b_i = c_j = e$, then edge e, which connects vertices $\pi_i = F_i - F_{i-1}$ and $\tau_j = G_j - G_{j-1}$, would have to be the first edge in the paths from both of these vertices to edge 1; this is impossible in a tree. We conclude that B and C are disjoint. Since $|B| = r+1$ and $|C| = n-r$, we have $C = B^{\perp}$.

g) For each i we have $x_{b_i} \approx w_{b_i}$, because the path from τ_i to vertex 0 – which is the path from τ_i to edge 1, with edge 1 possibly removed – starts with the largest edge b_i, so Lemma 2.3 gives $x_{b_i} = w_{b_i} \pm (\text{smaller terms}) \approx w_{b_i}$. Similarly $y_{c_i} \approx w_{c_i}$. Now, $(0, \mathsf{x}) \in \sigma_{\mathcal{F}}$ gives $x_{b_1} > \cdots > x_{b_{r+1}}$, which implies $w_{b_1} > \cdots > w_{b_{r+1}}$, which in turn gives

$$b_1 > \cdots > b_r > b_{r+1}; \qquad \text{and analogously,} \qquad c_1 > \cdots > c_{n-r-1} > c_{n-r} = 1.$$

The former implies that B is nbc in M by Lemma 2.11. The latter, combined with c), implies that $c_1 > \cdots > c_{n-r-1} > 0$ respectively are the minimum elements of the flats $G_1^{\bullet}, \ldots, G_{n-r-1}^{\bullet}, G_{n-r}^{\bullet} = E$ that they sequentially generate, so $C \cup 0 \setminus 1 = B^{\perp} \cup 0 \setminus 1$ is nbc in $M^{\perp}$. It follows that B is β-nbc in M.

We conclude that B is β-nbc in M, $\mathcal{F} = \mathcal{F}_M(B)$, and $\mathcal{G} = \mathcal{F}_N(B^{\perp})$, as desired. □

Combinatorial proof of Theorem 1.2. This follows from the previous two lemmas. □

[2] It follows that the edge labels decrease along any such path, but we will not use this in the proof.

4 A proof via torus-equivariant geometry

We sketch a proof of Theorem 1.3 using the framework of *tautological classes* of matroids of Berget, Eur, Spink, and Tseng. See [5] for details on what follows.

In this framework, one works with the permutohedral fan Σ_E, which is the Bergman fan of the Boolean matroid on E. Toric geometry equips Σ_E with two "cohomology" rings: The torus-equivariant Chow ring $A^\bullet_T(\Sigma_E)$, and the non-equivariant Chow ring $A^\bullet(\Sigma_E)$, which is a quotient of $A^\bullet_T(\Sigma_E)$ by an explicitly described ideal I.

First, one defines certain elements $[\Sigma_{(M,e)}]$ and $[-\Sigma_{(M/e)^\perp}]$ of $A^\bullet(\Sigma_E)$ by using the fact that the fans $\Sigma_{(M,e)}$ and $-\Sigma_{(M/e)^\perp}$ are subfans of Σ_E that satisfy a balancing condition in the sense of Minkowski weights [8]. These elements have the following property: The ring $A^\bullet(\Sigma_E)$ is equipped with a degree map $\deg_{\Sigma_E} : A^\bullet(\Sigma_E) \to \mathbb{Z}$, which agrees with the map deg in Theorem 1.3 in the sense that

$$\deg(\Sigma_{(M,e)} \cap -\Sigma_{(M/e)^\perp}) = \deg_{\Sigma_E}([\Sigma_{(M,e)}] \cdot [-\Sigma_{(M/e)^\perp}]).$$

For a survey of these facts, see [9, Section 4] or [6, Section 7.1].

Second, one notes that [6] provided a distinguished representative in $A^\bullet_T(\Sigma_E)$ of the class $[\Sigma_{(M,e)}] \in A^\bullet(\Sigma_E) = A^\bullet_T(\Sigma_E)/I$, and similarly for the class $[-\Sigma_{(M/e)^\perp}]$. In more detail, for a matroid M of rank $r+1$ on a ground set E of size $n+1$, [6, Definition 3.9] defined *tautological Chern classes* of M as certain elements $\{c_i^T(\mathcal{S}_M^\vee)\}_{i=0,\dots,r+1}$ and $\{c_j^T(\mathcal{Q}_M)\}_{j=0,\dots,n-r}$ in the equivariant Chow ring $A^\bullet_T(\Sigma_E)$, and established the following properties.

Lemma 4.1. *[6, Theorem 7.6, Propositions 5.11, 5.13] Let M be a matroid of rank $r+1$ on a ground set E of size $n+1$. Define elements $[\Sigma_{(M,e)}]^T$ and $[-\Sigma_{(M/e)^\perp}]^T$ in $A^\bullet_T(\Sigma_E)$ by $[\Sigma_{(M,e)}]^T = c^T_{n-r}(\mathcal{Q}_M)$ and $[-\Sigma_{(M/e)^\perp}]^T = c^T_r(\mathcal{S}^\vee_{M/e \oplus U_{0,e}})$. Then, their images in the quotient $A^\bullet(\Sigma_E)$ are exactly $[\Sigma_{(M,e)}]$ and $[-\Sigma_{(M/e)^\perp}]$, respectively.*

Finally, to prove Theorem 1.3, one begins with [6, Theorem 6.2] which states that

$$\deg_{\Sigma_E}\left([\Sigma_{(M,e)}] \cdot c_r(\mathcal{S}_M^\vee)\right) = \beta(M).$$

Thus, the desired statement $\deg_{\Sigma_E}([\Sigma_{(M,e)}] \cdot [-\Sigma_{(M/e)^\perp}]) = \beta(M)$ will follow once one shows that $[\Sigma_{(M,e)}] \cdot \left(c_r(\mathcal{S}_M^\vee) - [-\Sigma_{(M/e)^\perp}]\right) = 0$ in $A^\bullet(\Sigma_E)$. For this end, one considers the distinguished representative $[\Sigma_{(M,e)}]^T \cdot \left(c_r^T(\mathcal{S}_M^\vee) - [-\Sigma_{(M/e)^\perp}]^T\right)$ of this product in $A^\bullet_T(\Sigma_E)$. An explicit description of this representative straightforwardly displays that it belongs to the ideal I.

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