Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #29, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

Barrett–Johnson inequalities for totally nonnegative matrices

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Abstract. Given a matrix A, let $A_{I,J}$ denote the submatrix of A determined by rows I and columns J. The Barrett–Johnson Inequalities relate sums of products of principal minors of positive semidefinite (PSD) matrices, when orders of the minors are given by integer partitions $\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1, \dots, \mu_s)$ of n. Specifically, we have

$$\lambda_1! \cdots \lambda_r! \sum_{(I_1,\dots,I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}) \;\geq\; \mu_1! \cdots \mu_s! \sum_{(J_1,\dots,J_s)} \det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s}),$$

for all PSD $n \times n$ matrices A, where sums are over ordered set partitions of $\{1, \dots, n\}$ satisfying $|I_k| = \lambda_k$, $|J_k| = \mu_k$, if and only if λ is majorized by μ. We show that these inequalities hold for totally nonnegative matrices as well.

1 Introduction

A matrix $A \in \mathrm{Mat}_{n \times n}(\mathbb{C})$ is called *Hermitian* if it satisfies $A^* = A$ where $*$ denotes conjugate transpose. Such a matrix is called *Hermitian positive semi-definite* (HPSD) if we have $x^* A x \geq 0$ for all $x \in \mathbb{C}^n$. For $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$, the Hermitian property reduces to symmetry $A^\top = A$, and the positive semidefinite (PSD) property reduces to $x^\top A x \geq 0$ for all $x \in \mathbb{R}^n$. A matrix $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ is called *totally nonnegative* (TNN) if each minor (determinant of a square submatrix) is nonnegative. One can deduce that the entries of all $n \times n$ HPSD and/or TNN matrices satisfy certain polynomial inequalities. In some cases, inequalities for the two classes of matrices are precisely the same. (See [14, §1].)

Given an $n \times n$ matrix $A = (a_{i,j})$ and subsets $I, J \subseteq [n] := \{1, \dots, n\}$, define the submatrix $A_{I,J} = (a_{i,j})_{i \in I, j \in J}$, and define the set $I^c := [n] \smallsetminus I$. Hadamard [7] showed that for A HPSD we have

$$\det(A) \leq a_{1,1} \cdots a_{n,n}, \tag{1.1}$$

and Koteljanskii [9] showed that this holds for A TNN as well. Fischer [5] strengthened (1.1) by showing that for all $I \subseteq [n]$ we have

$$\det(A) \leq \det(A_{I,I}) \det(A_{I^c,I^c}), \tag{1.2}$$

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and Ky Fan showed that this holds for A TNN as well (unpublished; see [2]). Barrett and Johnson [1] showed that for A PSD, averages of the products of pairs of minors appearing in (1.2) increase as the cardinality difference between I and I^c decreases. Furthermore, they proved a similar result for averages of products of many minors: given integer partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$ of n, we have

$$\sum_{(I_1,\ldots,I_r)} \frac{\det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r})}{\binom{n}{\lambda_1,\ldots,\lambda_r}} \geq \sum_{(J_1,\ldots,J_s)} \frac{\det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s})}{\binom{n}{\mu_1,\ldots,\mu_s}}, \tag{1.3}$$

if and only if λ is majorized by μ, where the sums are over *ordered set partitions* of $[n]$ of types λ and μ, i.e., sequences of subsets of $[n]$ satisfying having cardinalities

$$I_1 \uplus \cdots \uplus I_r = J_1 \uplus \cdots \uplus J_s = [n], \qquad |I_k| = \lambda_k, \quad |J_k| = \mu_k. \tag{1.4}$$

We will show that these inequalities also hold for TNN matrices.

2 The symmetric group, its traces, and symmetric functions

The *symmetric group algebra* $\mathbb{C}[\mathfrak{S}_n]$ is generated over $\mathbb{C}$ by $s_1, \ldots, s_{n-1}$, subject to relations

$$\begin{aligned} s_i^2 &= e && \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j| = 1, \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2. \end{aligned}$$

We define the *one-line notation* $w_1 \cdots w_n$ of $w \in \mathfrak{S}_n$ by letting any expression for w act on the word $1 \cdots n$, where each generator $s_j = s_{[j,j+1]}$ acts on an n-letter word by swapping the letters in positions j and $j+1$, i.e., $s_j \circ v_1 \cdots v_n = v_1 \cdots v_{j-1} v_{j+1} v_j v_{j+2} \cdots v_n$. Whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced (short as possible) expression for $w \in \mathfrak{S}_n$, we call ℓ the *length* of w and write $\ell = \ell(w)$. It is known that $\ell(w)$ is equal to $\textsc{inv}(w)$, the number of inversions in $w_1 \cdots w_n$. It is also known that conjugacy classes in $\mathfrak{S}_n$ are precisely the set of permutations that have the same cycle type. We name cycle type by integer partitions of n, the weakly decreasing positive integer sequences $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ satisfying $\lambda_1 + \cdots + \lambda_\ell = n$. The $\ell = \ell(\lambda)$ components of λ are called its *parts*, and we let $|\lambda| = n$ and $\lambda \vdash n$ denote that λ is a partition of n. Given $\lambda \vdash n$, we define the *transpose* partition $\lambda^\top = (\lambda_1^\top, \ldots, \lambda_{\lambda_1}^\top)$ by

$$\lambda_i^\top = \#\{j \,|\, \lambda_j \geq i\}.$$

Sometimes it is convenient to name a partition with exponential notation, omitting parentheses and commas, so that $4^2 1^6 := (4,4,1,1,1,1,1,1)$. Sometimes it is convenient to partially order partitions of n by the *majorization* (or *dominance*) *order*,

$$\lambda \preceq \mu \quad \text{iff} \quad \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \tag{2.1}$$

for all i. It is known that if λ is covered by μ in this partial order then λ, μ have equal parts except in two positions $i < j$ where we have

$$\mu_i = \lambda_i + 1, \qquad \mu_j = \lambda_j - 1. \tag{2.2}$$

Let Λ be the ring of symmetric functions in $x = (x_1, x_2, \dots)$ having integer coefficients, and let Λ_n be the $\mathbb{Z}$-submodule of homogeneous functions of degree n. This submodule has rank equal to the number of partitions of n. Five standard bases of Λ_n consist of the monomial $\{m_\lambda \,|\, \lambda \vdash n\}$, elementary $\{e_\lambda \,|\, \lambda \vdash n\}$, (complete) homogeneous $\{h_\lambda \,|\, \lambda \vdash n\}$, power sum $\{p_\lambda \,|\, \lambda \vdash n\}$, and Schur $\{s_\lambda \,|\, \lambda \vdash n\}$ symmetric functions. (See, e.g., [15, Ch. 7] for definitions.) The change-of-basis matrix relating $\{e_\lambda \,|\, \lambda \vdash n\}$ and $\{m_\lambda \,|\, \lambda \vdash n\}$ is given by the equations

$$e_\lambda = \sum_{\mu \preceq \lambda^\top} M_{\lambda,\mu} m_\mu, \tag{2.3}$$

where $M_{\lambda,\mu}$ equals the number of column-strict Young tableaux of shape $\lambda^\top$ and content μ. That is, $M_{\lambda,\mu}$ is the number of histograms having λ_i boxes in column i for all i, filled with μ_1 ones, μ_2 twos, etc., with column entries strictly increasing from bottom to top.

Let $\mathcal{T}_n$ be the $\mathbb{Z}$-module of $\mathbb{Z}[\mathfrak{S}_n]$-*traces* (equivalently, $\mathfrak{S}_n$-class functions), linear functionals $\theta : \mathbb{Z}[\mathfrak{S}_n] \to \mathbb{Z}$ satisfying $\theta(gh) = \theta(hg)$ for all $g, h \in \mathbb{Z}[\mathfrak{S}_n]$. Like the $\mathbb{Z}$-module Λ_n, the trace space $\mathcal{T}_n$ has dimension equal to the number of integer partitions of n. The Frobenius $\mathbb{Z}$-module isomorphism (2.4)

$$\begin{aligned} \mathrm{Frob} : \mathcal{T}_n &\to \Lambda_n \\ \theta &\mapsto \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \theta(w) p_{\mathrm{ctype}(w)} \end{aligned} \tag{2.4}$$

defines bijections between standard bases of Λ, and $\mathcal{T}_n$. Schur functions correspond to irreducible characters, while elementary and homogeneous symmetric functions correspond to induced sign and trivial characters,

$$s_\lambda \leftrightarrow \chi^\lambda, \qquad e_\lambda \leftrightarrow \epsilon^\lambda = \mathrm{sgn}\big\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}, \qquad h_\lambda \leftrightarrow \eta^\lambda = \mathrm{triv}\big\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n},$$

where $\mathfrak{S}_\lambda$ is the Young subgroup of $\mathfrak{S}_n$ indexed by λ. The power sum and monomial bases of Λ_n correspond to bases of $\mathcal{T}_n$ which are not characters. We call these the *power sum* $\{\psi^\lambda \,|\, \lambda \vdash n\}$ and *monomial* $\{\phi^\lambda \,|\, \lambda \vdash n\}$ traces, respectively. These are the bases related to the irreducible character bases by the same matrices of character evaluations and inverse Kostka numbers that relate power sum and monomial symmetric functions to Schur functions,

$$\begin{aligned} p_\lambda &= \sum_\mu \chi^\mu(\lambda) s_\mu, & \psi^\lambda &= \sum_\mu \chi^\mu(\lambda) \chi^\mu, \\ m_\lambda &= \sum_\mu K^{-1}_{\lambda,\mu} s_\mu, & \phi^\lambda &= \sum_\mu K^{-1}_{\lambda,\mu} \chi^\mu, \end{aligned} \tag{2.5}$$

where $\chi^\mu(\lambda) := \chi^\mu(w)$ for any $w \in \mathfrak{S}_n$ having $\mathrm{ctype}(w) = \lambda$.

3 Immanants and totally nonnegative polynomials

Each of the inequalities stated in Section 1 may be stated in terms of a polynomial in matrix entries. In particular, let $x = (x_{i,j})_{i,j\in[n]}$ be a matrix of n^2 indeterminates, and for $p(x) \in \mathbb{C}[x] := \mathbb{C}[x_{i,j}]_{i,j\in[n]}$ and $A = (a_{i,j})$ an $n \times n$ matrix, define $p(A) = p(a_{1,1}, a_{1,2}, \ldots, a_{n,n})$. While few of the polynomial inequalities in Section 1 specifically mention the symmetric group, all of them involve polynomials which are linear combinations of monomials of the form $\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in \mathfrak{S}_n\}$. Following [10], [16], we call such polynomials *immanants*. Specifically, given $f : \mathfrak{S}_n \to \mathbb{C}$ define the *f-immanant* to be the polynomial

$$\mathrm{Imm}_f(x) := \sum_{w\in\mathfrak{S}_n} f(w) x_{1,w_1} \cdots x_{n,w_n} \in \mathbb{C}[x]. \tag{3.1}$$

The sign character ($w \mapsto (-1)^{\ell(w)}$) immanant and trivial character ($w \mapsto 1$) immanant are the determinant and permanent,

$$\det(x) = \sum_{w\in\mathfrak{S}_n} (-1)^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}, \qquad \mathrm{per}(x) = \sum_{w\in\mathfrak{S}_n} x_{1,w_1} \cdots x_{n,w_n}.$$

Simple formulas for the induced sign and trivial character immanants are due to Littlewood–Merris–Watkins [10], [12]: for $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, we have

$$\begin{aligned} \mathrm{Imm}_{\epsilon^\lambda}(x) &= \sum_{(I_1,\ldots,I_r)} \det(x_{I_1,I_1}) \cdots \det(x_{I_r,I_r}), \\ \mathrm{Imm}_{\eta^\lambda}(x) &= \sum_{(I_1,\ldots,I_r)} \mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r}), \end{aligned} \tag{3.2}$$

where the sums are over all ordered set partitions of $[n]$ of type λ (1.4).

Some current interest in immanants and their connection to TNN matrices was inspired by Lusztig's work with canonical bases of quantum groups. (See, e.g., [11].) In particular, one quantum group has an interesting basis whose elements can be described in terms of immanants which evaluate nonnegatively on TNN matrices. Call a polynomial $p(x)$ *totally nonnegative* (TNN) if $p(A) \geq 0$ whenever A is a totally nonnegative matrix. There is no known procedure to decide if a given polynomial is TNN.

The formula (3.2) makes it obvious that $\mathrm{Imm}_{\epsilon^\lambda}(x)$ is a TNN polynomial for each partition $\lambda \vdash n$. A stronger result [17, Cor. 3.3] asserts that irreducible character immanants $\mathrm{Imm}_{\chi^\lambda}(x)$ are TNN as well. It is clear that the $\mathfrak{S}_n$-trace immanants

$$\{\mathrm{Imm}_\theta(x) \,|\, \theta \in \mathcal{T}_n, \mathrm{Imm}_\theta(x) \text{ is TNN }\}$$

form a cone, i.e., are closed under real nonnegative linear combinations. Stembridge has conjectured [18, Conj. 2.1] that the extreme rays of this cone are generated by the monomial trace immanants

$$\{\mathrm{Imm}_{\phi^\lambda}(x) \,|\, \lambda \vdash n\}, \tag{3.3}$$

and has shown [18, Prop. 2.3] that the cone of TNN $\mathfrak{S}_n$-trace immanants lies inside of the cone generated by (3.3).

Proposition 3.1. *Each immanant of the form* $\mathrm{Imm}_\theta(x)$ *with* $\theta \in \mathcal{T}_n$ *is a totally nonnegative polynomial only if it is equal to a nonnegative linear combination of monomial trace immanants.*

Thus it is conjectured that an $\mathfrak{S}_n$-trace immanant is TNN if and only if it is equal to a nonnegative linear combination of monomial trace immanants. Indeed it is known that some monomial trace immanants generate extremal rays of the cone of TNN $\mathfrak{S}_n$-trace immanants [3, Thm.İ10.3]. (See Theorem 4.6.)

4 The Temperley-Lieb algebra and 2-colorings

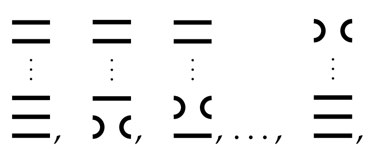
Given a complex number ξ, we define the *Temperley-Lieb algebra* $T_n(\xi)$ to be the $\mathbb{C}$-algebra generated by elements $t_1, \ldots, t_{n-1}$ subject to the relations

$$\begin{aligned} t_i^2 &= \xi t_i, && \text{for } i = 1, \ldots, n-1, \\ t_i t_j t_i &= t_i, && \text{if } |i-j| = 1, \\ t_i t_j &= t_j t_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

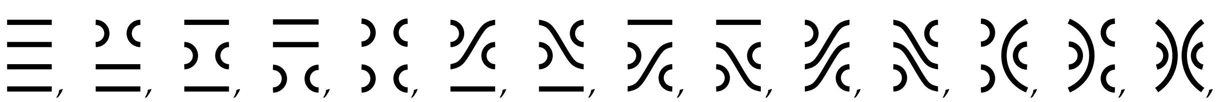
When $\xi = 2$ we have the isomorphism $T_n(2) \cong \mathbb{C}[\mathfrak{S}_n]/(1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1)$. (See e.g. [4], [6, Sec. 2.1, Sec. 2.11], [19, Sec. 7].) Specifically, the isomorphism is given by

$$\begin{aligned} \sigma : \mathbb{C}[\mathfrak{S}_n] &\to T_n(2), \\ s_i &\mapsto t_i - 1. \end{aligned} \tag{4.1}$$

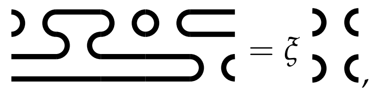
Let $\mathcal{B}_n$ be the multiplicative monoid generated by $t_1, \ldots, t_{n-1}$ when $\xi = 1$, also called the standard basis of $T_n(\xi)$. It is known that $|\mathcal{B}_n|$ is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Diagrams of the basis elements of $T_n(\xi)$, made popular by Kauffman [8, Sec. 4] are (undirected) graphs with $2n$ vertices and n edges. The identity and generators $1, t_1, \ldots, t_{n-1}$ are represented by



and multiplication of these elements corresponds to concatenation of diagrams, with cycles contributing a factor of ξ. For instance, the fourteen basis elements of $T_4(\xi)$ are



and the equality $t_3t_2t_3t_3t_1 = \xi t_1t_3$ in $T_4(\xi)$ is represented by



(4.2)

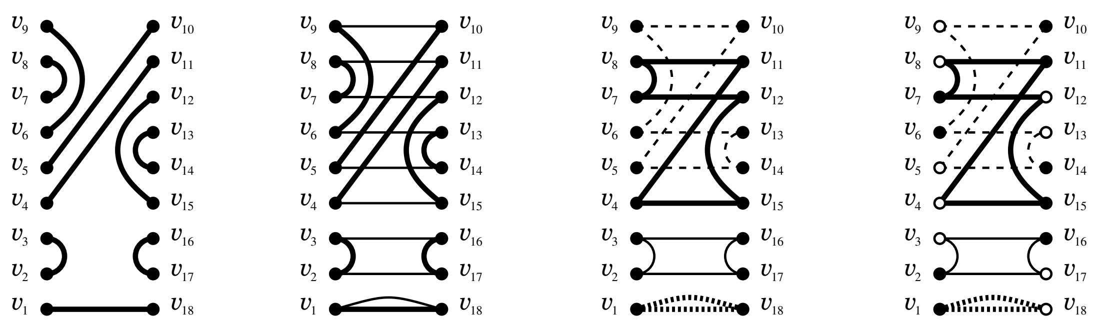
with the "bubble" becoming the scalar multiple ξ. We will identify each element $\tau \in \mathcal{B}_n$ with its Kauffman diagram, and will label vertices $v_1, \ldots, v_{2n}$, clockwise from the lower left. We define the height of a vertex by

$$\mathrm{hgt}(v_i) = \begin{cases} i & \text{if } 1 \leq i \leq n, \\ 2n+1-i & \text{if } n+1 \leq i \leq 2n. \end{cases}$$

For instance, (4.4) shows the element $t_7t_6t_8t_5t_7t_4t_6t_5t_2 \in \mathcal{B}_9$ on the left, with vertex labels. Vertices have heights $1, \ldots, 9$, from bottom to top. It is easy to see that for all $\tau \in \mathcal{B}_n$, each edge (v_i, v_j) satisfies

$$\mathrm{hgt}(v_i) - \mathrm{hgt}(v_j) = \begin{cases} 1 \pmod 2 & \text{if } i,j \leq n \text{ or } i,j \geq n+1, \\ 0 \pmod 2 & \text{otherwise.} \end{cases} \tag{4.3}$$

Define $\hat{\tau}$ to be the graph obtained from τ by adding edges $(i, 2n+1-i)$ for all i (even if such an edge already exists). Since each vertex in $\hat{\tau}$ has degree 2, it is clear that this graph is a disjoint union of cycles. For example, corresponding to the element τ on the left of (4.4) we have $\hat{\tau}$ to its right, the decomposition of this graph into four disjoint cycles, and a proper 2-coloring of this graph.



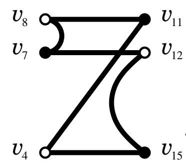
τ $\hat{\tau}$ cycles of $\hat{\tau}$ 2-coloring of $\hat{\tau}$ (4.4)

The Temperley-Lieb algebra $T_n(2)$ sometimes arises in the 2-coloring of combinatorial objects. Define a *principal coloring* of $\tau \in \mathcal{B}_n$ to be a map $\kappa : \mathrm{vertices}(\tau) \rightarrow \{\mathrm{black}, \mathrm{white}\}$ which is a *proper* coloring of $\hat{\tau}$, i.e.,

$$\begin{aligned} &\mathrm{color}(v_i) \neq \mathrm{color}(v_{2n+1-i}) && \text{for } i = 1, \ldots, n, \\ &\mathrm{color}(v_i) \neq \mathrm{color}(v_j) && \text{if } v_i \text{ and } v_j \text{ are adjacent in } \tau. \end{aligned}$$

Let (τ, κ) denote the graph τ with its vertices colored by κ.

Principal colorings of τ are closely related to cycles in $\hat{\tau}$. It is clear that colors must alternate along any one cycle of $\hat{\tau}$. It is also true that vertex colors alternate as one views the vertices in clockwise order, ignoring the edges of that cycle. For example, consider a 2-coloring of the cycle $(v_4, v_{11}, v_8, v_7, v_{12}, v_{15})$ of $\hat{\tau}$ in (4.4),



Proposition 4.1. *If $\hat{\tau}$ is a single cycle, then there are two principal colorings of τ. In each, vertices of odd index and of even index have opposite colors.*

Proof. Omitted. $\square$

Clearly if κ is a principal coloring of $\tau \in \mathcal{B}_n$ and if $\hat{\tau}$ is a single cycle, then we have

$$\left| \# \begin{pmatrix} \text{white vertices on} \\ \text{left of } (\tau, \kappa) \end{pmatrix} - \# \begin{pmatrix} \text{white vertices on} \\ \text{right of } (\tau, \kappa) \end{pmatrix} \right| = \begin{cases} 0 & \text{if } n \text{ even,} \\ 1 & \text{if } n \text{ odd.} \end{cases} \tag{4.5}$$

In this situation we call (τ, κ) *balanced* if n is even, and *unbalanced* otherwise. More specifically, we call (τ, κ) *left-unbalanced* (*right-unbalanced*) if it has more white vertices on the left (right). Now consider $\tau \in \mathcal{B}_n$ with $\hat{\tau}$ a disjoint union of cycles $C_1, \ldots, C_d$, and κ a principal coloring of τ. Define

$$\begin{aligned} \alpha = \alpha(\tau, \kappa) &:= \#\{i \mid (\tau_{C_i}, \kappa) \text{ right unbalanced}\}, \\ \beta = \beta(\tau, \kappa) &:= \#\{i \mid (\tau_{C_i}, \kappa) \text{ left unbalanced}\}. \end{aligned} \tag{4.6}$$

For example, in (4.4), the proper 2-coloring κ of $\hat{\tau}$ corresponds to a principal coloring of τ with $\alpha(\tau, \kappa) = 1$, $\beta(\tau, \kappa) = 2$, and $d = 4$. Also note that there is one balanced cycle.

It is easy to characterize the colorings κ of a given Temperley-Lieb basis element τ for which the numbers α, β (4.6) are constant.

Lemma 4.2. *Let (τ, κ), (τ, κ') be principal colorings with j white vertices on the left, for some j, $0 \le j \le \lfloor \frac{n}{2} \rfloor$. Then we have $\alpha(\tau, \kappa) = \alpha(\tau, \kappa')$ and $\beta(\tau, \kappa) = \beta(\tau, \kappa')$.*

Proof. Omitted. $\square$

By this lemma, we may write

$$\alpha(\tau, j) := \alpha(\tau, \kappa), \qquad (\beta(\tau, j) := \beta(\tau, \kappa)) \tag{4.7}$$

if there exists a principal coloring of τ in which j vertices on the left are white.

Just as $T_n(2)$ is related to 2-coloring, it is related to total nonnegativity of polynomials in the subspace

$$\mathrm{span}_{\mathbb{R}}\{\det(x_{I,I})\det(x_{I^c,I^c}) \mid I \subseteq [n]\} \tag{4.8}$$

of immanants. We define an immanant $\mathrm{Imm}_\tau(x)$ for each $\tau \in \mathcal{B}_n$ in terms of the function

$$\begin{aligned} f_\tau : \mathbb{C}[\mathfrak{S}_n] &\to \mathbb{R} \\ w &\mapsto \text{ coefficient of } \tau \text{ in } \sigma(w), \end{aligned} \tag{4.9}$$

(extended linearly). To economize notation, we will write Imm_τ instead of Imm_{f_τ},

$$\mathrm{Imm}_\tau(x) = \sum_{w \in \mathfrak{S}_n} f_\tau(w) x_{1,w_1} \cdots x_{n,w_n}.$$

For example, consider the case $n = 3$ and $\tau = t_1 \in \mathcal{B}_3$. Extracting the coefficients of t_1 in the expressions

$$\begin{gathered} \sigma(e) = 1, \qquad \sigma(s_1) = t_1 - 1, \qquad \sigma(s_2) = t_2 - 1, \\ \sigma(s_1 s_2) = (t_1 - 1)(t_2 - 1) = t_1 t_2 - t_1 - t_2 + 1, \\ \sigma(s_2 s_1) = (t_2 - 1)(t_1 - 1) = t_2 t_1 - t_1 - t_2 + 1, \\ \sigma(s_1 s_2 s_1) = (t_1 - 1)(t_2 - 1)(t_1 - 1) = t_1 + t_2 - t_1 t_2 - t_2 t_1 - 1, \end{gathered}$$

(where we have used $t_1 t_2 t_1 = t_1$ and $t_1^2 = 2t_1$ to obtain the last expression), we have $f_{t_1}(e) = 0$, $f_{t_1}(s_1) = 1$, $f_{t_1}(s_2) = 0$, $f_{t_1}(s_1 s_2) = -1$, $f_{t_1}(s_2 s_1) = -1$, $f_{t_1}(s_1 s_2 s_1) = 1$, and

$$\mathrm{Imm}_{t_1}(x) = x_{1,2}x_{2,1}x_{3,3} - x_{1,3}x_{2,1}x_{3,2} - x_{1,2}x_{2,3}x_{3,1} + x_{1,3}x_{2,2}x_{3,1}.$$

Note that in the special case $\tau = 1$, the function f_τ maps a permutation w to $(-1)^{\mathrm{INV}(w)}$. Thus the determinant is a Temperley-Lieb immanant,

$$\det(x) = \mathrm{Imm}_1(x).$$

It was shown in [13] that Temperley-Lieb immanants are a basis of the space (4.8), and that they are TNN. Furthermore, they are the extreme rays of the cone of TNN immanants in this space [13, Thm. 10.3].

Proposition 4.3. *Each immanant of the form*

$$\mathrm{Imm}_f(x) = \sum_{\substack{I,J \subseteq [n] \\ |I| = |J|}} c_{I,J} \det(x_{I,J}) \det(x_{I^c,J^c}) \tag{4.10}$$

is a totally nonnegative polynomial if and only if it is equal to a nonnegative linear combination of Temperley-Lieb immanants.

In fact each complementary product of minors is a 0-1 linear combination of Temperley-Lieb immanants [13, Prop. 4.4].

Theorem 4.4. *For $I \subseteq [n]$ we have*

$$\det(x_{I,I}) \det(x_{I^c,I^c}) = \sum_{\tau \in \mathcal{B}_n} b_\tau \mathrm{Imm}_\tau(x), \tag{4.11}$$

where

$$b_\tau = \begin{cases} 1 & \text{if there is a principal coloring of } \tau \text{ with } \{v_i \,|\, i \in I\} \text{ white, } \{v_i \,|\, i \in [n] \smallsetminus I\} \text{ black,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

By (3.2) we have that for each two-part partition $\lambda = (n-j, j)$ of n, the corresponding induced sign character immanant belongs to (4.8). Furthermore, we have the following explicit expansion of these in terms of the Temperley-Lieb immanant basis.

Theorem 4.5. *For $j = 0, \dots, \lfloor \frac{n}{2} \rfloor - 1$, we have*

$$\mathrm{Imm}_{\epsilon^{n-j,j}}(x) = \sum_{\tau \in \mathcal{B}_n} d_{j,\tau} \mathrm{Imm}_\tau(x),$$

where $d_{j,\tau}$ is the number of principal colorings of τ having j white vertices on the left. Explicitly, assuming such a coloring exists, this is $2^{d-\alpha-\beta}\binom{\alpha+\beta}{\alpha}$, where $d =$ the number of cycles of $\hat{\tau}$, and $\alpha = \alpha(\tau, j)$, $\beta = \beta(\tau, j)$ are defined as in (4.7).

Proof. Omitted. □

Combining (2.3) and (3.2), we see that monomial immanants $\mathrm{Imm}_{\phi^\mu}(x)$ indexed by partitions of the form $\mu = 2^c 1^d \vdash n$ belong to the space (4.8) as well. To expand these in the Temperley–Lieb immanant basis, we define for each $\mu = 2^c 1^d \vdash n$ the set $P(\mu)$ of all $\tau \in \mathcal{B}_n$ such that there exists a principal coloring of τ with $c + d$ white vertices on the left and no principal coloring of τ with $c + d + 1$ white vertices on the left.

Theorem 4.6. *For $\mu = 2^c 1^d \vdash n$, we have that $\mathrm{Imm}_{\phi^\mu}(x)$ is a totally nonnegative polynomial. In particular we have*

$$\mathrm{Imm}_{\phi^\mu}(x) = \sum_{\tau \in P(\mu)} b_{\mu,\tau} \mathrm{Imm}_\tau(x),$$

where $b_{\mu,\tau} = 2^{\#\text{cycles of } \hat{\tau} \text{ of cardinality } 0 \pmod 4}$.

Proof. Omitted. □

5 Main results

Fischer's inequalities (1.2) naturally lead to the questions of how products

$$\det(A_{I,I})\det(A_{I^c,I^c}) \tag{5.1}$$

of complementary pairs of minors compare to one another, and of whether a greater cardinality difference $|I^c| - |I|$ tends to make the product (5.1) greater or smaller. This second question led Barrett and Johnson [1] to consider the average value of such products when cardinalities are fixed,

$$\frac{1}{\binom{n}{k}} \sum_{\substack{I \subseteq [n] \\ |I|=k}} \det(A_{I,I})\det(A_{I^c,I^c}). \tag{5.2}$$

They found that for PSD matrices, a smaller cardinality difference makes the average product of minors greater [1, Thm. 1]. We give two proofs that the same is true for TNN matrices.

Theorem 5.1. *For all TNN matrices A and for $k = 0, \dots, \lfloor \frac{n}{2} \rfloor - 1$, we have*

$$\frac{1}{\binom{n}{k}} \sum_{|I|=k} \det(A_{I,I})\det(A_{I^c,I^c}) \leq \frac{1}{\binom{n}{k+1}} \sum_{|I|=k+1} \det(A_{I,I})\det(A_{I^c,I^c}). \tag{5.3}$$

First proof (idea). By (3.2) it is equivalent to show that the polynomial

$$\frac{\mathrm{Imm}_{\epsilon^{n-k-1,k+1}}(x)}{\binom{n}{k+1}} - \frac{\mathrm{Imm}_{\epsilon^{n-k,k}}(x)}{\binom{n}{k}} \tag{5.4}$$

is totally nonnegative. By Theorem 4.5, this difference belongs to the span of the Temperley-Lieb immanants. Multiplying by $n!/(k!(n-k-1)!)$ we obtain

$$(k+1)\mathrm{Imm}_{\epsilon^{n-k-1,k+1}}(x) - (n-k)\mathrm{Imm}_{\epsilon^{n-k,k}}(x) = \sum_{\tau \in \mathcal{B}_n} c_\tau \mathrm{Imm}_\tau(x), \tag{5.5}$$

where $c_\tau = (k+1)d_{k+1,\tau} - (n-k)d_{k,\tau}$, and $d_{k+1,\tau}$, $d_{k,\tau}$ are defined in terms of proper colorings of τ as in Theorem 4.5. Straightforward computations show that $c_\tau \geq 0$. □

Second proof (idea). Again we multiply the polynomial (5.4) by $n!/(k!(n-k-1)!)$. Expanding in the monomial immanant basis of the trace immanant space, we have

$$(n-k)\mathrm{Imm}_{\epsilon^{(k,n-k)}}(x) - (k+1)\mathrm{Imm}_{\epsilon^{(k+1,n-k-1)}}(x) = \sum_{\mu \vdash n} c_\mu \mathrm{Imm}_{\phi^\mu}(x) \tag{5.6}$$

where the integers $\{c_\mu \mid \mu \vdash n\}$ satisfy $(n-k)e_{(k,n-k)} - (k+1)e_{(k+1,n-k-1)} = \sum_{\mu \vdash n} c_\mu m_\mu$. The special case of (2.3) for two-part partitions, implies that each partition μ appearing with nonzero coefficient in (5.6) has the form $\mu = 2^a 1^{n-2a}$. Straightforward computations show that we have $c_\mu \geq 0$, and thus Theorem 4.6 completes the proof. □

Now we may state and prove our main theorem.

Theorem 5.2. *Fix $n > 0$ and partitions $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, $\mu = (\mu_1 \ldots, \mu_s) \vdash n$ with $\lambda \preceq \mu$. For all TNN matrices we have*

$$\lambda_1! \cdots \lambda_r! \sum_{(I_1,\ldots,I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}) \;\geq\; \mu_1! \cdots \mu_s! \sum_{(J_1,\ldots,J_s)} \det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s}), \quad (5.7)$$

Proof. By (2.2) it suffices to consider λ, μ having equal parts except $\mu_i = \lambda_i + 1$ and $\mu_j = \lambda_j - 1$ for some $i < j$. (Thus we may assume $s = r$ and will allow $\mu_s = 0$.) Let $\nu = (\nu_1, \ldots, \nu_{r-2})$ be the partition of $n - \lambda_i - \lambda_j$ consisting of all other parts.

As in the proofs of Theorem 5.1, we observe that each sum of products of minors is an induced sign character immanant (3.2). Thus the inequality (5.7) is equivalent to the total nonnegativity of the polynomial

$$\lambda_1! \cdots \lambda_r! \,\mathrm{Imm}_{\epsilon^\lambda}(x) - \mu_1! \cdots \mu_r! \,\mathrm{Imm}_{\epsilon^\mu}(x). \quad (5.8)$$

Dividing this by $\nu_1! \cdots \nu_{r-2}! \lambda_i! \mu_j!$ and collecting terms, we obtain

$$\begin{aligned}
&\lambda_j \sum_{\substack{J \subseteq [n] \\ |J| = |\nu|}} \mathrm{Imm}_{\epsilon^\nu}(x_{J,J}) \mathrm{Imm}_{\epsilon^{(\lambda_i,\lambda_j)}}(x_{J^c,J^c}) - \mu_i \sum_{\substack{J \subseteq [n] \\ |J| = |\nu|}} \mathrm{Imm}_{\epsilon^\nu}(x_{J,J}) \mathrm{Imm}_{\epsilon^{(\lambda_i+1,\lambda_j-1)}}(x_{J^c,J^c}) \\
&\quad = \sum_{\substack{J \subseteq [n] \\ |J| = |\nu|}} \mathrm{Imm}_{\epsilon^\nu}(x_{J,J}) \left(\lambda_j \mathrm{Imm}_{\epsilon^{(\lambda_i,\lambda_j)}}(x_{J^c,J^c}) - \mu_i \mathrm{Imm}_{\epsilon^{(\lambda_i+1,\lambda_j-1)}}(x_{J^c,J^c}) \right).
\end{aligned}$$

By Theorem 5.1, or more precisely (5.5), this polynomial is TNN and so is (5.8). □

It would be interesting to extend the Barrett–Johnson inequalities (1.3) to all HPSD matrices, and to state a permanental analog as originally suggested in [1]. Using (3.2) we may state these problems as follows.

Problem 5.3. *Characterize the pairs (λ, μ) of partitions of n for which we have*

1. $\lambda_1! \cdots \lambda_r! \,\mathrm{Imm}_{\epsilon^\lambda}(A) \leq \mu_1! \cdots \mu_r! \,\mathrm{Imm}_{\epsilon^\mu}(A)$ *for all A HPSD,*
2. $\lambda_1! \cdots \lambda_r! \,\mathrm{Imm}_{\eta^\lambda}(A) \geq \mu_1! \cdots \mu_r! \,\mathrm{Imm}_{\eta^\mu}(A)$ *for all A HPSD or PSD,*
3. $\lambda_1! \cdots \lambda_r! \,\mathrm{Imm}_{\eta^\lambda}(A) \geq \mu_1! \cdots \mu_r! \,\mathrm{Imm}_{\eta^\mu}(A)$ *for all A TNN.*

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