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Pop, Crackle, Snap (and Pow): Some Facets of Shards

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Abstract. Reading cut the hyperplanes in a real central arrangement $\mathcal{H}$ into pieces called *shards*, which reflect order-theoretic properties of the arrangement. We show that shards have a natural interpretation as certain generators of the fundamental group of the complement of the complexification of $\mathcal{H}$. Taking only positive expressions in these generators yields a new poset that we call the *pure shard monoid*.

When $\mathcal{H}$ is simplicial, its poset of regions is a lattice, so it comes equipped with a pop-stack sorting operator Pop. In this case, we use Pop to define an embedding Crackle of Reading's shard intersection order into the pure shard monoid. When $\mathcal{H}$ is the reflection arrangement of a finite Coxeter group, we also define a poset embedding Snap of the shard intersection order into the positive braid monoid; in this case, our three maps are related by $\mathsf{Snap} = \mathsf{Crackle} \cdot \mathsf{Pop}$.

Résumé. Reading a coupé les hyperplans dans un arrangement central réel $\mathcal{H}$ en morceaux appelés *shards*, qui reflètent les propriétés ordinales de l'arrangement. Nous montrons que les shards ont une interprétation naturelle en tant que certains générateurs du groupe fondamental du complément de la complexification de $\mathcal{H}$. En ne prenant que des expressions positives dans ces générateurs, on obtient un nouvel ensemble ordonné que nous appelons le *monoïde shard pur*.

Lorsque $\mathcal{H}$ est simplicial, son ensemble de régions est un treillis, et il est équipé d'un pop-stack opérateur de tri Pop. Dans ce cas, nous utilisons Pop pour définir une injection Crackle de l'ordre d'intersection des shards de Reading dans le monoïde shard pur. Lorsque $\mathcal{H}$ est l'arrangement de réflexion d'un groupe de Coxeter fini, nous définissons également une injection d'ensemble ordonné Snap de l'ordre d'intersection des shards dans le monoïde de tresses positives; dans ce cas, nos trois fonctions sont reliées par $\mathsf{Snap} = \mathsf{Crackle} \cdot \mathsf{Pop}$.

Keywords: hyperplane arrangement, fundamental group, shard, shard intersection order, Salvetti complex, pure shard monoid

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1 Introduction

Throughout this extended abstract of the article [9], we let $\mathcal{H}$ be a finite central irreducible real hyperplane arrangement in $\mathbb{R}^n$. Salvetti introduced a certain CW complex associated to $\mathcal{H}$ and used it to provide a presentation of the fundamental group of the complement of the complexification of $\mathcal{H}$. We prove that Salvetti's generating set is parameterized by shards, which Reading introduced and used to define his shard intersection order in the case when $\mathcal{H}$ is simplicial. We introduce the *pure shard monoid*, which is the monoid generated by Salvetti's generators; it comes equipped with a natural partial order that we believe deserves further attention. We prove that the interval from the identity element to the *full twist* in the pure shard monoid is self-dual. In the case when $\mathcal{H}$ is an arrangement of rank 2 with m hyperplanes, we prove that this interval is a planar lattice with rank generating function $1 + \left(\sum_{k=1}^{m-1} \left(2\binom{m}{k} - 2\right) q^k\right) + q^m$ and with $m2^{m-2}$ maximal chains.

We then assume $\mathcal{H}$ is simplicial. In this case, there is a known characterization of the shard intersection order involving the pop-stack sorting operator Pop on the poset of regions of $\mathcal{H}$. We introduce a new map Crackle, and we prove that it is a poset embedding of the shard intersection order into the pure shard monoid.

Next, we specialize further to the case when $\mathcal{H}$ is the reflection arrangement of a finite Coxeter group W. We introduce another map Snap, and we prove that it is a poset embedding of the shard intersection order on W into the weak order on the positive braid monoid. The restriction of Snap to the set $\mathrm{Sort}(W, c)$ of c-sortable elements of W originally arose in connection with Deodhar decompositions of noncrossing Catalan varieties. In this setting, we obtain as a corollary that Snap restricts to a poset embedding of the shard intersection order on $\mathrm{Sort}(W, c)$—which is isomorphic to the noncrossing partition lattice of W—into the weak order on the positive braid monoid.

Finally, we turn back to arbitrary finite central irreducible real arrangements and define a fourth map Pow. We prove that Pow is a poset embedding of the poset of regions of $\mathcal{H}$ into the pure shard monoid.

2 The Salvetti Complex

We write $\mathcal{R}$ for the set of regions (connected components of the complement) of $\mathcal{H}$, and we fix a base region $B \in \mathcal{R}$ and a point $x_B \in B$. Let $\mathcal{H}_{\mathbb{C}}$ be the union of the complexifications of the hyperplanes in $\mathcal{H}$. The *complexified hyperplane complement* is $\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}$; write $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}, x_B)$ for its fundamental group with base point x_B. For $C, D \in \mathcal{R}$, we write $D \leq C$ if every hyperplane in $\mathcal{H}$ that separates D from B also separates C from B. The resulting poset $\mathrm{Weak}(\mathcal{H}, B) := (\mathcal{R}, \leq)$ is the *poset of regions* of $\mathcal{H}$ with respect to B.

Following [10, 15], we construct a CW complex $\mathsf{Sal}(\mathcal{H})$ by gluing together oriented dual zonotopes for $\mathcal{H}$ along compatible faces—one zonotope for each choice of base region B, oriented from B to $-B$. The resulting CW complex $\mathsf{Sal}(\mathcal{H})$ has the same fundamental group as the complexified hyperplane complement:

$$\pi_1(\mathsf{Sal}(\mathcal{H}), B) = \pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}, x_B).$$

The 1-skeleton of $\mathsf{Sal}(\mathcal{H})$ is given by orienting all edges of $\mathsf{Weak}(\mathcal{H}, B)$ away from B, and then for each edge e, adding in a reversed edge e^*. An illustration is given in Figure 1.

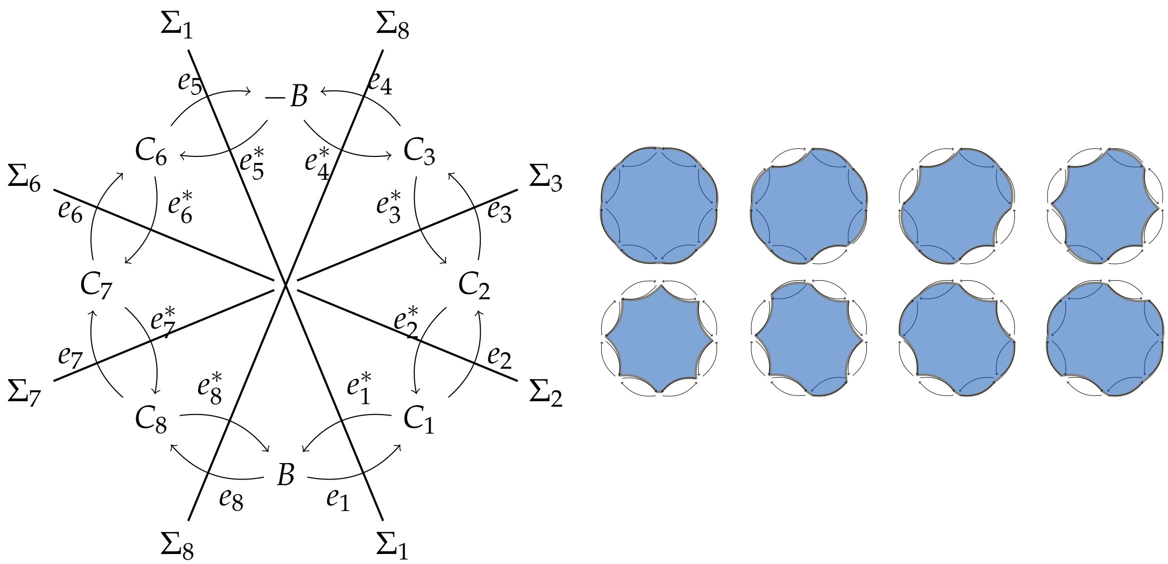

Figure 1: *Left:* the hyperplane arrangement for the dihedral group $I_2(4)$ has eight regions, and its four hyperplanes are cut into six shards. *Right:* the eight 2-cells of $\mathsf{Sal}(\mathcal{H})$, indicated in blue. One 2-cell is attached for each of the eight homotopies $e_1e_2e_3e_4 \cong e_8e_7e_6e_5, e_2e_3e_4e_5^* \cong e_1^*e_8e_7e_6, \ldots, e_8^*e_1e_2e_3 \cong e_7e_6e_5e_4^*$.

Given arbitrary regions $C, C' \in \mathcal{R}$, we define a *gallery* from C' to C to be a sequence of edges (of the form e, e^*, e^{-1}, and $(e^*)^{-1}$) that starts at C' and ends at C. A gallery from C' to C is *positive* if it only uses edges of the form e and e^* (not e^{-1} or $(e^*)^{-1}$). The gallery is *minimal* if its length is equal to the number of hyperplanes separating C' from C, and it is called a *loop* if $C = C'$. Let us fix a positive minimal gallery $\mathsf{gal}(C', C)$ from C' to C in $\mathsf{Sal}(\mathcal{H})$; any two such galleries from C' to C are homotopic. If e is the edge $C' \xrightarrow{e} C$, we define the corresponding loop $\ell_e \in \pi_1(\mathsf{Sal}(\mathcal{H}), B)$ by

$$\ell_e := \mathsf{gal}(B, C') \cdot ee^* \cdot \mathsf{gal}(B, C')^{-1} \in \pi_1(\mathsf{Sal}(\mathcal{H}), B). \tag{2.1}$$

Because of homotopies, this definition of the loop does not depend on the choice of the gallery. Write $\mathcal{L}_{\mathrm{edge}} = \mathcal{L}_{\mathrm{edge}}(\mathcal{H}, B)$ for the set of all such loops ℓ_e. The group $\pi_1(\mathsf{Sal}(\mathcal{H}), B)$ is generated by $\mathcal{L}_{\mathrm{edge}}$.

3 Shards

We recall some constructions and results from [13, 14]. A subarrangement $\mathcal{A}$ of $\mathcal{H}$ is called *full* if it consists of all hyperplanes of $\mathcal{H}$ containing a particular subset of $\mathbb{R}^n$. For a full rank-2 subarrangement $\mathcal{A}$ of $\mathcal{H}$, let $B_{\mathcal{A}}$ be the region of $\mathcal{A}$ containing B. We say a hyperplane $H \in \mathcal{A}$ is *basic* if its intersection with the boundary of $B_{\mathcal{A}}$ has dimension $n-1$. For $H, H' \in \mathcal{A}$, we say H *cuts* H' if H is basic in $\mathcal{A}$ but H' is not. Each hyperplane $H \in \mathcal{H}$ is broken into a number of connected pieces if we remove all points in H contained the the hyperplanes of $\mathcal{H}$ that cut H—a *shard* of H is then the closure of one of these connected pieces. We write H_Σ for the unique hyperplane containing a shard Σ. Let $Ш(\mathcal{H}, B)$ denote the set of shards.

Each cover relation $C' \lessdot C$ in $\mathrm{Weak}(\mathcal{H}, B)$ can be labeled by a shard $\Sigma(C' \lessdot C)$, which is the unique shard separating the region C' from the region C; in this case, we call $\Sigma(C' \lessdot C)$ a *lower shard* of C. Let $\mathrm{cov}_Ш(C)$ be the set of lower shards of C.

Now assume $\mathcal{H}$ is simplicial. Then $\mathrm{Weak}(\mathcal{H}, B)$ is a semidistributive lattice [4], and the set of shards forms an elegant geometric realization of the set of join-irreducible elements. Furthermore, shard intersections encode the canonical join representations of $\mathrm{Weak}(\mathcal{H}, B)$. The map $C \mapsto \bigcap \mathrm{cov}_Ш(C)$ defines a bijection from $\mathcal{R}$ to the set of arbitrary intersections of shards. In [14], Reading introduced another poset $\mathrm{Shard}(\mathcal{H}, B) := (\mathcal{R}, \preceq)$ called the *shard intersection order*, which is defined by

$$C \preceq D \quad \text{if and only if} \quad \bigcap \mathrm{cov}_Ш(C) \supseteq \bigcap \mathrm{cov}_Ш(D).$$

As with $\mathrm{Weak}(\mathcal{H}, B)$, the poset $\mathrm{Shard}(\mathcal{H}, B)$ is a lattice—but while $\mathrm{Weak}(\mathcal{H}, B)$ is "tall and slender" (with height equal to the number of hyperplanes and with the number of atoms equal to the dimension), $\mathrm{Shard}(\mathcal{H}, B)$ is "short and wide" (with height equal to dimension and with the number of atoms equal to the number of shards). When $\mathcal{H}$ is the reflection arrangement of a finite Coxeter group W, the relationship between noncrossing partitions and sortable elements allowed Reading to embed the W-noncrossing partition lattice into the shard intersection order, thereby giving a uniform proof that the noncrossing partition lattice is indeed a lattice. An example is illustrated in Figure 2.

4 Shards and the Salvetti Complex

Whenever we have an edge $C' \xrightarrow{e} C$ in $\mathrm{Sal}(\mathcal{H})$, we will write $\Sigma(e)$ for the associated shard $\Sigma(C' \lessdot C)$. Although the generators in $\mathcal{L}_{\mathrm{edge}}$ are a priori indexed by cover relations in $\mathrm{Weak}(\mathcal{H}, B)$, our first theorem says that they are really indexed by the much smaller set of shards.

Theorem 1 ([9]). *Let $\mathcal{H}$ be a central real hyperplane arrangement. Given edges $C' \xrightarrow{e} C$ and $D' \xrightarrow{f} D$ in $\mathrm{Sal}(\mathcal{H})$, the loops ℓ_e and ℓ_f are homotopic if and only if $\Sigma(e) = \Sigma(f)$.*

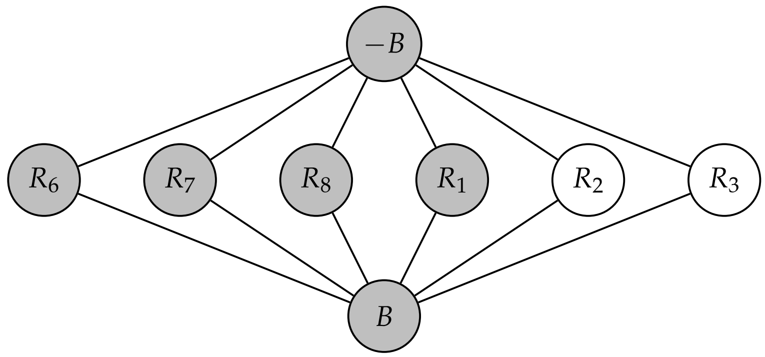

Figure 2: The poset Shard$(\mathcal{H}, B)$ for the arrangement of Figure 1, which is the reflection arrangement of the dihedral group $I_2(4)$. Gray indicates elements in the image of Reading's embedding of the $I_2(4)$-noncrossing partition lattice (with respect to a certain Coxeter element); see [9, Section 7.5].

By Theorem 1, it makes sense write $\mathcal{L}_{\mathrm{III}} = \mathcal{L}_{\mathrm{edge}}$, indexing the loops in $\mathcal{L}_{\mathrm{edge}}$ by shards. Thus, for any shard Σ, we define the *shard loop* $\ell_\Sigma = \ell_e$, where e is any edge such that $\Sigma(e) = \Sigma$.

Let us define the *pure shard monoid*, denoted $\mathbf{P}^+(\mathcal{H}, B)$, to be the submonoid of $\pi_1(\mathsf{Sal}(\mathcal{H}), B)$ generated by $\mathcal{L}_{\mathrm{III}}$. That is, an element of $\pi_1(\mathsf{Sal}(\mathcal{H}), B)$ is in $\mathbf{P}^+(\mathcal{H}, B)$ if and only if it can be represented by a word in the alphabet $\mathcal{L}_{\mathrm{III}}$. This allows us to endow $\mathbf{P}^+(\mathcal{H}, B)$ with a partial order $\leq$ by declaring that $p \leq p'$ if there is a word over $\mathcal{L}_{\mathrm{III}}$ representing p' that contains a prefix representing p.

The *full twist* is the element Δ^2 of $\pi_1(\mathsf{Sal}(\mathcal{H}), B)$ defined by

$$\Delta^2 := \mathrm{gal}(B, -B) \cdot \mathrm{gal}(-B, B).$$

When $\mathcal{H}$ is a finite irreducible simplicial arrangement, it is known that the center of $\pi_1(\mathsf{Sal}(\mathcal{H}), B)$ is an infinite cyclic group generated by Δ^2 [6]. We are especially interested in $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$, the interval between the identity element $\mathbb{1}$ and the full twist Δ^2 in $\mathbf{P}^+(\mathcal{H}, B)$. This interval is "tall and wide," but it is *not* a lattice in general, preventing the use of Garside theory to study $\pi_1(\mathsf{Sal}(\mathcal{H}), B)$. Figure 3 shows this interval when $\mathcal{H}$ is the arrangement from Figure 1.

The combinatorics of the pure shard monoid can be quite involved. However, we can prove the following general result (which is stated as a conjecture in [9]).

Theorem 2. *If $\mathcal{H}$ is a central arrangement, then the interval $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$ is self-dual.*

For a rank-2 arrangement with m hyperplanes, we have the following precise theorem, which tells us that there are exactly $m2^{m-2}$ words over the alphabet $\mathcal{L}_{\mathrm{III}}$ representing Δ^2.

Theorem 3 ([9]). *Let $\mathcal{H}$ be hyperplane arrangement of rank 2 with m hyperplanes. The interval $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$ is a planar lattice with rank generating function $1 + \left(\sum_{k=1}^{m-1} \left(2\binom{m}{k} - 2\right) q^k\right) + q^m$ and with $m2^{m-2}$ maximal chains.*

5 Pop

In the next three sections, we discuss three incarnations of the shard intersection order given by three maps Pop, Crackle, and Snap. The first one, defined using the map Pop, is not new, but it inspired our terminology for the other two. For these three sections, we assume that $\mathcal{H}$ is simplicial so that $\mathrm{Weak}(\mathcal{H}, B)$ is a semidistributive lattice.

Let L be a locally finite meet-semilattice with meet operation denoted by $\wedge$. Motivated by work on *pop-stacks* from enumerative combinatorics and theoretical computer science [1, 5, 17], the first author defined the *pop-stack sorting operator* $\mathsf{Pop}\colon L \to L$ in [7] (see also [8]) by

$$\mathsf{Pop}(x) := x \wedge \bigwedge\{y \in L : y \lessdot x\}. \tag{5.1}$$

In the case when $L = \mathrm{Weak}(\mathcal{H}, B)$, we can use Pop to characterize $\mathrm{Shard}(\mathcal{H}, B)$. For $C \in \mathcal{R}$, let $\Sigma([\mathsf{Pop}(C), C])$ be the set of shards that label the cover relations in the interval $[\mathsf{Pop}(C), C]$; that is,

$$\Sigma([\mathsf{Pop}(C), C]) := \left\{\Sigma(D' \lessdot D) : \mathsf{Pop}(C) \leq D' \lessdot D \leq C\right\}.$$

By [14, Proposition 5.7], we have

$$C' \preceq C \quad \text{if and only if} \quad \Sigma([\mathsf{Pop}(C'), C']) \subseteq \Sigma([\mathsf{Pop}(C), C]). \tag{5.2}$$

6 Crackle

In this section, we continue to assume $\mathcal{H}$ is simplicial. Let us define the *crackle map* $\mathsf{Crackle}\colon \mathcal{R} \to \pi_1(\mathrm{Sal}(\mathcal{H}), B)$ by

$$\mathsf{Crackle}(C) := \mathrm{gal}(B, \mathsf{Pop}(C)) \cdot \mathrm{gal}(\mathsf{Pop}(C), C) \cdot \mathrm{gal}(C, \mathsf{Pop}(C)) \cdot \mathrm{gal}(B, \mathsf{Pop}(C))^{-1}. \tag{6.1}$$

This map generalizes the shard loops ℓ_Σ of Equation (2.1) and Theorem 1: if J is a join-irreducible region of $\mathrm{Weak}(\mathcal{H}, B)$, then $\mathsf{Pop}(J)$ is the unique region covered by J, so $\mathsf{Crackle}(J) = \ell_{\Sigma(\mathsf{Pop}(J) \lessdot J)}$.

Just as Equation (5.2) characterized $\mathrm{Shard}(\mathcal{H}, B)$ using Pop, we can give a characterization of $\mathrm{Shard}(\mathcal{H}, B)$ using Crackle. Recall that if P and Q are posets, then a map $\psi\colon P \to Q$ is called a *poset embedding* if it is a poset isomorphism from P to its image $\psi(P) \subseteq Q$.

Theorem 4. *The map* Crackle *is a poset embedding from* $\mathrm{Shard}(\mathcal{H}, B)$ *into the interval* $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$.

Theorem 4 is illustrated in Figure 3.

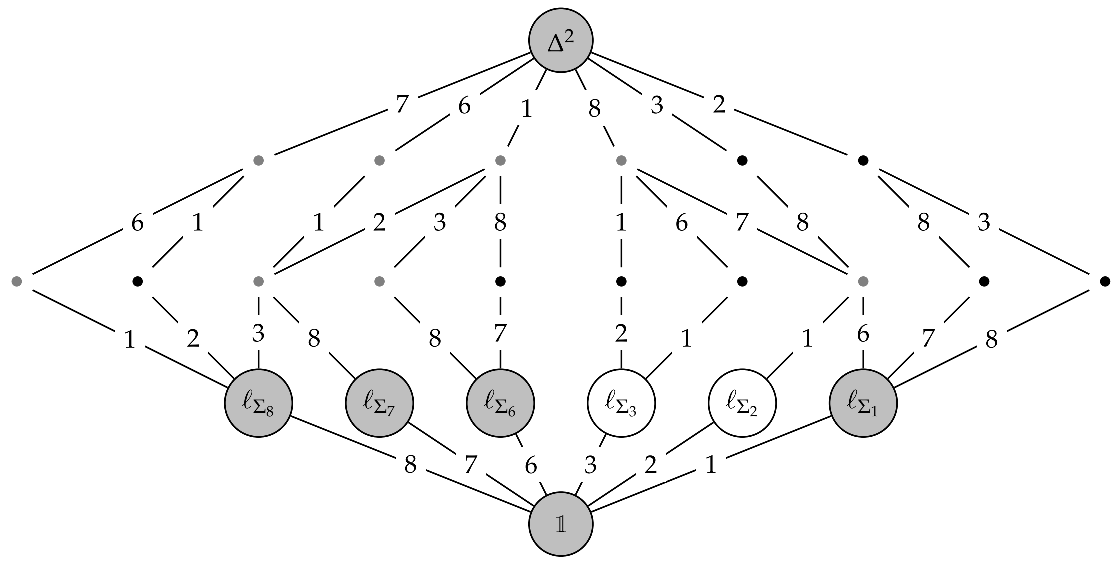

Figure 3: The interval $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$ in the pure shard monoid $\mathbf{P}^+(\mathcal{H}, B)$ between the identity element and the full twist Δ^2, where $(\mathcal{H}, B)$ is as in Figure 1. An edge $p \lessdot p'$ is labeled i when $p' = p \cdot \ell_{\Sigma_i}$. Circled elements are in the image of Crackle. An element is colored gray if it appears as a prefix of a word for Δ^2 using only generators corresponding to noncrossing shards; see [9, Section 7.5].

7 Snap

We now specialize to the case when $\mathcal{H}$ is the reflection arrangement of a finite Coxeter group W. We identify the base region B with the identity element of W; the free transitive action of W on $\mathcal{R}$ then allows us to identity regions of $\mathcal{H}$ with elements of W. We write $\mathrm{Shard}(W) := \mathrm{Shard}(\mathcal{H}, B)$, $\mathrm{Weak}(W) := \mathrm{Weak}(\mathcal{H}, B)$, $\mathbf{P}^+(W) = \mathbf{P}^+(\mathcal{H}, B)$, etc.

In this setting, the group $\mathbf{P}(W) := \pi_1(\mathbb{C}^n \setminus \mathcal{H}_\mathbb{C}, x_B)$ is called the *pure braid group* of W, while the group $\mathbf{B}(W) := \pi_1((\mathbb{C}^n \setminus \mathcal{H}_\mathbb{C})/W, x_B)$ is called the *braid group* of W. The Coxeter group W fits into the following well-known exact sequence with its braid and pure braid groups:

$$1 \to \mathbf{P}(W) \to \mathbf{B}(W) \xrightarrow{\varphi} W \to 1.$$

Let $\mathbf{S}$ be the set of simple generators of $\mathbf{B}(W)$ obtained by lifting the set S of simple reflections of W. The generators in $\mathbf{S}$ satisfy the same braid relations as the corresponding simple reflections of W; the difference is that W also includes the relations stating that the simple reflections are involutions. Thus, the projection $\varphi \colon \mathbf{B}(W) \to W$ is the quotient map that sends each generator $\mathbf{s} \in \mathbf{S}$ to the corresponding $s \in S$ and imposes these additional relations. The submonoid of $\mathbf{B}(W)$ generated by $\mathbf{S}$ is called the *positive braid monoid* of W and is denoted by $\mathbf{B}^+(W)$. The weak order $(W, \leq)$ is defined by saying

$u \leq v$ if and only if any reduced word for u appears as a prefix of some reduced word for v. Analogously, the weak order $\mathrm{Weak}(\mathbf{B}^+(W)) := (\mathbf{B}^+(W), \leq)$ is defined by saying $\mathbf{u} \leq \mathbf{v}$ if and only if any word over $\mathbf{S}$ representing $\mathbf{u}$ appears as a prefix of some word over $\mathbf{S}$ representing $\mathbf{v}$. In this setting, the full twist Δ^2 is equal to the lift $\mathbf{w}_\circ^2$ of the long element $w_\circ$ of W; we will also write $\Delta = \mathbf{w}_\circ$.

We can rephrase Theorem 1 when $\mathcal{H}$ is the reflection arrangement of W as follows.

Corollary 1. *Suppose $u, v \in W$ and $s, t \in S$ are such that $u \lessdot us$ and $v \lessdot vt$. Let $\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}$ be the lifts of u, v, s, t, respectively, to $\mathbf{B}^+(W)$. We have $\mathbf{usu}^{-1} = \mathbf{vtv}^{-1}$ if and only if $\Sigma(w \lessdot ws) = \Sigma(u \lessdot ut)$.*

For $u, v \in W$, we let $u^v = v^{-1}uv$. Write $\mathbf{c} = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ for an ordering of the elements of $\mathbf{S}$. For $\mathbf{s} \in \mathbf{S}$ and a positive braid $\mathbf{w} = \mathbf{s}_{i_1} \cdots \mathbf{s}_{i_k} \in \mathbf{B}^+(W)$ with projection $w = \varphi(\mathbf{w}) \in W$, let $\mathbf{s}^{\mathbf{w}} = (t, j)$, where $t = s^w$ and j counts the number of times t appears in the sequence $s^{s_{i_k}}, s^{s_{i_k} s_{i_{k-1}}}, \ldots, s^{s_{i_k} s_{i_{k-1}} \cdots s_1}$.

In [11], a new set of noncrossing W-Catalan objects was introduced as the set of subwords of $\mathbf{c}^{h+1}$ that represent the full twist $\Delta^2 = \mathbf{w}_\circ^2$ and satisfy an additional *Deodhar condition*. When interpreted in the positive braid monoid $\mathbf{B}^+(W)$, this Deodhar condition is equivalent to restricting to those c-sortable elements $\mathbf{w}$ in the interval $[\mathbb{1}, \mathbf{w}_\circ^2]_{\mathbf{B}^+}$ with the property that for each descent $\mathbf{s}$ of $\mathbf{w}$, we have $\mathbf{s}^{\mathbf{w}} = (t, j)$ with j even—this is a nonstandard *Deodhar embedding* of the c-sortable elements into the interval $[\mathbb{1}, \mathbf{w}_\circ^2]_{\mathbf{B}^+}$ (different from the usual lift of W into $\mathbf{B}^+(W)$). The second author speculated that restricting the 2nd c-Fuss–Cambrian lattice to the image of this Deodhar embedding would recover the noncrossing partition lattice $\mathrm{NC}(W, c)$. As the 2nd c-Fuss–Cambrian lattice is a subposet of $\mathrm{Weak}(\mathbf{B}^+(W))$, it makes sense to generalize this Deodhar embedding of c-sortable elements to all elements of W.

The *snap map* $\mathsf{Snap}\colon W \to \mathbf{B}^+(W)$ is our generalization of the Deodhar embedding. We write Pop for the pop-stack sorting operators on the lattice $\mathrm{Weak}(W)$ and the meet-semilattice $\mathrm{Weak}(\mathbf{B}^+(W))$, relying on the argument of the operator to indicate the context. For $w \in W$, let $\mathrm{des}(w)$ denote the right descent set of w, let $w_\circ(\mathrm{des}(w))$ be the longest element of the parabolic subgroup of W generated by $\mathrm{des}(w)$, and write $\mathbf{w}$ and $\mathbf{w}_\circ(\mathrm{des}(w))$ for the usual lifts of w and $w_\circ(\mathrm{des}(w))$ to $\mathbf{B}^+(W)$. Define

$$\mathsf{Snap}(w) := \mathsf{Pop}(\mathbf{w}) \cdot (\mathbf{w}_\circ(\mathrm{des}(w)))^2. \tag{7.1}$$

Since $\mathsf{Crackle}(w) = \mathsf{Pop}(\mathbf{w}) \cdot (\mathbf{w}_\circ(\mathrm{des}(w)))^2 \cdot \mathsf{Pop}(\mathbf{w})^{-1}$ in $\mathbf{P}^+(W) \subseteq \mathbf{B}(W)$, it follows that

$$\mathsf{Snap}(w) = \mathsf{Crackle}(w)\mathsf{Pop}(\mathbf{w}).$$

Just as the shard intersection order was characterized via Pop in Equation (5.2) and via $\mathsf{Crackle}$ in Theorem 4, it is also characterized via Snap.

Theorem 5. *The map Snap is a poset embedding from $\mathsf{Shard}(W)$ into $[\mathbb{1}, \Delta^2]_{\mathbf{B}^+}$.*

Theorem 5 is illustrated in Figure 4.

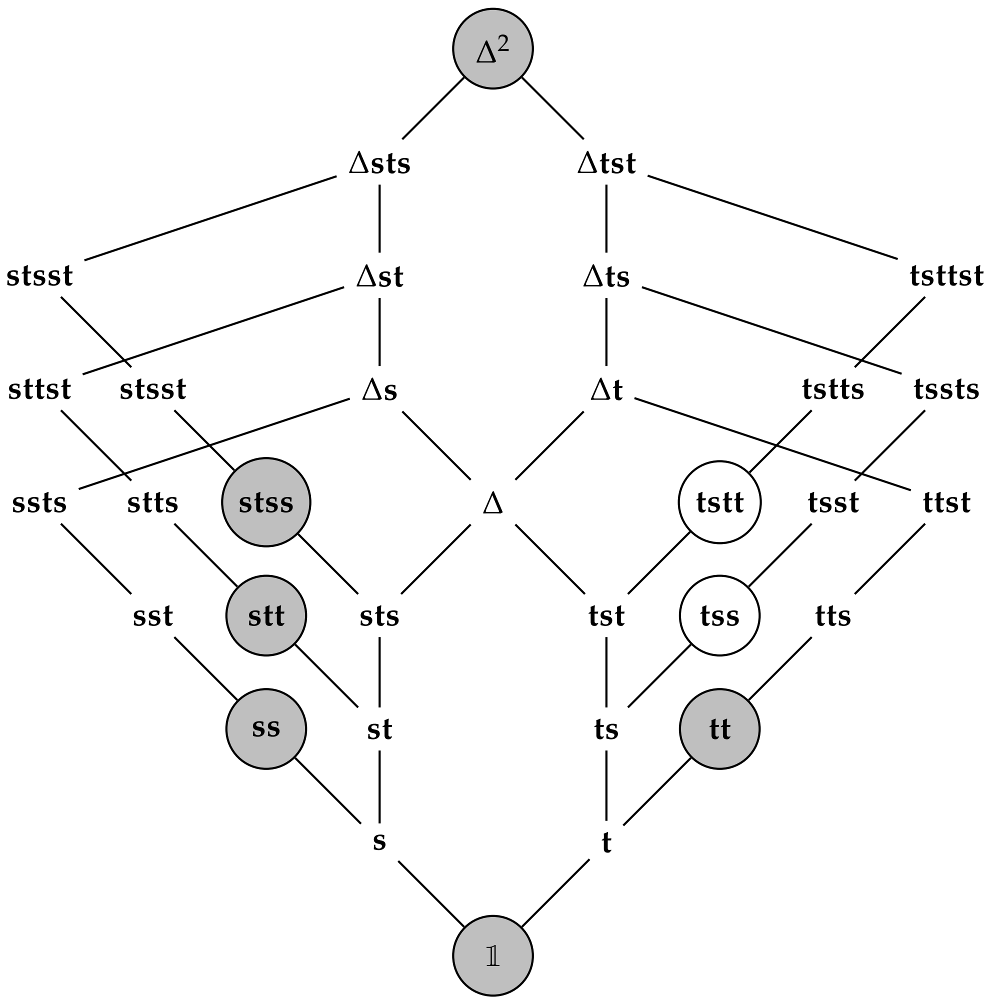

Figure 4: The interval $[\mathbb{1}, \Delta^2]_{\mathbf{B}^+}$ in $\mathrm{Weak}(\mathbf{B}^+(I_2(4)))$, where $I_2(4)$ is the dihedral group of order 8 with simple reflections s and t. The reflection arrangement of $I_2(4)$ is shown in Figure 1. Circled elements are in the image of Snap. Gray indicates that the element is the image of a c-sortable element under Snap, where $c = st$; see [9, Section 7.5].

8 Pow

In her history "The Untold Tale of Pow!, the Fourth Rice Krispies Elf: A look into the era when the cereal mascots were more than just Snap!, Crackle! and Pop!" [16], Smith writes:

> "Lost in the shuffle, however, was a fourth Rice Krispies elf named Pow!
> His short life is a time-capsule of an era when everyone was dreaming big."

Inspired by this fourth mascot, we introduce a map $\mathsf{Pow} : \mathcal{R} \to \mathbf{P}^+(\mathcal{H}, B)$ in the general setting when $\mathcal{H}$ is a central irreducible real hyperplane arrangement. Given a region $C \in \mathcal{R}$ and a positive minimal gallery

$$B = C_0 \xrightarrow{e_1} C_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} C_{k-1} \xrightarrow{e_k} C_k = C,$$

we let

$$\mathsf{Pow}(C) := \ell_{\Sigma(e_k)} \ell_{\Sigma(e_{k-1})} \cdots \ell_{\Sigma(e_1)}.$$

(We prove in [9] that Pow is well-defined.) Just as Crackle embeds the "short and wide" poset $\mathrm{Shard}(\mathcal{H}, B)$ into the "tall and wide" interval $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$, the map Pow embeds the "tall and slender" poset $\mathrm{Weak}(\mathcal{H}, B)$ into $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$.

Theorem 6. *The map* Pow *is a poset embedding from* $\mathrm{Weak}(\mathcal{H}, B)$ *into* $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$.

9 Future Work

9.1 Noncrossing Pure Braid Presentations

In future work, we will combine Theorem 1 with Salvetti's presentation of $\pi_1(\mathrm{Sal}(\mathcal{H}), B)$, Coxeter–Catalan combinatorics, Cambrian lattices, and noncrossing shards to write explicit presentations of the pure braid groups of finite Coxeter groups. In the special case of the symmetric group and the Tamari lattice, our method will recover Artin's original presentation of the pure braid group [2, 3].

9.2 The Pure Shard Monoid

The pure shard monoid is an interesting algebraic and order-theoretic structure that deserves further study—in particular, we would like to better understand the elements and the maximal chains in the interval $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$. These maximal chains correspond to $\mathcal{L}_{\mathrm{III}}$-words representing the full twist Δ^2. When $\mathcal{H}$ is an arrangement of rank 2, we characterized these words in [9, Proposition 4.7]. It would already be interesting to better understand $[\mathbb{1}, \Delta^2]_{\mathbf{P}^+}$ for special cases, such as rank-3 arrangements or reflection arrangements of type-A Coxeter groups.

9.3 Infinite Arrangements

We have assumed throughout that $\mathcal{H}$ is finite. It is natural to ask what aspects of the above theory generalize to arrangements with infinitely many hyperplanes.

9.4 Bubbles, Blossom, Buttercup (and Bliss)

We view the bubble sort operator Bubbles as the 0-Hecke action of any reduced word for the long element $w_\circ$ in the symmetric group. The *higher Bruhat order* is a partial order defined on these reduced words [12]. We wonder if there are similar "higher Bruhat orders" built from the $\mathcal{L}_{\mathrm{III}}$-words for Δ^2. One might expect to find relevant maps Blossom and Buttercup in this theory. We recommend this subsection's title as the logical name for this proposed work.

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