Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #31, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

Refined canonical stable Grothendieck polynomials and their duals

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Abstract. In this extended abstract we introduce refined canonical stable Grothendieck polynomials and their duals with two infinite sequences of parameters. These polynomials unify several generalizations of Grothendieck polynomials including canonical stable Grothendieck polynomials due to Yeliussizov, refined Grothendieck polynomials due to Chan and Pflueger, and refined dual Grothendieck polynomials due to Galashin, Liu, and Grinberg. We give Jacobi–Trudi-like formulas, combinatorial models, Schur expansions, Schur positivity, and dualities of these polynomials. We also consider flagged versions of Grothendieck polynomials and their duals with skew shapes.

Keywords: Grothendieck polynomials, symmetric functions, Jacobi–Trudi formulas, Schur positivity

1 Introduction

1.1 History of Grothendieck polynomials

Grothendieck polynomials were introduced by Lascoux and Schützenberger [8] for studying the Grothendieck ring of vector bundles on a flag variety. When focusing on the Grothendieck ring of vector bundles on Grassmannians rather than that of flag varieties, the stable Grothendieck polynomials are indexed by partitions. These stable Grothendieck polynomials form a basis for the (connective) K-theory ring of Grassmannians, see [12, Section 2.3] and references therein. In this context, Lenart [9] gave the Schur expansion of $G_\lambda(x)$ and showed that $\{G_\lambda(x) : \lambda \text{ is a partition}\}$ is a basis for (a completion of) the space of symmetric functions. Buch [1] showed that the stable

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Grothendieck polynomial $G_\lambda(\boldsymbol{x})$ is equal to a generating function for semistandard set-valued tableaux of shape λ.

After Buch's work [1], many generalizations of $G_\lambda(\boldsymbol{x})$ have been studied from various viewpoints. In [7], Lam and Pylyavskyy introduced the dual stable Grothendieck polynomials $g_\lambda(\boldsymbol{x})$, and found a combinatorial interpretation for them in terms of reverse plane partitions. Similar to the Grothendieck polynomial case, the dual Grothendieck polynomials form a basis for the space of symmetric functions.

Yeliussizov [15] introduced the canonical stable Grothendieck polynomials $G_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$ with two parameters α and β, and defined the dual canonical stable Grothendieck polynomials $g_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$ by the relation $\langle G_\mu^{(-\alpha,-\beta)}(\boldsymbol{x}), g_\lambda^{(\alpha,\beta)}(\boldsymbol{x})\rangle = \delta_{\lambda,\mu}$, where $\langle -,- \rangle$ is the Hall inner product. Yeliussizov [15] showed that these polynomials behave as nicely as Schur functions under the involution ω. Moreover, he found combinatorial interpretations for $G_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$ and $g_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$.

There are also generalizations of Grothendieck polynomials and their duals with infinite parameters; see [2, 3, 6, 13]. In this extended abstract, motivated by these previous works, we introduce refined canonical stable Grothendieck polynomials $G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta})$ and their duals $g_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta})$ with infinite parameters $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\dots)$ and $\boldsymbol{\beta} = (\beta_1,\beta_2,\dots)$, and furthermore their flagged versions with skew shapes. Our generalization unifies all generalizations of Grothendieck polynomials mentioned above.

1.2 Refined canonical Grothendieck polynomials and their duals

Let $\boldsymbol{x} = (x_1, x_2, \dots)$, $\boldsymbol{x}_n = (x_1,\dots,x_n)$ be sequences of variables, and $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\dots)$, $\boldsymbol{\beta} = (\beta_1,\beta_2,\dots)$ sequences of parameters.

We generalize the canonical stable Grothendieck polynomials as follows. See Section 2 for precise definitions of the notations.

Definition 1.1. For a partition λ with at most n parts, the *refined canonical stable Grothendieck polynomial* $G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta})$ is defined by

$$G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{\det\left(x_j^{\lambda_i+n-i} \dfrac{(1-\beta_1 x_j)\cdots(1-\beta_{i-1}x_j)}{(1-\alpha_1 x_j)\cdots(1-\alpha_{\lambda_i}x_j)} \right)_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)}. \tag{1.1}$$

Using the notation $\ominus$ we can rewrite (1.1) as follows:

$$G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{\det\left(h_{\lambda_i+n-i}[x_j \ominus (A_{\lambda_i} - B_{i-1})]\right)_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)}, \tag{1.2}$$

where $A_k = \alpha_1 + \cdots + \alpha_k$ and $B_k = \beta_1 + \cdots + \beta_k$ for $k \ge 1$, and $A_k = B_k = 0$ for $k \le 0$.

We define our generalization of dual Grothendieck polynomials via the following bialternant formula.

Definition 1.2. For a partition λ with at most n parts, the *refined dual canonical stable Grothendieck polynomial* $g_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta})$ is defined by

$$g_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{\det(h_{\lambda_i+n-i}[x_j - A_{\lambda_i-1} + B_{i-1}])_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i - x_j)}. \tag{1.3}$$

Our generalizations $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ generalize several well-studied variations of Grothendieck polynomials as follows. Here $\mathbf{0} = (0,0,\dots)$, $\mathbf{1} = (1,1,\dots)$, $\boldsymbol{\alpha}_0 = (\alpha,\alpha,\dots)$ and $\boldsymbol{\beta}_0 = (\beta,\beta,\dots)$.

Variations of G or g	introduced in	how to specialize
$G_{\nu/\lambda}(\boldsymbol{x})$	Buch [1]	$G_{\nu/\lambda}(\boldsymbol{x};\mathbf{0},\mathbf{1})$
$g_{\lambda/\mu}(\boldsymbol{x})$	Lam–Pylyavskyy [7]	$g_{\lambda/\mu}(\boldsymbol{x};\mathbf{0},\mathbf{1})$
$G_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$	Yeliussizov [15]	$G_\lambda(\boldsymbol{x};\boldsymbol{\alpha}_0,-\boldsymbol{\beta}_0)$
$g_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$	Yeliussizov [15]	$g_\lambda(\boldsymbol{x};-\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0)$
$RG_\sigma(\boldsymbol{x};\boldsymbol{\beta})$	Chan–Pflueger [2]	$G_\sigma(\boldsymbol{x};\mathbf{0},\boldsymbol{\beta})$
$\tilde{g}_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\beta})$	Galashin–Grinberg–Liu [3]	$g_{\lambda/\mu}(\boldsymbol{x};\mathbf{0},\boldsymbol{\beta})$
$G_{\lambda/\mu,f/g}(\boldsymbol{x})$	Matsumura [11]	$G_{\lambda/\mu}^{\mathrm{row}(g,f)}(\boldsymbol{x};\mathbf{0},-\boldsymbol{\beta}_0)$
$\tilde{g}_{\lambda/\mu}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\beta})$	Grinberg [4], Kim [6]	$g_{\lambda/\mu}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\mathbf{0},\boldsymbol{\beta})$

The remainder of the extended abstract is organized as follows. In Section 2, we briefly recall necessary definitions. In Section 3, we give Schur expansions and Jacobi–Trudi-like formulas for $G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta})$ and prove their duality. In Section 4, we give tableau models for $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and obtain Schur positivity of $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},-\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};-\boldsymbol{\alpha},\boldsymbol{\beta})$. In Sections 5 and 6, we extend refined canonical stable Grothendieck polynomials and their duals to skew shapes with certain flag conditions. In Section 7, we give skew Schur expansions for $G_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$, and they behave nicely under the involution ω. The full version of this extended abstract is [5], where the missing proofs can be found.

2 Preliminaries

In this section, we set up notations and give the necessary background. Throughout this extended abstract, we denote by $\mathbb{Z}$ (resp. $\mathbb{Z}_+$) the set of integers (resp. positive integers).

2.1 Partitions and tableaux

A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1,\dots,\lambda_\ell)$ of positive integers. Each λ_i is called a *part*, and the *length* $\ell(\lambda) = \ell$ of λ is the number of parts in λ. We denote by Par_n the set of partitions with at most n parts, and by Par the set of all partitions.

We will identify a partition λ with its Young diagram. Each element $(i,j) \in \lambda$ is called a *cell* of λ. The Young diagram of λ is visualized by placing a square (i,j) for each cell (i,j) in λ using the matrix coordinates. The *transpose* λ' of λ is the partition whose Young diagram is obtained from that of λ by reflecting across the diagonal. The *content* $c(i,j)$ of a cell $(i,j) \in \lambda$ is $j-i$.

For two partitions $\lambda, \mu \in \mathrm{Par}$ with $\mu \subseteq \lambda$, the *skew shape* λ/μ is defined to be $\lambda - \mu = \{(i,j) : 1 \le i \le \ell(\lambda), \mu_i < j \le \lambda_i\}$. We denote by $|\lambda/\mu|$ the number of cells in λ/μ.

A *semistandard Young tableau* of shape λ/μ is a filling T of the cells in λ/μ with positive integers such that the integers are weakly increasing in each row and strictly increasing in each column. Let $\mathrm{SSYT}(\lambda/\mu)$ be the set of semistandard Young tableaux of shape λ/μ. A *reverse plane partition* of shape λ/μ is a filling T of the cells in λ/μ with positive integers such that the integers are weakly increasing in each row and in each column. Let $\mathrm{RPP}(\lambda/\mu)$ be the set of reverse plane partitions of shape λ/μ.

2.2 Symmetric functions

Let Λ be the $\mathbb{Q}$-algebra of all symmetric functions in $\mathbb{Q}[[x_1, x_2, \dots]]$. The *completion* $\hat{\Lambda}$ of Λ is the set of symmetric functions with possibly unbounded degree. For $n \ge 1$, the *complete homogeneous symmetric function* $h_n(\boldsymbol{x})$ and the *elementary symmetric function* $e_n(\boldsymbol{x})$ are defined to be

$$h_n(\boldsymbol{x}) = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}, \quad \text{and} \quad e_n(\boldsymbol{x}) = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}.$$

Define an endomorphism $\omega : \Lambda \longrightarrow \Lambda$ of the $\mathbb{Q}$-algebra Λ by $\omega(h_n(\boldsymbol{x})) = e_n(\boldsymbol{x})$.

An important basis is the Schur function basis. For a partition $\lambda \in \mathrm{Par}_n$, the *Schur polynomial* $s_\lambda(\boldsymbol{x}_n)$ of shape λ is defined by

$$s_\lambda(\boldsymbol{x}_n) = \frac{\det(x_j^{\lambda_i+n-i})_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i - x_j)}.$$

The *Schur function* $s_\lambda(\boldsymbol{x})$ is defined to be the coefficient-wise limit of $s_\lambda(\boldsymbol{x}_n)$ as $n \to \infty$. The Schur function $s_{\lambda/\mu}(\boldsymbol{x})$ of shape λ/μ is defined by $s_{\lambda/\mu}(\boldsymbol{x}) = \sum_{T\in\mathrm{SSYT}(\lambda/\mu)} \boldsymbol{x}^T$, where $\boldsymbol{x}^T = x_1^{m_1} x_2^{m_2} \cdots$ and m_i is the number of appearances of i in T.

The *Hall inner product* is the inner product on Λ defined by $\langle s_\lambda(\boldsymbol{x}), s_\mu(\boldsymbol{x})\rangle = \delta_{\lambda,\mu}$. This inner product is naturally extended to $\hat{\Lambda} \times \Lambda$ because every element $f \in \hat{\Lambda}$ can be written as $f = \sum_{\lambda\in\mathrm{Par}} c_\lambda s_\lambda(\boldsymbol{x})$.

The Jacobi–Trudi formula and its dual give expansions of Schur functions in terms of $h_n(\boldsymbol{x})$ and $e_n(\boldsymbol{x})$ as determinants: for partitions $\mu \subseteq \lambda$,

$$s_{\lambda/\mu}(\boldsymbol{x}) = \det\left(h_{\lambda_i-\mu_j-i+j}(\boldsymbol{x})\right)_{i,j=1}^{\ell(\lambda)} \quad \text{and} \quad s_{\lambda'/\mu'}(\boldsymbol{x}) = \det\left(e_{\lambda_i-\mu_j-i+j}(\boldsymbol{x})\right)_{i,j=1}^{\ell(\lambda)}.$$

In this extended abstract, we consider the completion of $\Lambda_{\mathbb{Q}[\boldsymbol{\alpha},\boldsymbol{\beta}]} = \mathbb{Q}[\boldsymbol{\alpha},\boldsymbol{\beta}] \otimes \Lambda$ instead of Λ.

2.3 Plethystic substitutions

We define plethystic substitution via the *power sum symmetric functions* $p_n(\boldsymbol{x})$. For a formal power series $Z \in \mathbb{Q}[[z_1, z_2, \dots]]$, the *plethystic substitution* $p_n[Z]$ *of* Z *into* $p_n(\boldsymbol{x})$ is defined to be the formal power series obtained from Z by replacing each z_i by z_i^n. By definition, $f[z_1 + z_2 + \cdots] = f(z_1, z_2, \dots)$. For more properties of plethystic substitution, we refer the reader to [10].

For integers r, s, and n, we define

$$X = x_1 + x_2 + \cdots, \qquad X_n = x_1 + x_2 + \cdots + x_n, \qquad X_{[r,s]} = x_r + x_{r+1} + \cdots + x_s,$$

$$A_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \qquad B_n = \beta_1 + \beta_2 + \cdots + \beta_n,$$

where $X_n = A_n = B_n = 0$ if $n \le 0$ and $X_{[r,s]} = 0$ if $r > s$.

In this extended abstract, we only consider the plethystic substitution into $h_n(\boldsymbol{x})$ and $e_n(\boldsymbol{x})$. We will use the following properties, which follows from [10, Theorem 6].

Proposition 2.1. *For* $n \ge 0$ *and* $Z \in \mathbb{Q}[[z_1, z_2, \dots]]$, *we have*

$$h_n[-Z] = (-1)^n e_n[Z] \quad \text{and} \quad e_n[-Z] = (-1)^n h_n[Z].$$

For $Z_1, Z_2 \in \mathbb{Q}[[z_1, z_2, \dots]]$,

$$h_n[Z_1 + Z_2] = \sum_{a+b=n} h_a[Z_1] h_b[Z_2] \quad \text{and} \quad e_n[Z_1 + Z_2] = \sum_{a+b=n} e_a[Z_1] e_b[Z_2].$$

We introduce a novel notation $\ominus$ as follows.

Definition 2.2. For $n \in \mathbb{Z}$ and $Z_1, Z_2 \in \mathbb{Q}[[z_1, z_2, \dots]]$ with no constant terms,

$$h_n[Z_1 \ominus Z_2] = \sum_{a-b=n} h_a[Z_1] h_b[Z_2] \quad \text{and} \quad e_n[Z_1 \ominus Z_2] = \sum_{a-b=n} e_a[Z_1] e_b[Z_2].$$

3 Schur expansions, Jacobi–Trudi-like formulas, and duality

In this section, we give Schur expansions and Jacobi–Trudi-like formulas for $G_\lambda(\boldsymbol{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ using the Cauchy–Binet theorem. We also give the duality between $G_\lambda(\boldsymbol{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ with respect to the Hall inner product.

For $\lambda, \mu \in \mathrm{Par}_n$, define

$$C_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\mu_i-\lambda_j-i+j}[A_{\lambda_j} - B_{j-1}]\right)_{i,j=1}^{n},$$
$$c_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\lambda_i-\mu_j-i+j}[-A_{\lambda_i-1} + B_{i-1}]\right)_{i,j=1}^{n}.$$

We expand $G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta})$ into Schur functions as follows.

Theorem 3.1. *Let $\lambda \in \mathrm{Par}_n$. We have*

$$G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\mu\supseteq\lambda} C_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) s_\mu(\boldsymbol{x}_n),$$
$$g_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\mu\subseteq\lambda} c_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) s_\mu(\boldsymbol{x}_n).$$

We can apply the Cauchy–Binet theorem again to the Schur expansions in Theorem 3.1 to obtain Jacobi–Trudi-like formulas.

Theorem 3.2. *For $\lambda \in \mathrm{Par}_n$, we have*

$$G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\lambda_i-i+j}[X_n \ominus (A_{\lambda_i} - B_{i-1})]\right)_{i,j=1}^{n},$$
$$g_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\lambda_i-i+j}[X_n - A_{\lambda_i-1} + B_{i-1}]\right)_{i,j=1}^{n}.$$

Using the Schur expansions of $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$, we show the following duality with respect to the Hall inner product, which justifies the name "refined dual canonical stable Grothendieck polynomial" for $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$.

Theorem 3.3. *For $\lambda, \mu \in \mathrm{Par}$, we have*

$$\langle G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}), g_\mu(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})\rangle = \delta_{\lambda,\mu}.$$

4 Combinatorial models and Schur positivity

In this section we give combinatorial models for the Schur coefficients $C_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $c_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta})$ of $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$. As a consequence we show that $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},-\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};-\boldsymbol{\alpha},\boldsymbol{\beta})$ are Schur-positive. In order to state our results we need the following tableaux, which are modified versions of elegant tableaux [7, 9].

Definition 4.1. A *$\mathbb{Z}$-elegant tableau* of shape λ/μ is a filling T of the cells in λ/μ with integers such that the rows are weakly increasing, the columns are strictly increasing, and

$$\min(i-\mu_i, 1) \le T(i,j) < i \qquad \text{for all } (i,j) \in \lambda/\mu.$$

Let $\mathrm{ET}_{\mathbb{Z}}(\lambda/\mu)$ denote the set of all $\mathbb{Z}$-elegant tableaux of shape λ/μ.

Definition 4.2. A *$\mathbb{Z}$-inelegant tableau* of shape μ/λ is a filling T of the cells in μ/λ with integers such that the rows are weakly decreasing, the columns are strictly decreasing, and

$$\min(\mu_i - i, 0) < T(i,j) \le \lambda_i \qquad \text{for all } (i,j) \in \mu/\lambda.$$

Let $\mathrm{IET}_{\mathbb{Z}}(\mu/\lambda)$ denote the set of all $\mathbb{Z}$-inelegant tableaux of shape μ/λ.

We now give combinatorial interpretations for the Schur coefficients $C_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $c_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta})$ of $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ in terms of tableaux.

Theorem 4.3. *Let $\lambda,\mu \in \mathrm{Par}$ with $\lambda \subseteq \mu$. We have*

$$C_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in \mathrm{IET}_{\mathbb{Z}}(\mu/\lambda)} \prod_{(i,j)\in\mu/\lambda} (\alpha_{T(i,j)} - \beta_{T(i,j)-c(i,j)}),$$

$$c_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in \mathrm{ET}_{\mathbb{Z}}(\lambda/\mu)} \prod_{(i,j)\in\lambda/\mu} (-\alpha_{T(i,j)+c(i,j)} + \beta_{T(i,j)}),$$

where $\alpha_m = \beta_m = 0$ for $m \le 0$. In particular, $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},-\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};-\boldsymbol{\alpha},\boldsymbol{\beta})$ are Schur-positive, that is, $C_{\lambda,\mu}(\boldsymbol{\alpha},-\boldsymbol{\beta})$ and $c_{\lambda,\mu}(-\boldsymbol{\alpha},\boldsymbol{\beta})$ are polynomials in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with nonnegative integer coefficients.

For the proof of Theorem 4.3, we express $C_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $c_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta})$ in terms of nonintersecting lattice paths using the Lindström–Gessel–Viennot lemma. The nonintersecting lattice paths are then easily transformed into the desired tableaux.

5 Flagged Grothendieck polynomials

In this section we give a combinatorial model for $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ using marked multiset-valued tableaux. More generally, we consider two flagged versions of $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and extend the partition λ to a skew shape. To begin with, we introduce the following definition.

Definition 5.1. A *marked multiset-valued tableau* of shape λ/μ is a filling T of λ/μ with multisets such that

- $T(i,j)$ is a nonempty (finite) multiset $\{a_1 \le \cdots \le a_k\}$ of positive integers, in which each integer a_i may be marked if $i \ge 2$ and $a_{i-1} < a_i$, and
- $\max(T(i,j)) \le \min(T(i,j+1))$ and $\max(T(i,j)) < \min(T(i+1,j))$ if $(i,j),(i,j+1) \in \lambda/\mu$ and $(i,j),(i+1,j) \in \lambda/\mu$, respectively.

Let $\mathrm{MMSVT}(\lambda/\mu)$ denote the set of marked multiset-valued tableaux of shape λ/μ.

	$1,2^*,2$	$2,2,4^*$
1	$3,3$	

Figure 1: An example of $T \in \mathrm{MMSVT}((3,2)/(1))$, where the marked integers are indicated with $*$. The weight of T is given by $\mathrm{wt}(T) = x_1^2 x_2^4 x_3^2 x_4 \alpha_2^2 \alpha_3 (-\beta_1)^2$.

For $T \in \mathrm{MMSVT}(\lambda/\mu)$, let $\boldsymbol{x}^{T(i,j)} = x_1^{m_1} x_2^{m_2} \cdots$, where m_k is the total number of appearances of k and k^* in $T(i,j)$, and let $\mathrm{unmarked}(T(i,j))$ (resp. $\mathrm{marked}(T(i,j))$) denote the number of unmarked (resp. marked) integers in $T(i,j)$. We define

$$\mathrm{wt}(T) = \prod_{(i,j)\in\lambda/\mu} \boldsymbol{x}^{T(i,j)} \alpha_j^{\mathrm{unmarked}(T(i,j))-1} (-\beta_i)^{\mathrm{marked}(T(i,j))}.$$

See Figure 1.

The main goal of this section is to prove the following combinatorial interpretation for $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$.

Theorem 5.2. *For a partition λ, we have*

$$G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in\mathrm{MMSVT}(\lambda)} \mathrm{wt}(T).$$

Our strategy for proving Theorem 5.2 is to introduce a flagged version which allows us to use induction. We note that a row-flagged version with partition shape is sufficient to prove this theorem, but for completeness we also consider a column-flagged version and we generalize the partition shape to any skew shape.

Definition 5.3. Let $\boldsymbol{r} = (r_1, r_2, \dots)$, $\boldsymbol{s} = (s_1, s_2, \dots)$ be in $\mathbb{Z}_+^n$. Let $\mathrm{MMSVT}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\lambda/\mu)$ denote the set of $T \in \mathrm{MMSVT}(\lambda/\mu)$ such that $r_i \le \min(T(i,j))$ and $\max(T(i,j)) \le s_i$ for all $(i,j) \in \lambda/\mu$. Similarly, $\mathrm{MMSVT}^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}(\lambda/\mu)$ denotes the set of $T \in \mathrm{MMSVT}(\lambda/\mu)$ such that $r_j \le \min(T(i,j))$ and $\max(T(i,j)) \le s_j$ for all $(i,j) \in \lambda/\mu$. We define the *row-flagged* and *column-flagged refined canonical stable Grothendieck polynomials* by

$$G_{\lambda/\mu}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in\mathrm{MMSVT}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\lambda/\mu)} \mathrm{wt}(T),$$

$$G_{\lambda/\mu}^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in\mathrm{MMSVT}^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}(\lambda/\mu)} \mathrm{wt}(T).$$

Now we give Jacobi–Trudi-like formulas for $G^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $G^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$.

Theorem 5.4. *Let $\lambda,\mu \in \mathrm{Par}_n$, $\boldsymbol{r},\boldsymbol{s} \in \mathbb{Z}^n_+$ with $\mu \subseteq \lambda$. If $r_i \le r_{i+1}$ and $s_i \le s_{i+1}$ whenever $\mu_i < \lambda_{i+1}$ for $1 \le i \le n-1$, then*

$$G^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = C \det\left(h_{\lambda_i-\mu_j-i+j}[X_{[r_j,s_i]} \ominus (A_{\lambda_i} - A_{\mu_j} - B_{i-1} + B_j)]\right)_{i,j=1}^{n}, \tag{5.1}$$

where $C = \prod_{i=1}^{n}\prod_{l=r_i}^{s_i}(1-\beta_i x_l)$.

We prove Theorem 5.4 by showing that the both sides of (5.1) satisfy the same recurrence relations and initial conditions. The idea is based on a proof of the Jacobi–Trudi-like formula for flagged Schur functions due to Wachs [14]. We prove the following theorem in a similar way.

Theorem 5.5. *Let $\lambda,\mu \in \mathrm{Par}_n$, $\boldsymbol{r},\boldsymbol{s} \in \mathbb{Z}^n_+$ with $\mu \subseteq \lambda$. If $r_i - \mu_i \le r_{i+1} - \mu_{i+1}$ and $s_i - \lambda_i \le s_{i+1} - \lambda_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$ for $1 \le i \le n-1$, then*

$$G^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}_{\lambda'/\mu'}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = D \det\left(e_{\lambda_i-\mu_j-i+j}[X_{[r_j,s_i]} \ominus (A_{i-1} - A_j - B_{\lambda_i} + B_{\mu_j})]\right)_{i,j=1}^{n},$$

where $D = \prod_{i=1}^{n}\prod_{l=r_i}^{s_i}(1-\alpha_i x_l)^{-1}$.

6 Flagged dual Grothendieck polynomials

In this section we give a combinatorial model for the refined dual canonical stable Grothendieck polynomials $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ using marked reverse plane partitions. To this end we introduce a flagged version of $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ using marked reverse plane partitions and prove a Jacobi–Trudi-like formula for this flagged version. More generally, we consider two flagged versions of $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and extend the partition λ to a skew shape.

Definition 6.1. A *left-marked reverse plane partition* (or simply *marked reverse plane partition*) of shape λ/μ is a reverse plane partition T of shape λ/μ in which every entry $T(i,j)$ with $T(i,j) = T(i,j+1)$ may be marked. We denote by $\mathrm{MRPP}(\lambda/\mu)$ the set of marked reverse plane partitions of shape λ/μ. For $T \in \mathrm{MRPP}(\lambda/\mu)$, define

$$\mathrm{wt}(T) = \prod_{(i,j)\in\lambda/\mu} \mathrm{wt}(T(i,j)),$$

where

$$\mathrm{wt}(T(i,j)) = \begin{cases} -\alpha_j & \text{if } T(i,j) \text{ is marked,} \\ \beta_{i-1} & \text{if } T(i,j) \text{ is not marked and } T(i,j) = T(i-1,j), \\ x_{T(i,j)} & \text{if } T(i,j) \text{ is not marked and } T(i,j) \neq T(i-1,j). \end{cases}$$

Here, the equality $T(i,j) = T(i-1,j)$ means that their underlying integers are equal. See Figure 2.

		1	2	4^*	4
	1^*	1	3	5	
	1	1			
3^*	3^*	3			

Figure 2: An example of $T \in \mathrm{MRPP}(\lambda/\mu)$, where $\lambda = (6,5,3,3)$ and $\mu = (2,1,1)$. We have the weight $\mathrm{wt}(T) = x_1x_2x_3^2x_4x_5(-\alpha_1)(-\alpha_2)^2(-\alpha_5)\beta_1\beta_2^2$.

The main goal of this section is to prove the following combinatorial interpretation for $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$.

Theorem 6.2. *We have*

$$g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in\mathrm{MRPP}(\lambda)} \mathrm{wt}(T). \tag{6.1}$$

Similar to our approach in the previous section, in order to prove Theorem 6.2 we introduce a flagged version. For completeness we again consider row- and column-flagged versions on skew shapes.

Definition 6.3. Let $\boldsymbol{r} = (r_1, r_2, \dots) \in \mathbb{Z}_+^n$ and $\boldsymbol{s} = (s_1, s_2, \dots) \in \mathbb{Z}_+^n$. We denote by $\mathrm{MRPP}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\lambda/\mu)$ (resp. $\mathrm{MRPP}^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}(\lambda/\mu)$) the set of $T \in \mathrm{MRPP}(\lambda/\mu)$ such that $r_i \le T(i,j) \le s_i$ (resp. $r_j \le T(i,j) \le s_j$) for all $(i,j) \in \lambda/\mu$. We define *row-flagged* and *column-flagged refined dual canonical stable Grothendieck polynomials* by

$$g_{\lambda/\mu}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in\mathrm{MRPP}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\lambda/\mu)} \mathrm{wt}(T),$$

$$g_{\lambda/\mu}^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in\mathrm{MRPP}^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}(\lambda/\mu)} \mathrm{wt}(T).$$

Now we give Jacobi–Trudi-like formulas for the two flagged versions of $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$.

Theorem 6.4. *Let $\lambda, \mu \in \mathrm{Par}_n, \boldsymbol{r}, \boldsymbol{s} \in \mathbb{Z}_+^n$ such that $r_i \le r_{i+1}$ and $s_i \le s_{i+1}$ whenever $\mu_i < \lambda_{i+1}$. Then we have*

$$g_{\lambda/\mu}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\lambda_i-\mu_j-i+j}[X_{[r_j,s_i]} - A_{\lambda_i-1} + A_{\mu_j} + B_{i-1} - B_{j-1}]\right)_{i,j=1}^{n},$$

$$g_{\lambda'/\mu'}^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(e_{\lambda_i-\mu_j-i+j}[X_{[r_j,s_i]} - A_{i-1} + A_{j-1} + B_{\lambda_i-1} - B_{\mu_j}]\right)_{i,j=1}^{n}.$$

The proof is based on the ideas in [6].

7 Schur expansions and the omega involution for skew shapes

When we set $r_i = 1$ and $s_i = \infty$ for each i in Theorems 5.4, 5.5 and 6.4, we have the refined canonical stable Grothendieck polynomial $G_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ of skew shape λ/μ, and its dual $g_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$.

We define a *generalized partition* of length n to be a sequence $(\lambda_1,\dots,\lambda_n) \in \mathbb{Z}^n$ satisfying $\lambda_1 \geq \cdots \geq \lambda_n$. Denote by GPar_n the set of generalized partitions of length n. We expand $G_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ in terms of skew Schur functions (up to some factor) using the Cauchy–Binet Theorem.

Theorem 7.1. *For $\lambda,\mu \in \mathrm{Par}_n$ with $\mu \subseteq \lambda$, we have*

$$G_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = C \sum_{\substack{\rho,\nu\in\mathrm{GPar}_n\\ \rho\subseteq\mu\subseteq\lambda\subseteq\nu}} C_{\lambda,\nu}(\boldsymbol{\alpha},\boldsymbol{\beta}) s_{\nu/\rho}(\boldsymbol{x}) C'_{\rho,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}), \tag{7.1}$$

where $C = \prod_{i=1}^{n}\prod_{l=1}^{\infty}(1-\beta_i x_l)$ and

$$C_{\lambda,\nu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\nu_i-\lambda_j-i+j}[A_{\lambda_j} - B_{j-1}]\right)_{i,j=1}^{n},$$
$$C'_{\rho,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\mu_i-\rho_j-i+j}[-A_{\mu_i} + B_i]\right)_{i,j=1}^{n}.$$

Theorem 7.2. *For $\lambda,\mu \in \mathrm{Par}_n$, we have*

$$g_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\substack{\rho,\nu\in\mathrm{Par}_n\\ \mu\subseteq\rho\subseteq\nu\subseteq\lambda}} c_{\lambda,\nu}(\boldsymbol{\alpha},\boldsymbol{\beta}) s_{\nu/\rho}(\boldsymbol{x}) c'_{\rho,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}),$$

where

$$c_{\lambda,\nu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\lambda_i-\nu_j-i+j}[-A_{\lambda_i-1} + B_{i-1}]\right)_{i,j=1}^{n},$$
$$c'_{\rho,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\rho_i-\mu_j-i+j}[A_{\mu_j} - B_{j-1}]\right)_{i,j=1}^{n}.$$

Finally, we obtain the effect of the involution ω on $G_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$.

Theorem 7.3. *For $\lambda,\mu \in \mathrm{Par}$ with $\mu \subseteq \lambda$, we have*

$$\omega(G_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})) = G_{\lambda'/\mu'}(\boldsymbol{x};-\boldsymbol{\beta},-\boldsymbol{\alpha}),$$
$$\omega(g_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})) = g_{\lambda'/\mu'}(\boldsymbol{x};-\boldsymbol{\beta},-\boldsymbol{\alpha}).$$

Acknowledgments

The authors are grateful to Travis Scrimshaw for many helpful comments and fruitful discussions. They also thank Darij Grinberg for useful comments.

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