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Quasisymmetric harmonics of the exterior algebra

Nantel Bergeron[*1], Kelvin Chan[†1], Farhad Soltani[‡1], and Mike Zabrocki[§1]

[1] *Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3 Canada*

Abstract. We study *fermionic* quasisymmetric polynomials in the polynomial ring R_n with n anticommuting variables. The main results of this paper are that the quasisymmetric polynomials in R_n form a commutative sub-algebra of R_n, there is a basis of the quotient of R_n by the ideal I_n generated by the quasisymmetric polynomials in R_n that is indexed by ballot sequences, and there is a basis of the ideal generated by quasisymmetric polynomials that is indexed by sequences that break the ballot condition.

Résumé. Nous étudions les polynômes quasisymétriques fermioniques de l'anneau des polynômes R_n à n variables anti-commutatives. Les principaux résultats de cet article sont que les polynômes quasi-symétriques dans R_n forment une sous-algèbre commutative de R_n; qu'il existe une base du quotient de R_n par l'idéal I_n engendré par les polynômes quasi symétriques dans R_n qui est indexée par des séquences de scrutin; et qu'il existe une base de l'idéal généré par les polynômes quasi symétriques qui est indexée par des séquences qui brisent la condition de scrutin.

Keywords: Quasisymmetric Polynomials, Fermionic Variables, Exterior Algebra, Ballot Sequences, Polynomial Harmonics

1 Introduction

The study of coinvariants of groups dates back to Shephard-Todd and Chevalley [26, 5] and has fruitfully produced many connections between algebra, combinatorics and physics. Motivated by recent developments in coinvariants of symmetric groups and symmetric functions theory incorporating anticommuting (also known as fermionic) variables, we study a coinvariant-like quotient of an exterior algebra obtained by the quotient of the ideal generated by quasisymmetric functions in anticommuting variables. The quotient has a dimension that can be interpreted as the number of ballot sequences (or other interpretations, see for instance the OEIS [25] sequences A008315 and A001405).

*bergeron@yorku.ca.
†ktychan@yorku.ca.
‡farhad.soltani91@gmail.com.
§zabrocki@yorku.ca.

Many coinvariant quotients feature rich combinatorial connections. One well-known example is the coinvariant ring of the symmetric group whose dimension is $n!$. It is the quotient of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ in commuting variables by the ideal generated by the symmetric polynomials with no constant term. As an $\mathcal{S}_n$ representation, this quotient is naturally graded and is well-known to be isomorphic to the regular representation. Its graded Hilbert series is the q-factorial. Many useful bases of this space have been found by studying combinatorics related to permutations. For more details, see the nice surveys of [3, 13, 23, 24].

This line of inquiry inspired Garsia and Haiman [12, 17] to considered the ring of diagonal harmonics whose dimension is $(n+1)^{n-1}$, the number of parking functions. It is a similar quotient in two sets of commuting variables as an $\mathcal{S}_n$ module. Haiman's work [18] showed that the diagonal harmonics have a deep connection to the theory of Macdonald polynomials. A combinatorial expression for the Frobenius image of the diagonal harmonics known as the Shuffle Conjecture [15] showed that the module structure is closely related to the combinatorics of parking functions and can be described in terms of certain labelled Catalan paths. This connection relating the symmetric functions and the combinatorial expression was proven in [4] and is now known as the Shuffle Theorem.

The connection between the combinatorics and the symmetric function expressions of the Shuffle Theorem have been generalized [16] and proven [8] to an expression known as the Delta Conjecture. The last author with the group at the Fields Institute [29] proposed a deformation of diagonal harmonics to two sets of commuting variables and one set of anticommuting variables. In this case, the connection of representation theoretic interpretation to the symmetric function expression remains open. The symmetric function expressions and representation theoretic interpretation was extended further to include the quotient of two sets of commuting and two sets of anticommuting variables in [7] to what is known as the Theta Conjecture. At present, this also remains an open conjecture, but progress has been made on some special cases [20, 21, 27, 28].

The ring of quasisymmetric polynomials $QSym_n$ contains the ring of symmetric polynomials Sym_n. Many combinatorial structures of $QSym_n$ parallel that of Sym_n. The Termperley-Lieb algebra TL_n can be obtained as a quotient of the symmetric group algebra. Hivert described a TL_n action on $\mathbb{Q}[x_1, \ldots, x_n]$ making $QSym_n$ an isotypic trivial representation of TL_n [19]. In 2003, Aval, F. Bergeron, and the first author studied $QSym$ coinvariant spaces obtained by replacing the ideal of non-constant symmetric functions with the ideal of non-constant quasisymmetric functions [1, 2]. Surprisingly they found that dimensions of $QSym$ coinvariants are equal to the Catalan numbers. At the heart of their argument is a recursion built from Catalan paths. Li extended this argument to study some components of coinvariant spaces of diagonally quasisymmetric functions [22].

Desrosiers, Lapointe, and Mathieu [10, 9] introduced symmetric functions with one set of commuting and one set of anticommuting variables known as symmetric functions in superspace. The aforementioned proposal [29] of a representation theoretic interpretation of the Delta Conjecture can be thought of as a generalization of the classic coinvariant quotients to superspace. Then a natural direction for exploration is to expand the study of "super" symmetric coinvariants to "super" quasisymmetric coinvariants.

Our study of quasisymmetric coinvariant spaces in one set of anticommuting variables is a first step in that study. We denote polynomials in a set of n anticommuting variables by R_n. The main results of this paper show the following interesting facts about symmetric and quasisymmetric functions in anticommuting variables:

1. The quasisymmetric polynomials in R_n form a commutative sub-algebra of R_n (Proposition 2.1).
2. That R_n is free over the ring of symmetric polynomials (Proposition 2.2).
3. There is a basis of the quotient of R_n by the ideal I_n generated by the quasisymmetric polynomials in R_n that is indexed by ballot sequences (Proposition 2.5). The Hilbert series of the quotient is given by

$$\mathrm{Hilb}_{R_n/I_n}(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} f^{(n-k,k)} q^k \,, \tag{1.1}$$

   where $f^{(n-k,k)}$ is the number of standard tableaux of shape $(n-k,k)$ (Corollary 4.2).
4. There is a basis of the ideal generated by quasisymmetric polynomials that is indexed by sequences that break the ballot condition (Theorem 4.1) and a minimal Gröbner basis that is a subset of this basis (Corollary 4.4).

2 Quasisymmetric invariants on the exterior algebra

Fix n a positive integer and let $R_n = \mathbb{Q}[\theta_1, \theta_2, \ldots, \theta_n]$ be the polynomial ring in anticommuting variables. The ring R_n is isomorphic to the exterior algebra of a vector space of dimension n. The variables of this ring satisfy the relations

$$\theta_i\theta_j = -\theta_j\theta_i \text{ if } 1 \le i \ne j \le n \qquad \text{and} \qquad \theta_i^2 = 0 \text{ for } 1 \le i \le n \,.$$

Since these conditions impose that a monomial in R_n has no repeated variables, the monomials are in bijection with subsets of $\{1, 2, \ldots, n\}$ and the dimension of R_n is therefore equal to 2^n.

Denote $[n] := \{1, 2, \ldots, n\}$ and let $A = \{a_1 < a_2 < \cdots < a_r\} \subseteq [n]$. We define $\theta_A := \theta_{a_1}\theta_{a_2}\cdots\theta_{a_r}$, then the set of monomials $\{\theta_A\}_{A \subseteq [n]}$ is a basis of R_n.

We define an action on monomials of R_n and extend this action linearly. For each integer $1 \leq i < n$, let π_i be an operator on R_n that is defined by

$$\pi_i(\theta_A) = \begin{cases} \theta_A & \text{if } i, i+1 \in A \text{ or } i, i+1 \notin A \\ \theta_{A \cup \{i+1\} \setminus \{i\}} & \text{if } i \in A \text{ and } i+1 \notin A \\ \theta_{A \cup \{i\} \setminus \{i+1\}} & \text{if } i+1 \in A \text{ and } i \notin A \end{cases} . \tag{2.1}$$

These operators instead of exchanging an i for an $i+1$ like the symmetric group action, have the effect of shifting the indices of the variables (if possible). They are therefore known as quasisymmetric operators. They were studied in depth by Hivert [19]. The operators are not multiplicative on R_n in general since, for example,

$$\pi_1(\theta_1\theta_2) = \theta_1\theta_2 = -\pi_1(\theta_1)\pi_1(\theta_2) .$$

The corresponding operators are also not multiplicative when they act on the polynomial ring in commuting variables.

A polynomial that is invariant under the action of quasisymmetric operators is said to be quasisymmetric invariant (or just 'quasisymmetric'). The quasisymmetric invariants of R_n are linearly spanned by the elements:

$$F_{1^r}(\theta_1, \theta_2, \ldots, \theta_n) := \sum_{\substack{A \subseteq [n] \\ |A| = r}} \theta_A . \tag{2.2}$$

The symbols F_{1^r} for the elements borrows the notation from the polynomial ring in commuting variable invariants known as the 'fundamental quasisymmetric polynomials.' The commuting polynomial quasisymmetric invariants are indexed by compositions. We choose to use F_{1^r} instead of M_{1^r} to maintain consistency with [2, 1] since our methods are similar to theirs.

2.1 Quasisymmetric functions generate a commutative subalgebra

In [11], the authors showed that the quasisymmetric functions in one set of commuting variables and one set of anticommuting variables forms a bi-graded Hopf algebra. This implies that the quasisymmetric functions in one set of anticommuting variables are closed under multiplication and the space is spanned by one element at each non-negative degree. The product is not commutative in general, but it is in the case of this subalgebra even though the variables are anticommuting.

In the notation of [11], $F_{1^r} = M_{\dot{0}^r} = L_{\dot{0}^r}$ where $\dot{0}^r = (\dot{0}, \dot{0}, \ldots, \dot{0})$ a composition of length r. The degree of F_{1^r} is exactly r.

Proposition 2.1. *The subalgebra generated by quasisymmetric invariants $\{F_{1^r} | r \geq 0\}$ is commutative. Moreover, $F_{1^r} F_{1^s} = a_{r,s} F_{1^{r+s}}$, where for $r + s \leq n$,*

$$a_{r,s} = \begin{cases} \binom{\lfloor \frac{r+s}{2} \rfloor}{\lfloor \frac{r}{2} \rfloor} & \text{if } rs \equiv 0 \pmod 2. \\ 0 & \text{otherwise} \end{cases} .$$

A remark brought to our attention by D. Grinberg [14] shows that $a_{r,s}$ is equal to the q-binomial coefficient $\left[{r+s \atop r} \right]_q$ evaluated at $q \to -1$.

2.2 The ideal generated by symmetric invariants

The symmetric invariants Sym_{R_n} of R_n are very small since a basis consists of only two elements 1 and $F_1(\theta_1, \theta_2, \ldots, \theta_n)$. Therefore the ideal generated by the invariants of non-zero degree, which we shall denote J_n, is generated by a single element F_1. We begin by considering the symmetric coinvariants of R_n, the quotient ring R_n/J_n. Because the ideal J_n is principal we can understand this quotient with much more detail. This quotient ring is a special case of the ring recently studied in [20, 21].

Recall that $\dim R_n = 2^n$, and if we consider the quotient R_n/J_n it is isomorphic to R_{n-1} since in this algebra $\theta_n = -\theta_1 - \theta_2 - \cdots - \theta_{n-1}$. Let $A \subseteq [n-1]$ and $A' = A \cup \{n\}$, then the map which sends $\theta_{A'}$ to

$$-\theta_A(\theta_1 + \theta_2 + \cdots + \theta_{n-1}) \otimes 1 + \theta_A \otimes F_1 \quad \in \; R_n/J_n \otimes \mathrm{Sym}_{R_n}$$

and θ_A to

$$\theta_A \otimes 1 \quad \in \; R_n/J_n \otimes \mathrm{Sym}_{R_n}$$

is an algebra isomorphism. The multiplication in the tensor algebra has a sign twist. For example, $(1 \otimes F_1)(\theta_2 \otimes 1) = -(\theta_2 \otimes F_1)$ because the θ_2 has to commute past F_1. Since this map describes the image for each monomial in R_n, we have the following proposition.

Proposition 2.2. *For each $n \geq 1$,*

$$R_n \cong R_n/J_n \otimes \mathrm{Sym}_{R_n},$$

as an algebra. That is, R_n is free over Sym_{R_n}.

2.3 The ideal generated by the quasisymmetric invariants

Define an ideal of R_n generated by the quasisymmetric invariants as

$$I_n := \langle F_{1^r}(\theta_1, \theta_2, \ldots, \theta_n) : 1 \leq r \leq n \rangle .$$

The *exterior quasisymmetric coinvariants*[1] are defined to be $EQC_n := R_n/I_n$. Similar to its commutative variant, the ideal I_n is not invariant under the action of π. So the quotient EQC_n is not closed under the action of π.

2.4 Differential operators on the exerior algebra

We can define a set of differential operators on R_n which will permit us to define the orthogonal complement to the ideal and a notion of quasisymmetric harmonics.

The operators ∂_{θ_i} act on monomials in R_n by

$$\partial_{\theta_i}(\theta_A) = \begin{cases} (-1)^{\#\{j \in A : j < i\}} \theta_{A \setminus \{i\}} & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases} .$$

The operators can equally be characterized by the action that $\partial_{\theta_i}(1) = 0$ and the commutation relations

$$\partial_{\theta_i}\partial_{\theta_j} = -\partial_{\theta_j}\partial_{\theta_i} \text{ if } 1 \leq i \neq j \leq n \qquad \text{and} \qquad \partial_{\theta_i}^2 = 0 \text{ for } 1 \leq i \leq n$$
$$\partial_{\theta_i}\theta_j = -\theta_j\partial_{\theta_i} \text{ if } 1 \leq i \neq j \leq n \qquad \text{and} \qquad \partial_{\theta_i}\theta_i = 1 \text{ for } 1 \leq i \leq n \ .$$

For a monomial $\theta_A = \theta_{a_1}\theta_{a_2}\cdots\theta_{a_r}$, let $\overline{\theta_A} = \theta_{a_r}\theta_{a_{r-1}}\cdots\theta_{a_1}$ represent reversing the order of the variables in the monomial. Extend this notation to both differential operators and polynomials (and polynomials of differential operators) by extending the notation linearly. We can define an inner product on R_n by setting for $p, q \in R_n$.

$$\langle p, q \rangle = \overline{p(\partial_{\theta_1}, \partial_{\theta_2}, \ldots, \partial_{\theta_n})} q(\theta_1, \theta_2, \ldots, \theta_n)|_{\theta_1=\theta_2=\cdots=\theta_n=0} \ .$$

The monomials of R_n form an orthonormal basis of the space with respect to this inner product.

Define the orthogonal complement to I_n with respect to the inner product as the set

$$EQH_n := \{q \in R_n : \langle p, q \rangle = 0 \text{ for all } p \in I_n\} \tag{2.3}$$
$$= \left\{q \in R_n : p(\partial_{\theta_1}, \partial_{\theta_2}, \ldots, \partial_{\theta_n})q = 0 \text{ for all } p \in I_n\right\} \ . \tag{2.4}$$

The second equality follows from the fact that I_n is an ideal and shows that EQH_n is also the solution space of a system of differential equations. We refer to EQH_n as the *exterior quasisymmetric harmonics*.[2] The inner product is positive definite. It follows that, as graded vector spaces, $EQC_n \simeq EQH_n$ for all $n \geq 1$.

[1] We borrow the name 'coinvariant' space even though the generators, and not the whole ideal, is invariant under the quasisymmetric operators.

[2] The harmonics and diagonal harmonics borrows the name from the physics literature because the harmonic operator $\partial_1^2 + \partial_2^2 + \cdots + \partial_n^2$ is symmetric in the differential operators. In the case of the exterior algebra, this operator acts as zero and yet we persist by borrowing the name from the analogous spaces of commuting variables.

We will conclude this section by constructing a set of linearly independent elements inside EQH_n, which will give us a lower bound on the dimension of EQC_n. In Section 4 we will see that this is also an upper bound, thus concluding that our set is in fact a basis. To compute EQH_n we need to solve the differential equations in Equation (2.4). Remark first that since I_n is an ideal, we do not need to take all $p \in I_n$, but it is enough to solve for the generators $p = F_{1^r}$ for $1 \leq r \leq n$. We can reduce that further using Proposition 2.1 as noted in the following lemma.

Lemma 2.3. *For $n \geq 2$ we have I_n is the ideal generated by F_1 and F_{1^2}.*

Proof. Clearly we have that the ideal generated by F_1, F_{1^2} is contained in I_n. For the converse we note that for each $k \geq 1$ there are non-zero coefficients a and a' such that

$$aF_{1^{2k}} = (F_{1^2})^k \quad \text{and} \quad a'F_{1^{2k+1}} = (F_{1^2})^k F_1$$

hence all of the generators of I_n are contained in the ideal generated by F_1, F_{1^2}. □

From this we conclude that

$$EQH_n = \left\{ q \in R_n : \sum_{1 \leq i \leq n} \partial_{\theta_i} q = 0 \quad \text{and} \quad \sum_{1 \leq i < j \leq n} \partial_{\theta_j} \partial_{\theta_i} q = 0 \right\} . \tag{2.5}$$

Given $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, a non-crossing pairing of length k is a list $(C_1, C_2, \ldots, C_k)$ with

$$\begin{gathered} C_r = (i_r, j_r) \text{ for } 1 \leq i_r < j_r \leq n \text{ for each } 1 \leq r \leq k \text{ and,} \\ \text{either } i_r < j_r < i_s < j_s \text{ or } i_s < i_r < j_r < j_s \text{ for any } 1 \leq r < s \leq k. \end{gathered}$$

Given a non-crossing pairing $C = (C_1, C_2, \ldots C_k)$, we define

$$\Delta_C = \left(\theta_{j_1} - \theta_{i_1}\right)\left(\theta_{j_2} - \theta_{i_2}\right) \cdots \left(\theta_{j_k} - \theta_{i_k}\right) . \tag{2.6}$$

Here $\Delta_C = 1$ if $k = 0$. Remark that $j_1 < j_2 < \cdots < j_k$. The following proposition shows that there is a relationship between the non-crossing partition condition and the differential equations from Equation (2.5).

Proposition 2.4. *The set*

$$\mathcal{D}'_n = \{\Delta_C : \ C = (C_1, C_2, \ldots, C_k) \textit{ non-crossing pairing and } 0 \leq k \leq \lfloor \frac{n}{2} \rfloor\}$$

is contained in EQH_n.

The set $\mathcal{D}'_n$ is not linearly independent, for example for $n = 3$ and $k = 1$, we have the following three non-crossing pairing: $((1,2))$, $((1,3))$ and $((2,3))$, but

$$\Delta_{((1,2))} - \Delta_{((1,3))} + \Delta_{((2,3))} = 0 .$$

We want to select a linearly independent subset of $\mathcal{D}'_n$. We proceed as follows: consider a sequence $\alpha = (a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ such that $\sum_{i=1}^r a_i \leq r/2$ for all $1 \leq r \leq n$. Such sequences are known as *ballot sequences*. If ever it is the case that $\sum_{i=1}^r a_i > r/2$ then we say that α *breaks the ballot condition* at position r.

Given a ballot sequence α we build a non-crossing pairing $C(\alpha)$ by first replacing all 0s by open parentheses $0 \mapsto$'(', and all 1s by close parentheses $1 \mapsto$')', and then do the natural maximal pairing of parenthesis. The positions of the pairings give us in lexicographic order a non-crossing pairing which we shall denote $C(\alpha)$. Since α is a ballot sequence, every closed parenthesis is matched and some open parentheses might remain unpaired. The natural pairing of parenthesis guarantees that the result will be non-crossing. For example,

$$\alpha = 0010001101 \quad \mapsto \quad (()((())() \quad \mapsto \quad C(\alpha) = ((2,3),(6,7),(5,8),(9,10))\,.$$

The total number of ballot sequences of size n is equal to the central binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$ (see [25, A001405]).

Given this construction we have the following Proposition.

Proposition 2.5. *The set*

$$\mathcal{D}_n = \left\{ \Delta_{C(\alpha)} : \alpha \in \{0,1\}^n \text{ a ballot sequence} \right\}$$

is contained in EQH_n and is linearly independent.

Proof. The first statement follows from Proposition 2.4 since $\mathcal{D}_n \subseteq \mathcal{D}'_n \subseteq EQH_n$. To show the linear independence, fix α a ballot sequence and let $C(\alpha) = ((i_1, j_1), \ldots, (i_k, j_k))$ be its non-crossing pairing. We remark that the sequence of numbers $j_1 < j_2 < \cdots < j_k$ corresponds to the position of the 1s in α. Using the monomial ordering described in Section 3 and by inspection of the product in Equation (2.6), we observe that the term $\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}$ is the smallest lexicographic monomial in $\Delta_{C(\alpha)}$. For different ballot sequences α we get different positions of the 1s in α and thus different smallest lexicographic monomials, which shows the independence of $\mathcal{D}_n$. $\square$

Ballot sequences with k number of 1's are in bijection with standard tableaux of shape $(n-k, k)$. It is tempting to try describe $\mathcal{D}_n$ in terms of polynomials directly analogous to Specht polynomials in commuting variables. However, it can be shown that such a direct analogy does not work.

3 A linear basis of the ring

Again let n be a fixed positive integer and $R_n = \mathbb{Q}[\theta_1, \theta_2, \ldots, \theta_n]$. We have thus far represented the basis for R_n as the elements θ_A with $A \subseteq [n]$. Define $\alpha(A) \in \{0,1\}^n$ to be the sequence $a_1a_2a_3\cdots a_n$ with $a_i = 1$ if $i \in A$ and $a_i = 0$ if $i \notin A$ so that

$$\theta_A = \theta_1^{a_1}\theta_2^{a_2}\cdots\theta_n^{a_n} := \theta^{\alpha(A)} .$$

For such a sequence $\alpha \in \{0,1\}^n$, let $m_1(\alpha) := \sum_{i=1}^n a_i$ represent the number of 1s in the string. This will also be the degree of the monomial θ^α.

For sequences $\alpha \in \{0,1\}^n$, define elements G_α by

$$G_{1^s0^{n-s}} = F_{1^s} \tag{3.1}$$

and if $\alpha \neq 1^s0^{n-s}$, then α is of the form $u01^s0^{n-k-s}$ for some string u of length $k-1$ and we recursively define

$$G_{u01^s0^{n-k-s}} = G_{u1^s0^{n-k-s+1}} - (-1)^{m_1(u)}\theta_k G_{u1^{s-1}0^{n-k-s+2}} . \tag{3.2}$$

The challenge in arriving at this definition was finding the correct recursion which yields a set of polynomials $G_\alpha \in I_n$ whose leading term with respect to some monomial order is θ^α.

We will show below that the recurrence for the G_α is defined so that they are S-polynomials [6] for elements of the ideal I_n. In commutative variables, similar polynomials were defined by Aval-Bergeron-Bergeron [1, 2] as a (complete) subset of S-polynomials needed to compute all possible S-polynomials in the Buchburger algorithm for a Gröbner basis. It is not given that one can easily describe such a set of S-polynomials and here we have adapted the definition for working in the exterior algebra.

Example 3.1. For $\alpha = 010110$ and $\beta = 001100$, we compute the elements G_α and G_β using the definition.

$$\begin{aligned} G_{010110} &= G_{011100} + \theta_3 G_{011000} = (G_{111000} - \theta_1 G_{110000}) + \theta_3(G_{110000} - \theta_1 G_{100000}) \\ &= \theta_2\theta_4\theta_5 + \theta_2\theta_4\theta_6 + \theta_2\theta_5\theta_6 + 2\theta_3\theta_4\theta_5 + 2\theta_3\theta_4\theta_6 + 2\theta_3\theta_5\theta_6 + \theta_4\theta_5\theta_6 \end{aligned}$$

and we have that

$$\begin{aligned} G_{001100} &= G_{011000} - \theta_2 G_{010000} = (G_{110000} - \theta_1 G_{100000}) - \theta_2(G_{100000} - \theta_1 G_{000000}) \\ &= \theta_3\theta_4 + \theta_3\theta_5 + \theta_3\theta_6 + \theta_4\theta_5 + \theta_4\theta_6 + \theta_5\theta_6. \end{aligned}$$

Proposition 3.2. *The largest lexicographic term in G_α is θ^α.*

The proof of Proposition 3.2 follows by induction on the length of α and from a lemma that is analogous to Lemma 3.3 of [2]. The recursion in this result is really the origin of the definition of G_α because Equation (3.2) was adapted so that this lemma holds. It follows that the set $\{G_\alpha\}_{\alpha\in\{0,1\}^n}$ is a basis for R_n.

4 A basis for the quotient

The elements G_α are defined so that we could use them to identify a nice basis of the ideal I_n. This is our main theorem.

Theorem 4.1. *The set $A_n := \big\{G_\alpha : \alpha \in \{0,1\}^n$ breaks the ballot condition$\big\}$ is a $\mathbb{Q}$–linear basis of the ideal I_n.*

The proof of this theorem uses our understanding of the harmonic space $EQH_n \cong EQC_n$. The challenge here was to show that $G_\alpha \in I_n$ if and only if α breaks the ballot condition. In Proposition 2.5 we found that $\dim(EQH_n) = \dim(EQC_n)$ and is at least the number of ballot sequences. We first establish a small lemma about a spanning set for the quotient EQC_n showing that the dimension is at most the number of ballot sequences. Therefore we have equality and the set $\mathcal{D}_n$ in Proposition 2.5 is in fact a basis of EQH_n.

There are several straightforward consequence of this theorem which we state here.

Corollary 4.2. *The Hilbert series of EQH_n is given by*

$$Hilb_{R_n/I_n}(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} f^{(n-k,k)} q^k,$$

where $f^{(n-k,k)}$ counts the number of standard Young tableaux of shape $(n-k,k)$.

Corollary 4.3. *The set $\mathcal{D}_n$ is a basis of EQH_n and of EQC_n.*

Corollary 4.4. *The set A_n is a (non-reduced, non-minimal) Gröbner basis of I_n. A minimal Gröbner basis for I_n is given by*

$$\big\{G_\alpha : \alpha \in \{0,1\}^n \text{ breaks the ballot condition only at the rightmost 1 of } \alpha\big\}.$$

These results give a fairly complete description of the ideal I_n and the quotient R_n/I_n. To demonstrate the results above, we present the following example for a small value of n.

Example 4.5. Let $n = 4$, then $\dim R_4 = 16$. There are 6 ballot sequences, {0000, 0001, 0010, 0011, 0100, 0101} and so R_4/I_4 is spanned by $\{1, \theta_2, \theta_3, \theta_4, \theta_3\theta_4, \theta_2\theta_4\}$. The ideal I_4 is spanned by the 10 elements,

$$\begin{aligned}
G_{0110} &= \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4 & G_{0111} &= \theta_2\theta_3\theta_4 \\
G_{1000} &= \theta_1 + \theta_2 + \theta_3 + \theta_4 & G_{1001} &= \theta_1\theta_4 + \theta_2\theta_3 + 2\theta_2\theta_4 + 2\theta_3\theta_4 \\
G_{1010} &= \theta_1\theta_3 + \theta_1\theta_4 + 2\theta_2\theta_3 + 2\theta_2\theta_4 + \theta_3\theta_4 & G_{1011} &= \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4 \\
G_{1100} &= \theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_3\theta_4 & G_{1101} &= \theta_1\theta_2\theta_4 + 2\theta_1\theta_3\theta_4 + 2\theta_2\theta_3\theta_4 \\
G_{1110} &= \theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4 & G_{1111} &= \theta_1\theta_2\theta_3\theta_4
\end{aligned}$$

A reduced Gröbner basis for I_4 consists of the elements $\{G_{0110}, G_{1000}\}$.

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