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Jack characters as generating series of bipartite maps and proof of Lassalle's conjecture

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Abstract. We give an explicit formula for the power-sum expansion of Jack polynomials. We deduce it from a more general formula that we provide here, that interprets Jack characters in terms of bipartite maps. Finally, we prove Lassalle's conjecture from 2008 on integrality and positivity of Jack characters in Stanley's coordinates.

Keywords: Jack polynomials, Jack characters, maps, Lassalle's conjecture, Kerov polynomials

1 Introduction

1.1 Jack polynomials and Jack characters

Jack polynomials $J_\lambda^{(\alpha)}$ are symmetric functions, indexed by an integer partition λ and a deformation parameter α. They interpolate, up to scaling factors, between Schur functions for $\alpha = 1$ and zonal polynomials for $\alpha = 2$. Stanley initiated in his seminal work [18] the combinatorial analysis of these symmetric functions with connection with various objects of algebraic combinatorics, such as partitions, tableaux, paths and maps [14, 7, 9, 3, 15, 2].

Knop and Sahi have given in [9] a combinatorial interpretation for the coefficients of the Jack polynomial $J_\lambda^{(\alpha)}$ in the monomial basis in terms of tableaux of shape λ. Existence of a combinatorial expression for the expansion of the Jack polynomial $J_\lambda^{(\alpha)}$ in the power-sum basis was conjectured by Hanlon [8]. In this paper, we give such a combinatorial interpretation in terms of bipartite maps.

Roughly speaking, a map is a graph drawn on a locally orientable surface (see Section 2.3 for a precise definition). The study of maps is a well developed area with strong connections with analytic combinatorics, mathematical physics and probability [11]. The

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relationship between generating series of maps and the theory of symmetric functions was first noticed via a character theoretic approach, and has then been developed to include other techniques such as matrix integrals and differential equations [7, 10, 4, 2].

In fact, the aforementioned combinatorial interpretation follows from a more general result that we prove: a combinatorial interpretation for Jack characters. For a partition μ, the Jack character $\theta_\mu^{(\alpha)}$ is the function on Young diagrams defined by:

$$\theta_\mu^{(\alpha)}(\lambda) := \begin{cases} 0 & \text{if } |\lambda| < |\mu|, \\ \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)}[p_{\mu,1^{n-m}}]J_\lambda^{(\alpha)} & \text{if } |\lambda| \geq |\mu|. \end{cases}$$

where $[p_\mu]$ is the extraction symbol with respect to the variable $\mathbf{p}$ and $m_1(\mu)$ is the number of parts equal to 1 in the partition μ.

Jack characters play an important role in studying asymptotic behaviour of random Young diagrams [3]. When $\alpha = 1$, they correspond to the normalized irreducible characters of the symmetric group, and have a combinatorial interpretation in terms of permutation factorizations which has been conjectured by Stanley and proved by Féray (see [5] for the details). An analogous combinatorial expression has also been established in [6] for Zonal characters, which correspond to $\theta_\mu^{(\alpha)}$ with $\alpha = 2$. In this paper, we give a combinatorial interpretation for $\theta_\mu^{(\alpha)}$ for general α in terms of maps. The expression we obtain interpolates between the expressions given in [5] and [6] for $\alpha = 1$ and $\alpha = 2$.

1.2 Multi-layer maps

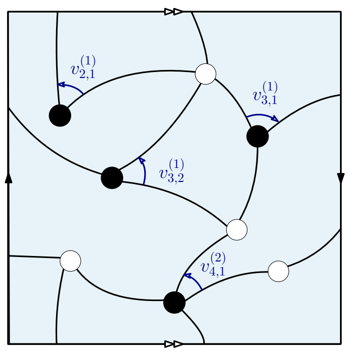
Definition 1.1. *Fix $k \geq 0$. We say that a bipartite map M is a k-layer map if its black vertices (resp. white vertices) are partitioned into k sets (which may be empty), called the layers of the map; $\mathcal{V}_\circ(M) = \bigcup_{1\leq i\leq k} \mathcal{V}_\circ^{(i)}(M)$ (resp. $\mathcal{V}_\bullet(M) = \bigcup_{1\leq i\leq k} \mathcal{V}_\bullet^{(i)}(M)$), which satisfy the following condition: if v is a white vertex in a layer i, then all its neighbors are in layers $j \leq i$, and it has at least one neighbor in the layer i.*

For $1 \leq i \leq k$, we define the partition $\nu_\bullet^{(i)}(M)$ obtained by reordering the degree distribution of the black vertices in the layer i. A k-layer map is labelled if:

- *in each layer $1 \leq i \leq k$, the black vertices having the same degree j are numbered by $1, 2, \ldots, m_j\left(\nu_\bullet^{(i)}(M)\right)$, where $m_j\left(\nu_\bullet^{(i)}(M)\right)$ denotes the number of parts of size j in $\nu_\bullet^{(i)}(M)$.*
- *each black vertex has a distinguished oriented corner.*

An example of a 2-layer map is given in the left part of Figure 1.

Note that a k-layer map can be seen as $(k+1)$-layer map with an empty layer $k+1$. We call then a *multi-layer* map a k-layer map for some $k \geq 1$. We denote by $\mathcal{M}^{(k)}$ (resp.



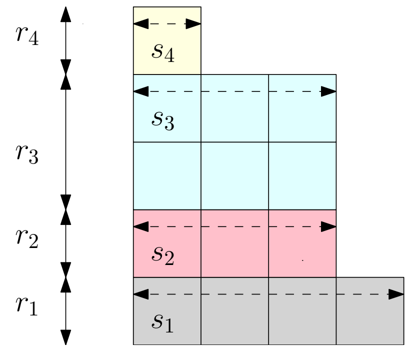


Figure 1: *Left:* A 2-layer map on the Klein bottle, represented here by a square whose left side should be glued to the right one (with a twist) and the top side should be glued to the bottom one (without a twist), as indicated by the arrows. $v_{j,k}^{(i)}$ denotes the black vertex of degree j numbered by k in the layer i. *Right:* The Young diagram of the partition $[4,3,3,3,1]$ as the union of 4 rectangles, with $\mathbf{s} = (4,3,3,1)$ and $\mathbf{r} = (1,1,2,1)$ as multirectangular coordinates.

$\mathcal{M}^{(\infty)}$) the set of all labelled k-layer maps (resp. labelled multi-layer maps). Similarly, we denote $\mathcal{M}_\mu^{(k)}$ and $\mathcal{M}_\mu^{(\infty)}$ those of face-type μ (see Section 2.3 for the definition). The following definition has been introduced by Goulden and Jackson.

Definition 1.2 ([7]). *A statistic of non-orientability on bipartite maps is a statistic ϑ with non-negative integer values, such that $\vartheta(M) = 0$ if and only if M is orientable.*

In practice, a statistic of non-orientability is supposed to "measure" the non-orientability of a map by counting the number of edges which contribute to its non-orientability, following a given algorithm of decomposition of the map. Several examples of such statistics have been introduced in previous works [10, 4, 2].

1.3 First main result

Theorem 1.3. *There exists a statistic of non-orientability ϑ on multi-layer maps such that for any partitions μ and λ, we have*

$$\theta_\mu^{(\alpha)}(\lambda) = (-1)^{|\mu|} \sum_{M \in \mathcal{M}_\mu^{(\infty)}} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}, \tag{1.1}$$

where b is the parameter related to α by $b := \alpha - 1$, and $z_{\nu_\bullet^{(i)}(M)}$ is the classical normalization factor (see Sections 2.1 and 2.3 for the precise definitions and notation).

Note that the product here is finite since each multi-layer map has a finite number of non empty layers. We give in Section 4 the key ingredients of the proof of Theorem 1.3. As a direct consequence of Theorem 1.3, we obtain the following interpretation of Jack polynomials in the basis of power-sum functions.

Theorem 1.4. *Let n be a positive integer and let λ be a partition of n. Then*

$$J_\lambda^{(\alpha)} = (-1)^n \sum_M p_{\nu_\diamond(M)} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}} \prod_{1\leq i\leq \ell(\lambda)} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

where the sum is taken over all $\ell(\lambda)$-layer maps M of size n, and $\nu_\diamond(M)$ denotes the face-type of M (see Section 2.3).

1.4 Lassalle's conjecture and second main result.

Stanley has used [19] the following definition of multirectangular coordinates.

Definition 1.5 ([19])**.** *Let $k \geq 1$ and let $s_1 \geq s_2 \cdots \geq s_k \geq 1$ and $r_1, \dots r_k$ be two sequences of nonnegative integers. We say that $(s_1, \dots, s_k)$ and $(r_1, \dots, r_k)$ are multirectangular coordinates for a partition λ and we denote $\lambda = \mathbf{s}^\mathbf{r}$, if λ is the union of k rectangles of sizes $s_i \times r_i$, or equivalently $\lambda = [s_1^{r_1} \dots s_k^{r_k}]$, see the right part of Figure 1 for an example.*

Since we do not require that the sequence $\mathbf{s}$ should be strictly decreasing, the multirectangular coordinates are not unique in general. If λ is a partition of multirectangular coordinates $(s_1, \dots, s_k)$ and $(r_1, \dots, r_k)$, we write for any partition μ

$$\tilde{\theta}_\mu^{(\alpha)}(\mathbf{s}, \mathbf{r}) := \theta_\mu^{(\alpha)}(\lambda),$$

where $\mathbf{r} = (r_1, \dots, r_k, 0 \dots)$ and $\mathbf{s} = (s_1, \dots, s_k, 0, \dots)$. In the case $b = 0$, which corresponds to the irreducible characters of the symmetric group, Stanley found [19] an explicit formula for $\tilde{\theta}_\mu^{(\alpha)}(\mathbf{s}, \mathbf{r})$ when λ is a rectangle, and he conjectured a formula for general $\mathbf{r}, \mathbf{s}$ that was soon after proved by Féray [5]. An analogous formula was found in the case $b = 1$ [6]. A conjecture that suggests positivity and integrality of Jack characters in multirectangular coordinates for arbitrary b was stated by Lassalle [12], and it remained unproven for nearly 15 years[1]. Our second main result is the proof of this conjecture.

Theorem 1.6. *The normalized Jack characters $(-1)^{|\mu|} z_\mu \tilde{\theta}_\mu^{(\alpha)}$ are polynomials in the variables $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ with non-negative integer coefficients, where $b := \alpha - 1$.*

[1]It has been proved in the particular case $\mathbf{s} = (s, 0, \dots)$ and $\mathbf{r} = (r, 0, \dots)$ in [1].

Interestingly, the proof of Theorem 1.6 consists of two parts that are proved using very different techniques. In the first part we deduce positivity as a consequence of the combinatorial expression of Jack characters obtained in Theorem 1.3 (see Section 5). In the second part, we obtain the integrality using integrable system of Nazarov–Sklyanin [16]. We relate their theory with Jack characters by proving an explicit combinatorial formula expressing certain basis of shifted symmetric functions in terms of normalized Jack characters. We conclude by showing that the transition matrix between these two bases is invertible over $\mathbb{Z}$. Consequently, we prove that Kerov polynomials for Jack characters have integer coefficients, which was an open problem posed by Lassalle in [13] (see Section 5.2 for details).

2 Notation

For the definitions and notation introduced in Sections 2.1 and 2.2 we refer to [18, 14].

2.1 Partitions

A *partition* $\lambda = [\lambda_1, ..., \lambda_\ell]$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq ... \geq \lambda_\ell > 0$. We denote by $\mathbb{Y}$ the set of all integer partitions. The integer ℓ is called the *length* of λ and is denoted $\ell(\lambda)$. The size of λ is the integer $|\lambda| := \lambda_1 + \lambda_2 + ... + \lambda_\ell$. If n is the *size* of λ, we say that λ is a partition of n and we write $\lambda \vdash n$. The integers λ_1,...,λ_ℓ are called the *parts* of λ. For $i \geq 1$, we denote $m_i(\lambda)$ the number of parts of size i in λ. We set then

$$z_\lambda := \prod_{i \geq 1} m_i(\lambda)! i^{m_i(\lambda)}.$$

We denote by $\leq$ the *dominance partial* ordering on partitions, defined by

$$\mu \leq \lambda \iff |\mu| = |\lambda| \text{ and } \quad \mu_1 + ... + \mu_i \leq \lambda_1 + ... + \lambda_i \text{ for } i \geq 1.$$

We identify a partition λ with its *Young diagram*, defined by

$$\lambda := \{(i, j), 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}.$$

2.2 Symmetric functions and Jack polynomials

We fix an alphabet $\mathbf{x} := (x_1, x_2, ..)$. We denote by $\mathcal{S}$ the algebra of symmetric functions in $\mathbf{x}$ with coefficients in $\mathbb{Q}$. For every partition λ, we denote m_λ the monomial function and p_λ the power-sum function associated with the partition λ. We consider the associated alphabet of power-sum functions $\mathbf{p} := (p_1, p_2, ..)$.

Let $\mathcal{S}_\alpha$ be the algebra of symmetric functions with coefficients in $\mathbb{Q}(\alpha)$. We denote by $\langle .,. \rangle_\alpha$ the α-deformation of the Hall scalar product defined by

$$\langle p_\lambda, p_\mu \rangle_\alpha = z_\lambda \alpha^{\ell(\lambda)} \delta_{\lambda,\mu}, \text{ for any partitions } \lambda, \mu.$$

Macdonald [14, Chapter VI.10] has proved that there exists a unique family of symmetric functions $(J_\lambda^{(\alpha)})_{\lambda \in \mathbb{Y}}$ in $\mathcal{S}_\alpha$ indexed by partitions, satisfying the following properties:

$$\begin{cases} \text{Orthogonality:} & \langle J_\lambda, J_\mu \rangle_\alpha = 0, \text{ for } \lambda \neq \mu, \\ \text{Triangularity:} & [m_\mu] J_\lambda = 0, \text{ unless } \mu \leq \lambda, \\ \text{Normalization:} & [m_{1^n}] J_\lambda = n!, \text{ for } \lambda \vdash n, \end{cases}$$

where $[m_\mu] J_\lambda$ denotes the coefficient of m_μ in J_λ, and 1^n is the partition with n parts equal to 1. These functions are known as the *Jack polynomials*.

2.3 Maps

A *connected map* is a connected graph embedded into a surface such that all the connected components of the complement of the graph are simply connected (see [11, Definition 1.3.7]). These connected components are called the *faces* of the map. We consider maps up to homeomorphisms of the surface. A connected map is *orientable* if the underlying surface is orientable. In this paper[2], a *map* is an unordered collection of connected maps. A map is orientable if each one of its connected components is orientable. Finally, the *size* of a map is its number of edges.

In this paper, all the maps considered are *bipartite*; *i.e* their vertices are colored in two colors, white and black, such that each edge connects two vertices of different colors. Note that in a bipartite map, all faces have even degree. We define then the *face-type* of a bipartite map M of size n, as the partition of n obtained by reordering the half degrees of the faces and we denote it $\nu_\diamond(M)$. We also denote its set of white and black vertices by $\mathcal{V}_\circ(M)$ and $\mathcal{V}_\bullet(M)$ respectively.

3 The combinatorial model

The purpose of this subsection is to define a family of statistic of non-orientability (see Definition 1.2) on multi-layer maps. We start by some general definitions related to non-orientable maps.

Let M be a bipartite map and let c_1 and c_2 be two corners of M of different colors. Then we have two ways to add an edge to M between these two corners (see Figure 2). We denote by e_1 and e_2 these edges. We say that the pair (e_1, e_2) is a *pair of twisted edges*

[2] This is not the standard definition of a map; usually a map is connected.

on the map M and we say that e_2 is obtained by twisting e_1. Note that if M is connected and orientable, then exactly one of the maps $M \cup \{e_1\}$ and $M \cup \{e_2\}$ is orientable. For a given map with a distinguished edge (M, e), we denote $(\widetilde{M}, \tilde{e})$ the map obtained by twisting the edge e.

We recall that b is the parameter related to the Jack parameter α by $b = \alpha - 1$. We now give the definition of a measure of non-orientability due to La Croix.

Definition 3.1. *[10, Definition 4.1] We call a measure of non-orientability* (MON) *a function ρ defined on the set of connected maps (M, e) with a distinguished edge, with values in $\{1, b\}$, satisfying the following conditions:*

- *if e connects two corners of the same face of $M \backslash \{e\}$, and the number of the faces increases by 1 by adding the edge e on the map $M \backslash \{e\}$, then $\rho(M, e) = 1$. In this case we say that e is a diagonal.*
- *if e connects two corners in the same face $M \backslash \{e\}$, and the number of the faces of $M \backslash \{e\}$ is equal to the number of faces of M, then $\rho(M, e) = b$. In this case we say that e is a twist.*
- *if e connects two corners of two different faces lying in the same connected component of $M \backslash \{e\}$, then ρ satisfies $\rho(M, e) + \rho(\widetilde{M}, \tilde{e}) = 1 + b$. In this case we say that e is a handle. Moreover, if M is orientable then $\rho(M, e) = 1$.*
- *if e connects two faces lying in two different connected components, then $\rho(M, e) = 1$. In this case, we say that e is a bridge.*

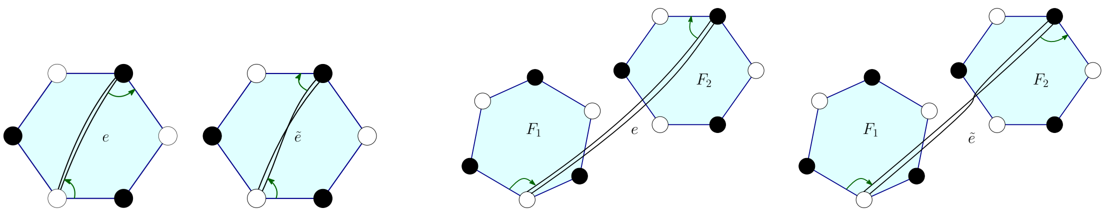
Let M be a bipartite map and let $e_1, e_2, \ldots, e_d$ be d distinct edges of M. For $1 \leq i \leq d$, we denote M_j be the map obtained by deleting the edges e_1, e_2,..., e_j from M. We define $\rho(M, e_1, e_2, \ldots, e_d)$ as the weight obtained by deleting the edges e_j successively:

$$\rho(M, e_1, e_2, \ldots, e_d) := \prod_{1 \leq j \leq d} \rho(M_j, e_j).$$

Given a MON ρ, one can associate a b-weight $\rho(M)$ to a multi-layer map M by choosing an order on the edges of M. We start by defining an order on black vertices.

We fix an integer $k \geq 1$ and a labelled k-layer map M. To each black vertex v of M we associate the triplet of positive integers (i, n, j), where i is the layer containing the vertex, n is its degree, and j is the number given to v by the labelling of the map. By definition, the triplets associated with two distinct black vertices are different. We define then a linear order $\prec_M$ on the black vertices of M given by the lexicographic order on the triplets $(-i, n, j)$. In other terms, the maximal black vertex with respect to $\prec_M$ is the vertex contained in the layer of smallest index, and having maximal degree and maximal label. Note that when we delete the maximal vertex from a k-layer map and all the edges incident to it, the map obtained is also a k-layer map.

We now introduce a statistic of non-orientability on k-layer labelled maps which will have a key role in this paper.



(a) (b)

Figure 2: The different ways of adding an edge on a map. The added edge is represented each time by a band, that can be twisted at most once. The arrows indicate how the added edge connects the two respective corners. Figure 2a: a pair of twisted edges $(e, \tilde{e})$ between two corners of the same face; e is a diagonal while $\tilde{e}$ is a twist. Figure 2b: A pair of twisted edges $(e, \tilde{e})$ between two corners of different faces F_1 and F_2. If F_1 and F_2 are on the same connected component then the edges e and $\tilde{e}$ are handles, otherwise they are bridges.

Definition 3.2 (Statistic of non-orientability on k-layer maps)**.** *Let ρ be a* MON *and let M be a connected labelled k-layer map of size n. We define the b-weight $\rho(M)$ of M, as the weight obtained by decomposing M recursively as following. If M is the empty map then $\rho(M) = 1$. Otherwise, let v be the maximal black vertex of M with respect to $\prec_M$, and let c be its root and d its degree. We denote by $e_1, \dots, e_d$ the edges incident to v as they appear when we turn around v starting from the root c. Let M' be the map obtained from M by deleting v and all the edges incident to it. Then, by definition $\rho(M) := \rho(M, e_1, \dots, e_d) \cdot \rho(M')$. We extend this definition to disconnected maps by multiplicativity; if M is a labelled k-layer map with m connected components $M_1, \dots, M_m$, then $\rho(M) := \prod_{1 \le i \le m} \rho(M_i)$. Finally, we define the statistic ϑ_ρ on labelled multi-layer maps with values in non-negative integers, defined for every M by $\rho(M) = b^{\vartheta_\rho(M)}$.*

It follows from the definitions that ϑ_ρ is a statistic of non-orientability over labelled bipartite maps. We prove that Theorem 1.3 holds for every statistic of the form ϑ_ρ.

4 Idea of the proof of Theorem 1.3

The starting point of the proof of Theorem 1.3 is a characterization of Jack characters due to Féray (see Section 4.1).

4.1 Féray's characterization of Jack characters

Roughly, an α-shifted symmetric function $f(s_1, s_2, \dots)$ (a shifted function for short) is a function which is symmetric in the variables $s_i - i/\alpha$. If f is a shifted function and

λ a partition, then we denote $f(\lambda) := f(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}, 0, \dots)$. A shifted function is completely determined by its evaluation on Young diagrams $(f(\lambda))_{\lambda \in \mathbb{Y}}$, which gives an identification between shifted functions and functions on Young diagrams. In particular, Jack characters defined as functions on Young diagrams in Section 1.1 can be seen as shifted functions. The starting point of the proof of the main theorem is the following characterization of the Jack characters $\theta_\mu^{(\alpha)}$ which is due to Valentin Féray (see [17, Theorem A.2]).

Theorem 4.1. *Fix a partition μ. The Jack character $\theta_\mu^{(\alpha)}$ is the unique α-shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu|-\ell(\mu)}/z_\mu \cdot p_\mu$, such that $\theta_\mu^{(\alpha)}(\lambda) = 0$ for any partition $|\lambda| < |\mu|$.*

4.2 The generating series of k-layer maps

Fix $k \geq 1$. We define the generating series of k-layer maps:

$$F^{(k)}\left(t, \mathbf{p}, s_1, \dots, s_k\right) := \sum_{M \in \mathcal{M}^{(k)}} \frac{(-t)^{|M|} p_{\nu_\diamond(M)} b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{1 \leq i \leq k} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}. \tag{4.1}$$

We also consider the projective limit $F^{(\infty)} := \varprojlim F^{(k)}$.

Note that the normalizing factors which appear in this definition are related to the definition of labelling we use here[3]. This labelling allows to deal with maps with trivial automorphism groups and also plays an important role in the definition of statistic of non-orientability Section 3.

To obtain Theorem 1.3, we prove that $[t^{|\mu|} p_\mu] F^{(\infty)}(\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, \dots)$ satisfies the conditions of Theorem 4.1. To this purpose we express the functions $F^{(k)}$ using the differential operators introduced in [2].

5 Lassalle's conjecture

5.1 Positivity - ideas

We consider two sequences of variables $\mathbf{s} = (s_1, s_2, \dots)$ and $\mathbf{r} = (r_1, r_2, \dots)$. We introduce a bivariate version of the generating series of k-layer maps

$$\widetilde{F}^{(\infty)}\left(t, \mathbf{p}, \begin{matrix} s_1 & s_2 \dots \\ r_1 & r_2 \dots \end{matrix}\right) = \sum_{M \in \mathcal{M}^{(\infty)}} \frac{(-t)^{|M|} p_{\nu_\diamond(M)} b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{r_i^{|\mathcal{V}_\bullet^{(i)}(M)|} (-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}. \tag{5.1}$$

[3]the factor $2^{|\mathcal{V}_\bullet(M)| - cc(M)}$ can be omitted by using a finer labelling

Hence the functions $F^{(\infty)}$ defined in Equation (4.1) are obtained as a specialization of $\widetilde{F}^{(k)}$:

$$F^{(\infty)}(t, \mathbf{p}, s_1, s_2, \dots) = \widetilde{F}^{(\infty)}\left(t, \mathbf{p}, \begin{matrix} s_1 & s_2 \dots \\ 1 & 1 \dots \end{matrix}\right).$$

Proposition 5.1. *Let λ be a partition, and let $\begin{pmatrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{pmatrix}$ be multirectangular coordinates of λ. Then*

$$\widetilde{F}^{(\infty)}\left(t, \mathbf{p}, \begin{matrix} s_1 & \dots & s_k & 0 & \dots \\ r_1 & \dots & r_k & 0 & \dots \end{matrix}\right) = F^{(\infty)}(t, \mathbf{p}, \lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, \dots).$$

As a consequence of Theorem 1.3 and Proposition 5.1, we obtain that for any partition μ

$$\tilde{\theta}^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r}) = [t^{|\mu|} p_\mu]\widetilde{F}^{(\infty)}\left(t, \mathbf{p}, \begin{matrix} s_1 & \dots & s_k & 0 & \dots \\ r_1 & \dots & r_k & 0 & \dots \end{matrix}\right), \tag{5.2}$$

where $\mathbf{s} = (s_1 \dots, s_k, 0, \dots)$ and $\mathbf{r} = (r_1, \dots, r_k, 0, \dots)$. We now prove the positivity in Lassalle's conjecture.

Proof of the positivity in Theorem 1.6. Comparing Equation (5.1) and Equation (5.2) and taking the limit on k we get that for every MON ρ

$$\tilde{\theta}^{(\alpha)}_\mu(\mathbf{s}, \mathbf{r}) = (-1)^{|\mu|} \sum_{M \in \mathcal{M}^{(\infty)}_\mu} \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{r_i^{|\mathcal{V}^{(i)}_\bullet(M)|} (-\alpha s_i)^{|\mathcal{V}^{(i)}_\circ(M)|}}{z_{\nu^{(i)}_\bullet(M)}}. \tag{5.3}$$

Since every connected component of a bipartite map component contains at least one white vertex, the α-term which appears in the denominator is compensated. This concludes the proof of the positivity. □

5.2 Integrality - ideas

Let $\mathrm{Ch}^{(\alpha)}_\mu := \frac{z_\mu}{\sqrt{\alpha}^{|\mu| - \ell(\mu)}} \tilde{\theta}^{(\alpha)}_\mu$. For a positive integer k define free cumulant $R^{(\alpha)}_{k+1}$ as

$$R^{(\alpha)}_{k+1} := 2k \sum_{\substack{T \in \mathcal{M}^{(\infty)}_{(k)} \\ |\mathcal{V}_\bullet(T)| + |\mathcal{V}_\circ(T)| = k+1}} \prod_{i \geq 1} \left(\frac{r_i}{2\sqrt{\alpha}}\right)^{|\mathcal{V}^{(i)}_\bullet(T)|} \frac{(-\sqrt{\alpha} s_i)^{|\mathcal{V}^{(i)}_\circ(T)|}}{z_{\nu^{(i)}_\bullet(T)}}.$$

The constraints for the summation index says that the sum runs over all bicolored trees with extra labelings and rootings. In particular, the normalization factors $2^{|\mathcal{V}^{(i)}_\bullet(T)|} z_{\nu^{(i)}_\bullet(T)}$ will be compensated by the number of labelings and rootings and in effect it can be shown that the free cumulants are polynomials in $-\sqrt{\alpha}s_1, -\sqrt{\alpha}s_2 \ldots, \frac{r_1}{\sqrt{\alpha}}, \frac{r_2}{\sqrt{\alpha}} \ldots$ with integer coefficients, see [4].

We prove the following theorem that is a weaker version of another conjecture of Lassalle about the structure of Kerov polynomials for Jack characters [13].

Theorem 5.2. *For every partition μ there exists a polynomial $K_\mu \in \mathbb{Z}[x_1, x_2, x_3, x_4, \ldots]$ such that $\mathrm{Ch}_\mu^{(\alpha)} = K_\mu(\sqrt{\alpha}^{-1} - \sqrt{\alpha}, R_2^{(\alpha)}, R_3^{(\alpha)}, \ldots)$. In particular $z_\mu \tilde{\theta}_\mu^{(\alpha)}$ is a polynomial in $b, -s_1, -s_2 \ldots, r_1, r_2, \ldots$ with integer coefficients.*

The polynomial K_μ is called *Kerov polynomial*, and the last sentence in the above theorem follows from the bound on the degree of K_μ and its parity proved in [3][Proposition 3.10].

We deduce Theorem 5.2 from an explicit combinatorial formula for the moments expressed as a linear combination of Jack characters. A classical relation between moments and free cumulants reads:

$$M_k^{(\alpha)} = \sum_{\pi \in \mathrm{NC}([1..k])} \prod_{B \in \pi} R_{|B|}^{(\alpha)}, \tag{5.4}$$

where we sum over all non-crossing partitions of the set $[1..k]$. For any partition μ define $\mu \pm 1 := (\mu_1 \pm 1, \ldots, \mu_{\ell(\mu)} \pm 1)$ and $M_\mu^{(\alpha)} := \prod_{i=1}^{\ell(\mu)} M_{\mu_i}^{(\alpha)}$. We prove the following formula:

$$M_{\mu+1}^{(\alpha)} = \mathrm{Ch}_\mu^{(\alpha)} + \sum_{\nu < \mu+1} a_{\nu-1}^\mu \,\mathrm{Ch}_{\nu-1}^{(\alpha)} + \sum_{i=1}^{|\mu|-\ell(\mu)} \sum_{\nu\colon |\nu+1| = |\mu+1|-i} \gamma^i a_\nu^\mu \,\mathrm{Ch}_\nu^{(\alpha)}, \tag{5.5}$$

where a_ν^μ are positive integers with an explicit combinatorial interpretation as numbers of some lattice paths (that we won't describe in details due to lack of space). In particular Equation (5.5) defines a unitriangular system of equations over $\mathbb{Z}$, therefore it can be inverted and then Theorem 5.2 follows applying (5.4). The proof of Equation (5.5) relies on the analysis of the integrable system developed by Nazarov–Sklyanin in [16].

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