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Limit shapes for skew Howe duality

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Abstract. We study large random partitions boxed into a rectangle and coming from skew Howe duality, or alternatively from dual Schur measures. As the sides of the rectangle go to infinity, we obtain: 1) limit shape results for the profiles generalizing the Vershik–Kerov–Logan–Shepp curve; and 2) universal edge asymptotic results for the first parts in the form of the Tracy–Widom distribution, as well as less-universal critical regime results introduced by Gravner, Tracy and Widom. We do this for a large class of Schur parameters going beyond the Plancherel or principal specializations previously studied in the literature, parametrized by two real valued functions f and g. Connections to a Bernoulli model of (last passage) percolation are explored.

Keywords: random partition, limit shape, skew Howe duality, Tracy–Widom distribution, q-Krawtchouk ensemble

1 Introduction

Motivation. The study of random partitions has been, for the past 50 years, a fruitful area of research at the interface of algebraic combinatorics, representation theory, probability, and mathematical physics. Two cornerstone results are the limit shape result of Vershik–Kerov–Logan–Shepp and the Baik–Deift–Johansson Theorem providing random matrix-like asymptotic fluctuations for the longest increasing subsequence of random permutations. See [10, Ch. 1 and 2] and references therein.

In this extended abstract we generalize both of the above by looking at a class of measures called dual Schur measures with arbitrary parameters. Such measures fall into the category of Schur measures first introduced by Okounkov [9], and we use his techniques along the way. The novelty here is twofold. On one hand, we derive our asymptotic results keeping the two parameter sets generic and settle on a class indexed by two real functions. On the other hand, because we are using dual Schur measures, the partitions are forced to be contained within a certain rectangle, and we see this introduces special edge behavior in the so-called critical scaling regime case (discovered

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by Gravner, Tracy, and Widom in [2]), while allowing for universal fluctuations of the Tracy–Widom GUE type away from this regime. As our parameters are reasonably generic, this can be seen as a form of universality.

Combinatorial description. The Cauchy identity is a classical identity in algebraic combinatorics that has a representation-theoretic interpretation as characters computed in two different ways on the $\mathrm{GL}_n \times \mathrm{GL}_k$ representation $\mathrm{Sym}(\mathbb{C}^n \boxtimes \mathbb{C}^k)$. Howe duality [4] is the statement that this has a multiplicity free decomposition indexed by a single partition (rather than a pair). Using this, we can form a probability measure on partitions, a form of Okounkov's Schur measure [9], coming from two positive specialization of Schur polynomials $s_\lambda(X)$ (see [11, Ch. 7] for more on them). The Schur measure and the associated combinatorics have been well-studied from various perspectives and with a broad range of applications (*e.g.*, [1, 9]). In particular, it is a quintessential example of a determinantal point process, where the joint correlation kernel is described by a determinant of pairwise correlations.

There is another $\mathrm{GL}_n \times \mathrm{GL}_k$ representation $\bigwedge(\mathbb{C}^n \boxtimes \mathbb{C}^k)$ that has a multiplicity free irreducible decomposition [4]. This is known as *skew Howe duality*. Taking characters yields the classical dual Cauchy identity, which can be described by applying the involution ω, defined by $\omega s_\lambda(X) = s_{\lambda'}(X)$, to the Cauchy identity. This leads to the central object of our study, the measure on partitions given by

$$\mu_{n,k}(\lambda | X, Y) = \prod_{i,j=(1,1)}^{(n,k)} (1 + x_i y_j)^{-1} s_\lambda(X) s_{\lambda'}(Y), \tag{1.1}$$

where $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_k)$. Now the partition λ is constrained to a $n \times k$ rectangle. Various specializations of this measure have appeared previously under different names: the q-Krawtchouk ensembles (see, *e.g.*, [5, 7, 8]) or oriented digital boiling (see, *e.g.*, [3, 2]).

Main results. In this paper we study the asymptotic behavior of $\mu_{n,k}(\lambda)$ in the following sense. We first study the limit shape profile of λ as $n, k \to \infty$ with $\lim k/n \in (0, \infty)$ and then the asymptotic behavior of the largest part λ_1 (or a closely associated quantity if $\lambda_1 = k$ by our constraints). Our main results depend on two real-valued functions $f, g \colon [0, 1] \to [0, \infty)$ — see (1.2), which we take smooth[1] and satisfying certain integrability conditions, see Section 3 and Theorem 3.1.

Theorem 1.1. *Let $k, n \to \infty$ with $k/n \to c \in (0, \infty)$ and consider parameters*

$$x_i = f(i/n), \qquad y_j = g(j/k). \tag{1.2}$$

[1] This condition can be severely loosened, to the point we can take f, g to be step functions.

Modulo extra technical assumptions detailed in Section 3, the rescaled Russian-notation profile of a random λ sampled from $\mu_{n,k}(\cdot|X,Y)$ converges, point-wise in probability, to an explicit 1-Lipschitz function $\Omega(t)$ supported on an explicit interval $[x_-, x_+] \subseteq [-1, c]$.[2]

Our form for X and Y roughly means their histograms converge to continuous densities: $n^{-1}\sum_i \delta_{x_i}$ converges to the absolutely continuous (with respect to Lebesgue) measure $f(s)\mathrm{d}s$ (and likewise for the case of the y's and $g(s)\mathrm{d}s$) as $k, n \to \infty$ with $k/n \to c \in (0, \infty)$, where δ is the Dirac point mass. A picture of a limit shape is depicted in Figure 1.

We now turn to fluctuations around the limit shape. We note that the version $g \equiv 1$ of Theorem 1.2 below appeared in [3] and the case $f \equiv \alpha, g \equiv 1$ of Theorem 1.3 appeared in [2], with the authors calling the latter the *critical regime*.

Theorem 1.2. *With setup as in Theorem 1.1, consider the case when $x_+ < c$ (not full support). Let $L = \begin{cases} \lambda_1, & \text{if } \Omega \text{ convex around } x_+, \\ n - |\{i \mid \lambda_i = k\}|, & \text{if } \Omega \text{ concave around } x_+. \end{cases}$ Then, with F_{GUE} the Tracy–Widom GUE distribution [12] and for some explicit constant σ, we have:*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L - x_+ n}{\sigma^{-1} n^{1/3}}\right) = F_{GUE}(s). \tag{1.3}$$

Theorem 1.3. *With the scaling of Theorem 1.1, in the critical full support case of $x_+ = c$ and for $\Delta \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \mathbb{P}(\lambda_1 - nc \leq -\Delta) = \det_{0\leq i,j\leq \Delta-1} \left(\delta_{i,j} - K_{\mathrm{crit}}(i,j)\right) \tag{1.4}$$

with the matrix $K_{\mathrm{crit}}(i,j) = \sum_{\ell=0}^{(\Delta-j-1)/2} \begin{cases} \frac{1}{2\pi}\frac{1}{\ell!}\sin\frac{\pi(j-i)}{2}\Gamma(\ell + \frac{j-i}{2}), & \text{if } \ell + \frac{j-i}{2} \notin \mathbb{Z}_{\leq 0}, \\ \frac{1}{2}\frac{(-1)^\ell}{\ell!(\frac{i-j}{2}-\ell)!}, & \text{if } \ell + \frac{j-i}{2} \in \mathbb{Z}_{\leq 0}. \end{cases}$

Connections to last passage percolation. The statistic λ_1 for λ from $\mu_{n,k}(\cdot|X,Y)$ has the following interpretation. Consider the discrete (i,j)-grid $\{1,\dots,n\} \times \{1,\dots,k\}$. At each point (i,j) place a Bernoulli $0,1$ random variable ω_{ij} distributed as $\mathbb{P}(\omega_{ij} = 1) = \frac{x_i y_j}{1+x_i y_j}$. In Figure 1 (bottom left) the non-zero 1's are represented as solid dots. Let G be the length of the maximizing up-right path from $(0,0)$ to $(n+1, k+1)$[3] in the following sense: the path is only allowed to go strictly up and weakly to the right, and has to maximize the number of 1's (solid dots in *fig. cit.*) encountered. Such a model is a variant of the Bernoulli last passage percolation in a random environment. Then the Robinson–Schensted–Knuth correspondence and Schensted's theorem [11] yields the following.

Proposition 1.4. *$G = \lambda_1$ in distribution, where λ is distributed according to $\mu_{n,k}(\cdot|X,Y)$.*

[2]It means that $\Omega(t)$ is a straight line with slope ± 1 depending on the parameters for t outside $[x_-, x_+]$.
[3]The start and end points are dummy locations for convenience.

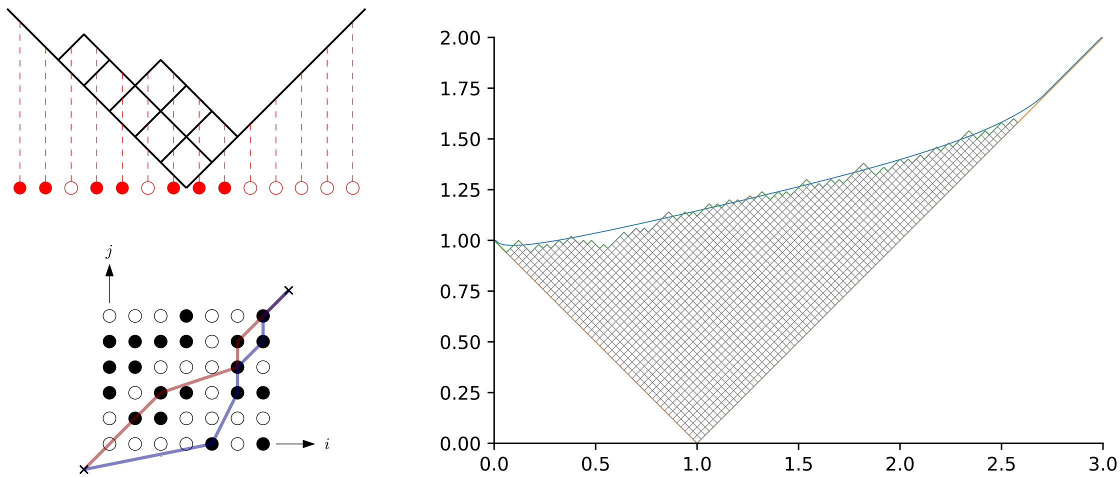

Figure 1: (Top left) Russian and Maya diagram for partition $(2^3, 1^2)$; (bottom left) two maximizing paths ($G = 5$) in the Bernoulli percolation model described in the introduction (the random variables $= 1$ are solid dots); (right) a random λ from $\mu_{n,k}(\lambda|X, Y)$, the limit shape (green) superimposed ($n = 50$, $k = 100$, $f(x) = 3x, g(x) = x$).

2 Free fermions and finite-size correlations

We briefly recall the free fermion Fock space and half vertex operators as a method to construct the skew Schur functions as matrix elements. Using this, we derive the correlation kernel though an application Wick's theorem.

A *partition* is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_i \geq \lambda_{i+1} \geq 0$ for all $i \geq 1$ with only finitely many positive entries. *Fermionic Fock space* (of charge 0) $\mathcal{F}_0$ is a span of certain vectors indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ defined by $|\lambda\rangle = v_{a_1} \wedge v_{a_2} \wedge v_{a_3} \cdots$, where $(a_i = \lambda_i - i + \frac{1}{2})_{i=1}^{\infty}$ is the *Maya sequence* of λ, in a semi-infinite wedge space. These vectors are obtained from the vacuum vector $|0\rangle = -\frac{1}{2} \wedge -\frac{3}{2} \wedge -\frac{5}{2} \wedge \cdots$ by the sequence of actions of the fermionic creation operators ψ_m that adjoin $m \in \mathbb{Z} + \frac{1}{2}$ to the Maya sequence with the proper sign, and fermionic annihilation operators ψ_m^* that remove m. They satisfy the canonical anti-commutation relations $\{\psi_m, \psi_\ell^*\} = \delta_{m,\ell}$.[4] Let $\langle\mu|$ denote the dual basis element with the natural pairing $\langle\mu|\lambda\rangle = \delta_{\mu,\lambda}$. From these operators we construct their generating series $\psi(z) = \sum_{m\in\mathbb{Z}+1/2} \psi_m z^m, \psi^*(w) = \sum_{m\in\mathbb{Z}+1/2} \psi_m^* w^{-m}$, as well as the half-vertex operators $\Gamma_\pm(X) = \exp \sum_{j\geq 1} \frac{p_j(x)\alpha_{\pm j}}{j}$ with $p_j(X) = \sum_i x_i^j$ and

[4] Here $\{a, b\} = ab + ba$, and all other anti-commutators vanish: $\{\psi_m, \psi_\ell\} = \{\psi_m^*, \psi_\ell^*\} = 0$.

$\alpha_j = \sum_{m\in\mathbb{Z}+1/2} \psi_{m-j}\psi_m^*$. The following properties are useful:

$$[\Gamma_+(X),\Gamma_+(Y)] = [\Gamma_-(X),\Gamma_-(Y)] = 1, \quad [\Gamma_+(X),\Gamma_-(Y)] = H(X;Y)$$
$$\Gamma_+(X)|0\rangle = |0\rangle, \qquad \langle 0|\Gamma_-(Y) = \langle 0|,$$
$$\mathrm{Ad}_{\Gamma_\pm(X)}\psi(z) = H(X;z^{\pm})\psi(z), \quad \mathrm{Ad}_{\Gamma_\pm(X)}\psi^\dagger(w) = H(X;w^{\pm 1})^{-1}\psi^\dagger(w), \tag{2.1}$$
$$\langle 0|\psi(z)\psi^\dagger(w)|0\rangle = \sum_{\ell=0}^{\infty} z^{-1/2-\ell}w^{1/2+\ell} = \frac{\sqrt{zw}}{z-w} \quad \text{for } |z|<|w|$$

where $H(X;Y) := \prod_{i,j}\frac{1}{1-x_iy_j}$, $[u,v] := uvu^{-1}v^{-1}$, and $\mathrm{Ad}_u x := uxu^{-1}$. We denote $\Gamma'_\pm(Y) := \Gamma_\pm^{-1}(-Y)$ and $E(X;Y) = \prod_{i,j}(1+x_iy_j) = H(X;-Y)^{-1}$.

Wick's theorem allows us to write matrix elements $\langle\mu|\Theta|\lambda\rangle$, for some special operators Θ on $\mathcal{F}_0$, as determinants. In particular, Schur polynomials (and functions) are matrix elements of the $\Gamma_\pm$ operators: $\langle\lambda|\Gamma_-(X)|0\rangle = \langle 0|\Gamma_+(X)|\lambda\rangle = s_\lambda(X)$ and $\langle\lambda|\Gamma'_-(X)|0\rangle = \langle 0|\Gamma'_+(Y)|\lambda\rangle = s_{\lambda'}(Y)$ (the determinants are the (dual) Jacobi–Trudi formulas for the Schur functions). Finally, using (2.1), the dual Cauchy identity becomes

$$\sum_\lambda s_\lambda(X)s_{\lambda'}(Y) = \langle 0|\Gamma_+(X)\Gamma'_-(Y)|0\rangle = E(X;Y). \tag{2.2}$$

Furthermore, again using Wick's theorem, we obtain the following form of Okounkov's correlations for the dual Schur measure:

Proposition 2.1 (Correlation kernel, [9])**.** *Let λ be random sampled from $\mu_{n,k}(\lambda|X,Y)$. Fix $a_i \in \mathbb{Z}+\frac{1}{2}$, $1\le i\le p$. The associated particles (Maya sequence) form a determinantal ensemble:*

$$\mu_{n,k}(\lambda \text{ has a particle at position } a_i, 1\le i\le p|X,Y) = \det\big[\mathcal{K}(a_i,a_j)\big]_{i,j=1}^p, \tag{2.3}$$

with the correlation kernel $\mathcal{K}(m,m')$ given by the integral representation

$$\mathcal{K}(m,m') = \oint\oint_{|w|<|z|} \frac{dz}{2\pi\mathbf{i}z}\frac{dw}{2\pi\mathbf{i}w}\frac{F(z)}{F(w)}\frac{w^{m'}}{z^m}\frac{\sqrt{zw}}{z-w}, \quad F(z) := \prod_{i=1}^{n}\frac{1}{1-x_iz}\prod_{j=1}^{k}\frac{1}{1+y_j/z} \tag{2.4}$$

where both contours encircle 0 and the z contour encircles all $-y_j$'s while excluding all $1/x_i$'s.

3 Asymptotics for the limit shape

We discuss how to obtain the limit shape alluded to in Theorem 1.1. We look at the behavior of $\mu_{n,k}(\lambda|X,Y)$ as we take the limit $n,k\to\infty$ such that $\lim\frac{k}{n} = c\in(0,\infty)$. The partition, drawn in Russian notation, becomes a piecewise-linear function with slope -1 (resp. 1) if a particle is in (resp. not in) the interval $[i,i+1]$. Rescaling by n^{-1}, our partition lies in the interval $[-1, c + o(1)]$ since $k \sim nc$. The limit density of particles turns out to be well-defined and gives the desired limit curve Ω.[5] Thus if we show that there exists a limiting density $\rho(t)$, then we can compute the limit shape by

$$\Omega(u) = 1 + \int_{-1}^{u} \mathrm{d}t \big(1 - 2\rho(t)\big). \tag{3.1}$$

We demonstrate this by showing that outside of an interval we call the support, the limit density is constant (0 or 1), while inside it converges to a discrete version of the random matrix sine kernel.

We make the following assumptions on the functions f, g: 1) $f, g \colon [0,1] \to [0,\infty)$ are smooth; 2) the Riemann integrals $\int_0^1 \frac{\mathrm{d}s}{(1-f(s)z)^2}$, $\int_0^1 \frac{\mathrm{d}s}{(z+g(s))^2}$ converge for $z \in \mathbb{C}$ away from the poles; and 3) the equation

$$\int_0^1 \mathrm{d}s \frac{f(s)z}{(1-f(s)z)^2} - c\int_0^1 \mathrm{d}s \frac{g(s)z}{(z+g(s))^2} = 0 \tag{3.2}$$

has two roots $z_\pm \in \mathbb{C} \cup \{\infty\}$. Furthermore we set

$$x_\pm = \int_0^1 \mathrm{d}s \frac{f(s)z_\pm}{1-f(s)z_\pm} + c\int_0^1 \mathrm{d}s \frac{g(s)}{z_\pm + g(s)}. \tag{3.3}$$

Theorem 3.1 (Bulk asymptotics). *With parameters $x_i = f(i/n), y_j = g(j/k)$, as $k, n \to \infty$ with $k/n \to c \in (0,\infty)$, and assuming the conditions 1)–3) above on f, g. Then for $t \in [x_-, x_+] \subseteq [-1, c]$ and $m, m' \in \mathbb{Z} + \frac{1}{2}$ we have*

$$\lim_{n\to\infty} \mathcal{K}(nt+m, nt+m') = \begin{cases} \dfrac{\sin\big(\varphi \cdot (m-m')\big)}{\pi(m-m')} & \text{if } m \neq m', \\ \dfrac{\varphi}{\pi} & \text{if } m = m', \end{cases} \tag{3.4}$$

where $\varphi(t) = \arg z(t)$ is given by the argument of the solution of

$$\int_0^1 \mathrm{d}s \frac{f(s)z}{1-f(s)z} + c\int_0^1 \mathrm{d}s \frac{g(s)}{z+g(s)} - t = 0. \tag{3.5}$$

Outside of $[x_-, x_+]$, $\lim_{n\to\infty} \mathcal{K}(nt+m, nt+m') \in \{0, 1\}$.

We then define $\rho(t) = \lim_{n\to\infty} \mathcal{K}(nt, nt)$ for $t \in [x_-, x_+]$; for $t < -1$ (resp. $t > c$), we have $\rho(t) = 1$ (resp. $\rho(t) = 0$); for $t \in [-1, x_-] \cup [x_+, c]$ the situation is more complicated as it depends on the functions and c, but $\rho(t)$ is still identically 0 or 1.

[5] The convergence of the corresponding (rescaled) functions to the limit shape is a more difficult question that we do not consider here; see *e.g.*, [10, Ch. 1] for the case of the Plancherel measure.

We prove Theorem 3.1 by using $\mathcal{K}(nt,nt) = \oint\oint_{|w|<|z|} \frac{dz}{2\pi\mathbf{i}z}\frac{dw}{2\pi\mathbf{i}w} e^{n(S(z)-S(w))}\frac{\sqrt{zw}}{z-w}$ for the action $S(z) = \frac{1}{n}\ln F(z) - t\ln z$, which is asymptotically

$$S(z) \approx -\int_0^1 \mathrm{d}s \ln(1-f(s)z) - c\int_0^1 \mathrm{d}s \ln(1+g(s)/z) - x\ln z. \tag{3.6}$$

Only critical points of the action contribute to the integral giving $\mathcal{K}(nt,nt)$. If we have two complex conjugate roots $z,\bar{z}$ of the equation

$$z\partial_z S(z) = \int_0^1 \mathrm{d}s \frac{f(s)z}{1-f(s)z} + c\int_0^1 \mathrm{d}s \frac{g(s)}{z+g(s)} - t = 0, \tag{3.7}$$

then $\rho(t) = \frac{1}{\pi}\arg z$. The support of the density $[x_-,x_+]$ is determined by finding the solutions $t = x_\pm$ when $z = \bar{z}$ is a double critical point of the action. There are two such z's by our assumption, called $z_\pm$, and obtained by solving

$$(z\partial_z)^2 S(z) = \int_0^1 \mathrm{d}s \frac{f(s)z}{(1-f(s)z)^2} - c\int_0^1 \mathrm{d}s \frac{g(s)z}{(z+g(s))^2} = 0. \tag{3.8}$$

Plugging into (3.7) and solving for t, we obtain the two ends of the support $t = x_\pm$.

Example 3.2 (Equal parameters)**.** As our first example we consider $x_i = \alpha, y_j = 1$ (and hence $f \equiv \alpha, g \equiv 1$) for all i,j with fixed $\alpha \in \mathbb{R}_{>0}$. The measure (1.1) becomes

$$\mu_{n,k}(\lambda|\alpha) = \alpha^{|\lambda|}(1+\alpha)^{-nk} \dim V_{\mathrm{GL}_n}(\lambda) \dim V_{\mathrm{GL}_k}(\lambda') \tag{3.9}$$

and was considered in [2] in relation to a stochastic growth process. We note the ensemble for λ is Johansson's Krawtchouk orthogonal polynomial ensemble [5]. Substituting f,g into (3.7), (3.8) we get $x_\pm = \frac{\alpha(c-1)\pm 2\sqrt{\alpha c}}{\alpha+1}$ and roots

$$z,\bar{z} = \frac{\alpha(c-1) + t(1-\alpha) \pm \sqrt{4\alpha(t+1)(t-c) + (\alpha(c-1)+t(1-\alpha))^2}}{2\alpha(t+1)}. \tag{3.10}$$

Therefore the limit density for $t \in [x_-,x_+]$ is $\rho(t) = \frac{1}{\pi}\arg z = \frac{1}{\pi}\arccos\left(\frac{\alpha(c-1)+t(1-\alpha)}{2\sqrt{\alpha(c-t)(t+1)}}\right)$.

Example 3.3 (q-weights I)**.** Consider now the principal specialization of characters $x_i = q^{i-1}, y_j = q^{j-1}$ related to q-dimensions of the corresponding representations. We have

$$\mu_{n,k}(\lambda|q) = \frac{q^{\|\lambda\|}\dim_q\left(V_{\mathrm{GL}_n}(\lambda)\right)\cdot q^{\|\overline{\lambda}'\|}\dim_q\left(V_{\mathrm{GL}_k}(\overline{\lambda}')\right)}{\prod_{i=1}^n\prod_{j=1}^k(q^{i-1}+q^{j-1})}, \tag{3.11}$$

where $\|\lambda\| = \sum_{i=1}^n (i-1)\lambda_i$ and $\overline{\lambda}$ means the complement of λ inside the $n\times k$ rectangle. The q-dimension $\dim_q$ of the irreducible GL_n representation is defined in, *e.g.*, [6, §10.10].

Using explicit formulas for q-dimensions, one can show that the point ensemble for λ is the so-called q-Krawtchouk orthogonal polynomial ensemble; see [7] for details. To compute the limit shape, we fix $\gamma > 0$ and set $q = e^{-\gamma/n} \to 1$ as $n, k \to \infty$ (*i.e.*, $f(s) = g(s) = e^{-\gamma s}$). The rescaled profile for a random λ from $\mu_{n,k}(\lambda | X, Y)$ converges point-wise in probability to the limit shape given by (3.1) with $\rho(t)$ and $x_\pm$ given by

$$\rho(t) = \frac{1}{\pi} \arccos \left(\operatorname{sgn}(-\gamma) \frac{e^{\gamma - \frac{\gamma(t+1)}{2}}}{2} \frac{1 - e^{\gamma(c-1)}}{\sqrt{(1 - e^{\gamma(t+1)})(1 - e^{\gamma(c-t)})}} \right), \tag{3.12}$$

$$x_\pm = -\frac{\operatorname{sgn}(\gamma)}{\gamma} \ln \frac{3e^{(c+1)\gamma} - e^{c\gamma} - e^{\gamma} + 3 \mp 2\sqrt{2}\sqrt{(e^{\gamma} - 1)(e^{c\gamma} - 1)(e^{(c+1)\gamma} + 1)}}{(1 + e^{c\gamma})^2}. \tag{3.13}$$

Example 3.4 (q-weights II)**.** We can also consider $x_i = q^{i-1}, y_j = q^{j-1}$; μ has a similar form as above: $\mu_{n,k}(\lambda | q, q^{-1}) = \frac{q^{\|\lambda\|} \dim_q\left(V_{\mathrm{GL}_n}(\lambda)\right) \cdot q^{-\|\overline{\lambda}'\|} \dim_{1/q}\left(V_{\mathrm{GL}_k}(\overline{\lambda}')\right)}{\prod_{i=1}^n \prod_{j=1}^k (q^{i-1} + q^{1-j})}$. In the same regime as above as $q \to 1$ the limit density and support are given by

$$\rho(t) = \frac{1}{\pi} \arccos \left(\operatorname{sgn}(-\gamma) \frac{e^{\frac{\gamma}{2}(t+1-c)}}{2} \frac{1 - e^{\gamma c} - e^{\gamma(c-t-1)} + e^{\gamma(c-t)}}{\sqrt{(1 - e^{\gamma(t+1)})(1 - e^{\gamma(c-t)})}} \right), \tag{3.14}$$

$$x_\pm = -1 - \frac{1}{\gamma} \ln 2 + \frac{1}{\gamma} \ln \left(1 + e^{\gamma(c+1)} \pm \sqrt{(e^{2\gamma c} - 1)(e^{2\gamma} - 1)} \right). \tag{3.15}$$

Example 3.5. Take $f(s) = \alpha s, g(s) = s$. We do not have closed-form formulas anymore since the equations (3.7), (3.8) take the form: $-\frac{1}{\alpha z} \ln(1 - \alpha z) - 1 + cz \ln\left(\frac{z}{1+z}\right) + c - t = 0$, $\frac{(\alpha z - 1)\ln(1 - \alpha z) - \alpha z}{\alpha z(\alpha z - 1)} + c\left(z \ln\left(\frac{z}{z+1}\right) + \frac{z}{z+1}\right) = 0$. Yet by solving these numerically we can still find the support $[x_-, x_+]$ and the limit shape as shown for $\alpha = 3$ in Figure 1 (right).

For Examples 3.2, 3.3 and 3.4, limit shapes for various values of γ are presented in Figure 2. For $\gamma = 0$ we recover the constant specialization $x_i = y_j = 1$.

4 Boundary asymptotics

To study boundary asymptotics, we zoom in around x_+ and see what happens to the process locally. In the asymptotic analysis described in the previous section, the two roots $z, \overline{z}$ coincide to a double critical point we called z_+.

Sketch of proof of Theorem 1.2. Before we proceed further, we assume for simplicity c is such that $\int_0^1 \mathrm{d}s\, f(s) < c \int_0^1 \mathrm{d}s\, \frac{1}{g(s)}$. This means, *a posteriori*, in the notation of Theorem 1.2, that we are looking at fluctuations for $\lambda_1 \ll k$. In other words, the limit shape

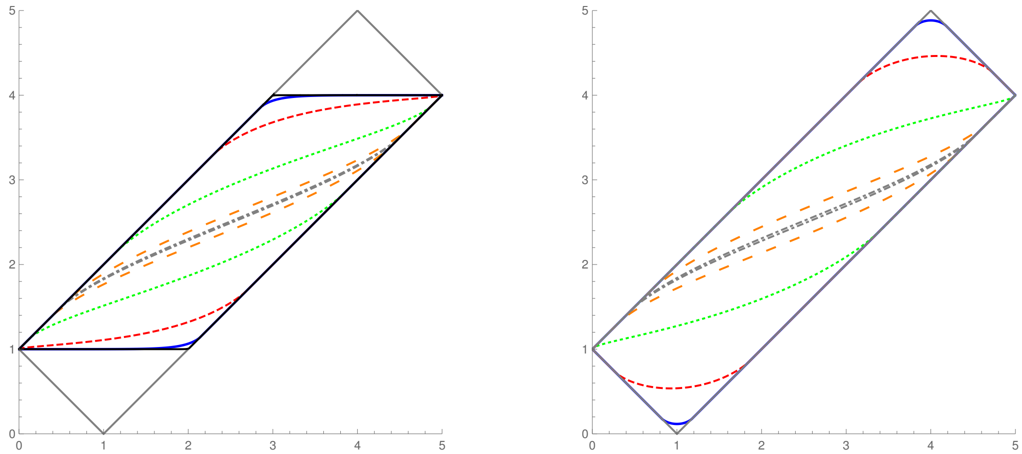

Figure 2: Plots of the limit shapes for Young diagrams corresponding to the densities (3.12) (on the left) and (3.14) (on the right) for $c = 4$ and the values of γ (bottom to top): -10 (solid blue), -2 (dashed red), -0.5 (dotted green), -0.1 (sparsely dashed orange), -0.01 (dot-dashed gray), 0.01 (dot-dashed gray), 0.1 (sparsely dashed orange), 0.5 (dotted green), 2 (dashed red), 10 (solid blue). Solid black lines on the left panel correspond to $\gamma = \pm\infty$ ($q = \text{const}$). We have shifted $t \mapsto t + 1$.

$\Omega(t)$ is convex around x_+, and in turn $x_+ < c$, so we do not have full support for Ω. In Figure 2, we are dealing with a limit shape *sitting under the diagonal of the rectangle*.

Near the double critical point $z_+ \in \mathbb{R} - \{0\}$ we have $S(z) = S'''(z_+)(z - z_+)^3/6 + \mathcal{O}[(z - z_+)^4]$. Scaling around $z = z_+ e^{\sigma\xi n^{-1/3}} \approx z_+(1 + \sigma\xi n^{-1/3} + \cdots)$ as $n \to \infty$, with σ a yet-to-be-determined constant, we see that we need to scale the matrix entries of $K(m, m')$ as $(m, m') \approx x_+ n + (\xi, \eta) n^{1/3}\sigma^{-1}$. Let $\sigma^{-1} = 2^{-1/3} S'''(z_+)^{1/3} z_+ > 0$. Then one can show

$$n^{1/3}\mathcal{K}(m, m') \to \mathcal{K}_{\text{Airy}}(\xi, \eta) := \frac{1}{(2\pi\mathbf{i})^2}\int_{i\mathbb{R}-\epsilon} \mathrm{d}\omega \int_{i\mathbb{R}+\epsilon} \mathrm{d}\zeta\, \frac{\exp(\zeta^3/3 - \xi\zeta - \omega^3/3 + \eta\omega)}{\zeta - \omega}, \tag{4.1}$$

where $0 < \epsilon \ll 1$ and $\mathcal{K}_{\text{Airy}}(\xi, \eta)$ is an integral representation of the Airy kernel [12].

For a quick justification notice that

$$n\big(S(z) - S(w)\big) \approx \frac{1}{6}\sigma^3 z_+^3 S'''(z_+)(\zeta^3 - \omega^3) = \frac{1}{6}\sigma^3 \left((z\partial_z)^3 S(z)\Big|_{z=z_+} \right)(\zeta^3 - \omega^3) \tag{4.2}$$

for $z = z_+ e^{\sigma\xi n^{-1/3}}, w = z_+ e^{\sigma\eta n^{-1/3}}$ in the vicinity of z_+, and we are therefore interested in

$$(z\partial_z)^3 S(z)\Big|_{z=z_+} = \int_0^1 \mathrm{d}s \left(\frac{2f^2(s) z_+^2}{(1 - f(s)z_+)^3} + \frac{2cg(s) z_+^2}{(z_+ + g(s))^3} \right), \tag{4.3}$$

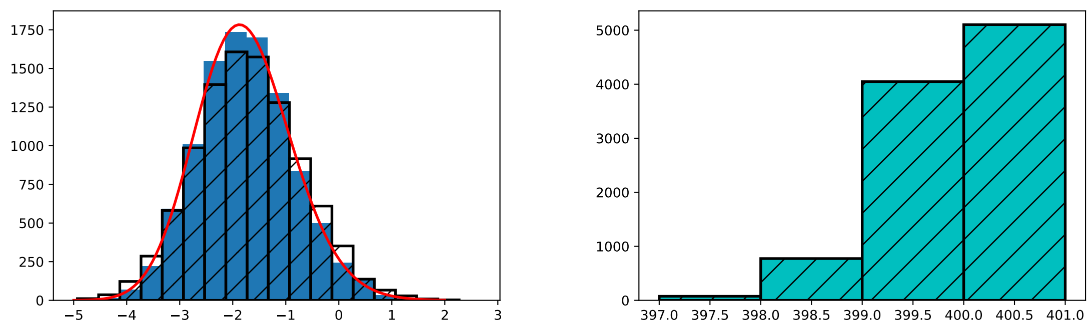

Figure 3: (Left) Illustration of Theorem 1.2: histogram of first row lengths in black ($n = 200$, $k = 400$, $f \equiv g \equiv 1$, 10000 samples), Tracy–Widom distribution (red), and Tracy–Widom histogram (blue, 10000 samples). (Right) Histogram of 10000 samples of first row lengths in the critical case of Theorem 1.3 ($n = 200$, $k = 400$, $f(x) = \frac{3}{2}x^2, g(x) = 2x^{-1}$).

to determine σ. Assuming this integral converges, we can conclude the convergence from (4.1). Further technical estimates along the same lines show that Fredholm determinants converge to Fredholm determinants, and the Tracy–Widom distribution is a Fredholm determinant of the Airy kernel on $\mathcal{L}^2(s, \infty)$. Therefore we conclude that the distribution of $\frac{\lambda_1 - x_+ n}{\sigma^{-1} n^{1/3}}$ is asymptotically the Tracy–Widom distribution GUE distribution. See Figure 3 (left) for an illustration.

Remark 4.1. In the case of Example 3.2 ($f \equiv \alpha, g \equiv 1$), we have $x_\pm = \frac{\alpha(c-1) \pm 2\sqrt{\alpha c}}{\alpha + 1}$ and $z_+ = \frac{\alpha(c+1) - (\alpha+1)\sqrt{\alpha c}}{\alpha(\alpha c - 1)}$. This yields (recall $c, \alpha > 0$) $\sigma = \frac{(\alpha+1)c^{\frac{1}{6}}}{\alpha^{\frac{1}{6}}(\sqrt{c}-\sqrt{\alpha})^{\frac{2}{3}}(1+\sqrt{\alpha c})^{\frac{2}{3}}}$. We see that $\sigma < \infty$ whenever $\alpha \neq c$.

Remark 4.2. Continuing on the previous remark, $\alpha = c$ implies $\sigma = \infty$. Formally this would yield an infinitely thin Tracy–Widom distribution; in practice, we obtain the critical case of Theorem 1.3 (first discussed in [2, Section 3.2], as the models are the same). This happens when λ_1 hits the east-most corner of the rectangle bounding it; from the point of view of asymptotic analysis, the critical point considered is $z_+ = 0$, a singularity of the action.

Sketch of proof of Theorem 1.3. Now we turn to Theorem 1.3. We land in this case when there is an obstruction to our asymptotic analysis of the generic case. This happens when the double critical point z_+ of the action under consideration is $z_+ = 0$. In this situation the constant $\sigma = \infty$, and the integral in (4.3) diverges. Combinatorially, it happens when $\lambda_1 = nc - o(1)$ (when λ_1 just hits the east-most corner of the rectangle bounding it). In terms of our functions f and g, we get the following condition:

$$\int_0^1 \mathrm{d}s\, f(s) = c \int_0^1 \frac{\mathrm{d}s}{g(s)}. \tag{4.4}$$

Notice that $x_+ = c$ is a root of (3.7) for $z = 0$, and then (3.8) gives the above condition.

Now we cannot pass the contours through the double critical point at 0 anymore (being a singularity of the action). This also implies that our limiting distribution will be discrete: we will not scale the m, m' coordinates continuously. Instead we take $m = cn - h, m' = cn - h'$ for some $h, h' \in \mathbb{N} + \frac{1}{2}$ and consider the asymptotics of $K(m, m')$ as such. The contours encircle 0 and satisfy $|w| < |z|$. The w contour can be made an arbitrary small circle around 0. Since the contour for z encircles all $-y_j$'s and does not contain all $1/x_i$'s, we can deform it to be from $-\mathbf{i}\infty$ to $\mathbf{i}\infty$ with a small bump to the right of 0 that contains contour for w. We first integrate over w. Since we have a critical point of the action at 0, $S'(0) = 0$, we can approximate the action as $-nS(w) \approx -nS(0) - \frac{n}{2}S''(0)w^2$; we thus need to compute the integral $\oint_{C_\varepsilon} e^{-\frac{n}{2}S''(0)w^2} w^{-h'} \frac{\sqrt{zw}}{z-w} \frac{\mathrm{d}w}{2\pi\mathbf{i}w}$. As $|w| < |z|$, we can expand $\frac{1}{z-w}$ and $e^{-\frac{n}{2}S''(0)w^2}$ into power series in w and obtain

$$\oint\limits_{C_\varepsilon} \frac{\mathrm{d}w}{2\pi\mathbf{i}w} e^{-\frac{n}{2}S''(0)w^2} w^{-h'} \frac{\sqrt{zw}}{z-w} = \oint\limits_{C_\varepsilon} \frac{\mathrm{d}w}{2\pi\mathbf{i}w} \left(\sum_{\ell=0}^{\infty} \frac{(-1)^\ell n^\ell (S''(0))^\ell}{2^\ell \ell!} w^{2\ell} \right) w^{-h'+\frac{1}{2}} \left(\sum_{j=0}^{\infty} \frac{w^j}{z^{j+\frac{1}{2}}} \right).$$

After the residue calculation we get $\sum_{\ell=0}^{\lfloor h'/2-1/4 \rfloor} \frac{(-1)^\ell n^\ell (S''(0))^\ell}{2^\ell \cdot \ell!} z^{-h'+2\ell}$. To integrate over z, we note that for purely imaginary z we have $\operatorname{Re} S(z) < 0$ and therefore as $n \to \infty$ main contribution comes from the bump around 0. We again can approximate $nS(z) \approx nS(0) + \frac{n}{2}S''(0)z^2$ and we cancel $nS(0)$ from the w-integral, so we need to compute the integral $\int_{-\mathbf{i}\infty}^{\mathbf{i}\infty} \frac{\mathrm{d}z}{2\pi\mathbf{i}z} e^{\frac{n}{2}S''(0)z^2} z^h \left(\sum_{\ell=0}^{\lfloor h'/2-1/4 \rfloor} \frac{(-1)^\ell n^\ell (S''(0))^\ell}{2^\ell \cdot \ell!} z^{-h'+2\ell} \right)$. We do a substitution $z = -\frac{\mathbf{i}\sqrt{2t}}{\sqrt{nS''(0)}}$; the new integration contour starts at $+\infty$, loops around 0 and then returns to $+\infty$ and we have:

$$\mathcal{K}_n(h, h') = \frac{1}{2\pi} \sum_{\ell=0}^{\lfloor h'/2-1/4 \rfloor} \frac{(-1)^{(h-h'+1)/2} 2^{(h-h')/2-1}}{(nS''(0))^{(h-h')/2} \ell!} \int_{\infty}^{0^+} \mathrm{d}t\, t^{(h-h')/2+\ell-1} e^{-t}. \tag{4.5}$$

If $(h - h') + l \geq 0$, last integral is Hankel's integral representation for the Γ-function $\Gamma(z) = -\frac{1}{2\mathbf{i}\sin \pi z} \int_{\infty}^{0^+} \mathrm{d}t\, (-t)^{z-1} e^{-t}$; otherwise we obtain $\frac{(-1)^{\frac{h-h'}{2}+l} 2\pi\mathbf{i}}{\left(\frac{h'-h}{2} - l\right)!}$ from the residue calculation. Thus the correlation kernel coincides with the formula [2, (3.19)] up to a normalization. To find the probability $\mathbb{P}(\lambda_1 - nc \leq -\Delta)$, we need to compute the determinant $\det(\delta_{i,j} - \mathcal{K}(i + 1/2, j + 1/2))_{i,j=0}^{\Delta-1}$ by inclusion-exclusion[6]; the prefactor powers with exponent proportional to $i - j$ cancel and we conclude our proof.

[6] We are looking at a gap probability.

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